IMPOSSIBILITY OF A WALRASIAN BARGAINING SOLUTION[1]

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Abstract

Is there a bargaining solution that pays out the Walrasian welfare for exchange economies? We show that there is none, for there are distinct exchange economies whose Walrasian equilibrium welfare payoffs disagree but which define the same bargaining problem and should have hence determined the same bargaining solution and its payoffs.

Keywords: Walrasian equilibrium, bargaining, implementation via bargaining.
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1 Introduction

It is well-known that the competitive (price-taking, or Walrasian) solution of an economy may fail to coincide with the Nash bargaining solution of the bargaining problem determined by this economy.[1] It seems to have escaped our records, however, that *no bargaining solution can guarantee a competitive welfare distribution*, so long as we insist on our definitions of 'bargaining problem' and 'bargaining solution' as modelled by Nash (1950). It is this impossibility (of a Walrasian bargaining solution) that we establish here.

We present a two-person economy where the space of possible utility improvements above autarky is symmetric with respect to the two agents, while the Walrasian solution allocates payoffs asymmetrically. In this economy, players have identical preferences. Hence, when we switch their initial endowments, the symmetric bargaining problem does not change, while the asymmetric Walrasian payoff vector does, showing that there cannot be a Walrasian bargaining solution, a function that maps each bargaining problem to a payoff vector.

In this example, while the initial endowments are symmetric and the agents have identical preferences, the trade-off between the two commodities differs from unity in the region where a tangency of the agents' indifference curves can occur with an exchange passing through autarky. This is visible to the Walrasian solution, but hidden from the bargaining solution. This suggests that bargaining models should permit specification of possible physical states over which bargaining actually takes place, and not just the image of this space under the utility functions of the bargainers — a point already made by other authors, such as Binmore (1987), in a debate going back to Harsanyi (1977). (In contrast with this position, only the payoffs are relevant in the models of Nash (1950) and Rubinstein (1982).)

2 Formalizing Basic Notions

For a sufficiently rich class E of n-person pure trade economies e possessing Walrasian equilibria, we will give precision to the idea of a bargaining problem $b(e) = (v, V)$ determined by e, where $V \subseteq \mathbb{R}^n$ is the set of feasible utility allocations for e and where

[1] E.g., Binmore (1987, pp.240) noted this for pure exchange economies. It was also a subject of homework and examinations at Boğaziçi University (Econ 321, Mathematical Economics, 1983 – 1994) (Sertel, 1983).

$v \in V$ is the no-trade utility allocation (of the initial endowments). By a bargaining solution we will mean any function s that picks a point $s(v, V) \in V$ in the bargaining set V of each such bargaining problem. Writing $W(e)$ for the (non-empty) set of utility payoffs at the Walrasian allocations of each economy $e \in E$, we ask whether there exists a bargaining solution s that satisfies

$$s(b(e)) \in W(e)$$

at every $e \in E$, a function that we would call a *Walrasian bargaining solution.*

In a sense, this is the question of whether W can be "*implemented*" via some bargaining solution s, i.e. whether there exists a bargaining solution s that makes Diagram 1 commute, satisfying the functional relation $s \circ b \in W$. Given bargaining problem $b(e)$, presumably there exists a bargaining environment in which $s\,(b(e))$ is the natural outcome. Hence, when Diagram 1 commutes, we can implement W by providing such a bargaining environment, thus obtaining the Walrasian payoffs as the natural outcome in that environment.

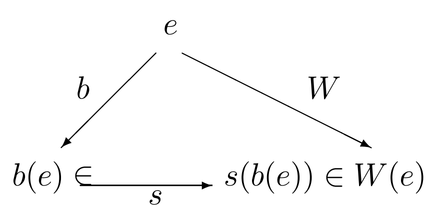

Diagram 1 (Bargaining) "Implementation" of an economic welfare correspondence W via a bargaining solution s.

We show that there is no such bargaining solution, and indeed we show this with three separate notions [$\beta(e)$, $\beta^{\varepsilon}(e)$ and $\beta^{\omega}(e)$, which we are about to define] of the bargaining problem $b(e)$ determined by an economy $e \in E$. The proof consists in presenting an example of two distinct pure trade economies e and e' in E for which the bargaining problems coincide, i.e. $b(e) = b(e')$, but for which $W(e) \cap W(e') = \emptyset$. For such two economies, the bargaining solution outcomes must coincide, while the Walrasian payoffs cannot agree. This is true in each of the three cases β, β^{ε} and β^{ω} of b.

Formally, let E be the set of all n-person pure trade economies $e = \{(u_i, \bar{x}_i)\}_{i \in N}$ with some Walrasian equilibrium $(p, \mathbf{x})$, where $N = \{1, \ldots, n\}$ is the agent set, $\bar{x}_i \in X$ is agent i's initial endowment in a consumption space $X = \mathbb{R}^m_+$ (the non-negative cone of some m-dimensional Euclidean space $\mathbb{R}^m$), $u_i : X \to \mathbb{R}$ is i's utility function ($i \in N$), $p \in \mathbb{R}^m_+$ is a strictly positive Walrasian price vector, and $\mathbf{x} = \{x_i\}_{i \in N}$ is a Walrasian allocation at p. Note that each agent i demands x_i at price p and the markets clear. At each economy $e \in E$, denote the utility (vector) of any allocation $\mathbf{x}$ by $\mathbf{u}(\mathbf{x}) = \{u_i(x_i)\}_{i \in N}$, and write

$$W(e) = \{\mathbf{u}(\mathbf{x}) | (p, \mathbf{x}) \text{ is a Walrasian equilibrium of } e \text{ for some price } p \in \mathbb{R}^m_+\}$$

for the set of Walrasian utility payoffs of e. This defines a non-empty set-valued function $W : E \to 2^{\mathbb{R}^n}$, regarded as the *Walrasian welfare* (correspondence).

Recall that by a bargaining problem we mean any ordered pair $b = (v, V)$ with $v \in V \subseteq \mathbb{R}^n$, and by a bargaining solution we mean any function s that picks a point $s(b)$ out of the bargaining set V of every bargaining problem b.

Toward defining our bargaining problems at each economy $e \in E$, write

$$X(e) = \{\mathbf{x} \in X^N | \sum_{i \in N} x_i = \sum_{i \in N} \bar{x}_i\}$$

for the set of feasible allocations for e. For every $e \in E$, we first consider a bargaining problem

$$\beta(e) = (\mathbf{u}(\bar{\mathbf{x}}), \mathbf{u}(X(e))),$$

where the threat point is the utility $\mathbf{u}(\bar{\mathbf{x}})$ of the initial endowment of e, and the bargaining set is the set $\mathbf{u}(X(e))$ of all feasible utility allocations. This also defines a map $\beta : E \to \mathbb{R}^n \times 2^{\mathbb{R}^n}$.

The bargaining problem $\beta(e)$ is not as interesting as the following two bargaining problems: the "Edgeworthian" bargaining problem $\beta^\varepsilon(e)$ arising when we equate the bargaining set with the space of utility possibilities *subject to trade,* and the "Walrasian" bargaining problem $\beta^\omega(e)$ arising when we equate the bargaining set with the space of utility possibilities *subject to trade among price-takers*. To define $\beta^\varepsilon(e)$ and $\beta^\omega(e)$, we limit ourselves to the case where $n = m = 2$. In this case, we define

$$X^\varepsilon(e) = \{\mathbf{x} \in X(e) | \mathbf{u}(\mathbf{x}) \geq \mathbf{u}(\bar{\mathbf{x}})\}$$

as the set of feasible allocations subject to trade, and we define

$$X^\omega(e) = \{\mathbf{x} \in H\{\text{med}\{\bar{\mathbf{x}}, y_1, y_2\}, \bar{\mathbf{x}}\} | (y_1, y_2) \in D_1(p) \times D_2(p), p \in \mathbb{R}^m_+\}$$

as the set of feasible allocations subject to trade among price-takers, where H takes the closed convex hull, med finds the median,[2] and $D_i(p)$ is the demand set of i at $p \in \mathbb{R}^m_+$.[3] We write

$$\beta^{\varepsilon}(e) = (\mathbf{u}(\bar{\mathbf{x}}), \mathbf{u}(X^{\varepsilon}(e))),$$

$$\beta^{\omega}(e) = (\mathbf{u}(\bar{\mathbf{x}}), \mathbf{u}(X^{\omega}(e))),$$

thus defining maps β^{ε} and β^{ω} from E to $\mathbb{R}^n \times 2^{\mathbb{R}^n}$, determining bargaining problems, $\beta^{\varepsilon}(e)$ and $\beta^{\omega}(e)$, respectively, at each economy $e \in E$.

Observe that $X^{\varepsilon}(e)$ just gives the set of allocations that neither trader finds inferior to his initial endowment, i.e., the set of all possible allocations after "acceptable" trade. On the other hand, $X^{\omega}(e)$ rations trade by allowing only as much trade to occur, at any given price, as each price-taking agent will agree to. Thus, whichever is short, supply or demand, becomes binding here as a quantity constraint on the trade among price-takers. We regard $X^{\varepsilon}(e)$ as Edgeworthian because trade here is subject only to acceptability by the traders, while we regard $X^{\omega}(e)$ as Walrasian because trade here is subject to price-taking behavior vis-à-vis some given price.

3 The Impossibility Result

Proposition (Impossibility): *There is no Walrasian bargaining solution, whether we regard the natural notion of bargaining problem for trade to be that of $b = \beta$, $b = \beta^{\varepsilon}$, or $b = \beta^{\omega}$.*

Proof: For a given notion b of bargaining problem, a necessary condition for the existence of a Walrasian bargaining solution is the following: for any bargaining problem $b(e^*)$, we must have

$$\bigcap_{e \in E^*} W(e) \neq \emptyset$$

where $E^* = b^{-1}(b(e^*))$ is the set of all economies $e \in E$ with $b(e) = b(e^*)$. Our Example below presents two economies $e, e' \in E$ for which $b(e) = b(e')$ in each of the three cases,

[2] Consider any $A \subset \mathbb{R}^2$ such that, for each $(x^1, x^2), (y^1, y^2) \in A$, $x^1 \geq y^1 \iff x^2 \leq y^2$. Let $A_1 = \{a^1 | \exists a^2 : (a^1, a^2) \in A\}$.Then, $\text{med}(A)$ is defined by $\text{med}(A) \in \{(a^1, a^2) \in A | a^1 = \text{med}(A_1)\}$, where $\text{med}(A_1)$ is the usual median of the set A_1 of scalers.

[3] We are thankful to Matthew Jackson for discussions (at BWED XVI, the XVI*th* Bosphorus Workshop on Economic Design, The Bay of Fethiye, August 1993) regarding the use of the operator med in the formulation of $X^{\omega}(e)$ for trade by price-takers. This idea was used also by Barberá and Jackson (1995), an earlier version was presented at BWED XVI.

β, β^{ε}, and β^{ω}, of b, but for which $W(e) \cap W(e') = \emptyset$. Thus, the necessary intersection property fails for each of β, β^{ε}, and β^{ω}, and we conclude that there exists no Walrasian bargaining solution in either case. ■

Example: Taking $m = n = 2$, denote quantities of the first and second goods by x and y, respectively. Consider the economies $e, e' \in E$ given by

$$e = ((u, (0, 10)), (u, (10, 0))),$$

$$e' = ((u, (10, 0)), (u, (0, 10))),$$

with

$$u(x, y) = (1/45) \min\{24x + 3y + 15, 9x + 18y, 4x + 23y + 5\}$$

at each $(x, y) \in \mathbb{R}^m_+$. Thus, in the Edgeworth box of Figure 1, e is positioned at the top left-hand corner, and e' at the bottom right-hand corner. The Walrasian equilibrium price (with x as numéraire) is $p = (1, 2)$ for both e and e'. The set of Walrasian allocations is A for e and A' for e', where

$$A = \{((t, 10 - t/2), (10 - t, t/2)) | t \in [6, 22/3]\},$$

$$A' = \{((10 - t, t/2), (t, 10 - t/2)) | t \in [6, 22/3]\}.$$

Therefore, we have $W(e) = \{(4, 2)\} \neq \{(2, 4)\} = W(e')$ and, thus, the two economies disagree in their Walrasian welfare. In Figure 1, $X(e) = X(e')$ is the region of the entire box, $X^{\varepsilon}(e) = X^{\varepsilon}(e')$ is the region enclosed by $ecc'e'd'de$, while $X^{\omega}(e)$ is the region enclosed by $ecde$ and $X^{\omega}(e')$ is the region enclosed by $e'c'd'e'$. The welfare images of $X^{\omega}(e)$ and $X^{\omega}(e')$ also coincide, as displayed in Figure 2. Thus, we see that the bargaining problems $b(e)$ and $b(e')$ coincide in each of our three cases of b:

$$\beta(e) = \beta(e') = ((1, 1), H\{(0, 6), (6, 0)\} \cup H\{(1/5, 29/5), (1, 1), (29/5, 1/5)\}),$$

$$\beta^{\varepsilon}(e) = \beta^{\varepsilon}(e') = ((1, 1), \{u \in \mathbb{R}^2_+ | (1, 1) \leq u, u_1 + u_2 \leq 6\}) = \beta^{\omega}(e') = \beta^{\omega}(e).$$

To sum up, e and e' have disjoint sets of Walrasian payoffs, but they determine identical bargaining problems under either of our three notions, β, β^{ε} or β^{ω}, of bargaining problem b.

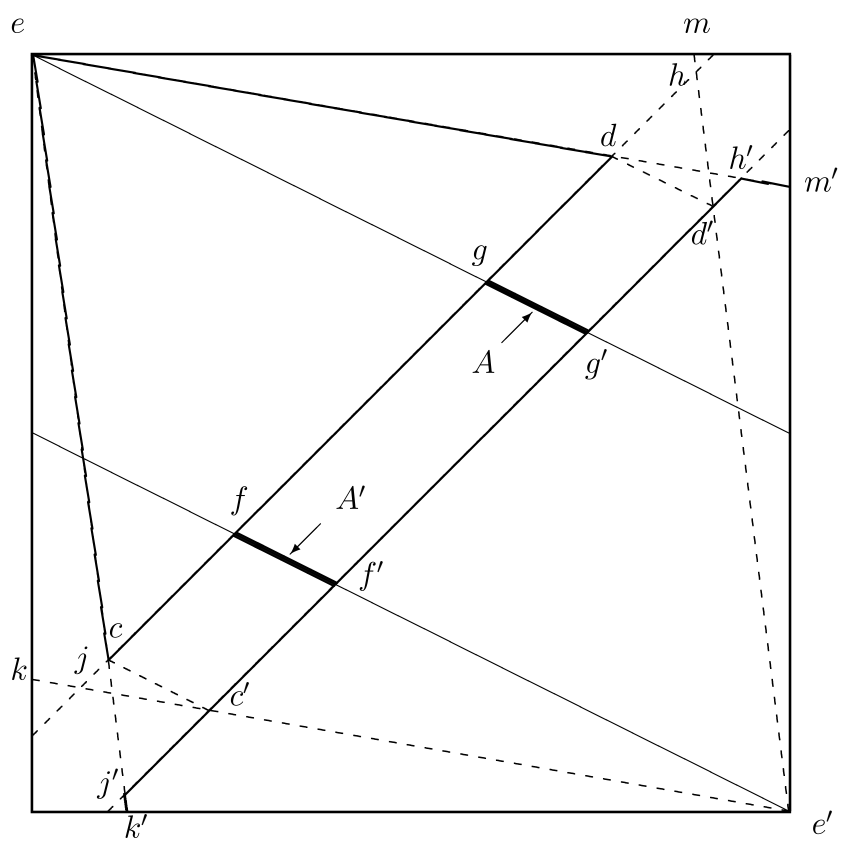

Figure 1: The Edgeworth box with e and e'. The offer curves (i.e., the consumption bundles demanded at some price) of Agent 1 and Agent 2 in economy e are the line segments connecting the points $[e, c, g, g', h', m']$ and $[e, d, g, g', j', k']$, respectively. In economy e' the offer curves of Agent 1 and Agent 2 are the line segments connecting the points $[e', c', f', f, h, m]$ and $[e', d', f', f, j, k]$, respectively. The sets of Walrasian allocations for economies e and e' are $A = H\{g, g'\}$ and $A' = H\{f, f'\}$ respectively.

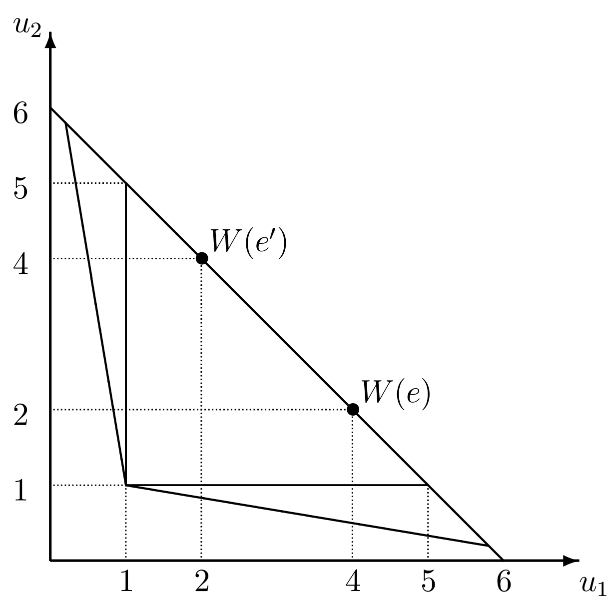

Figure 2: The welfare image of trade and bargaining at e and e'. $\mathbf{u}(X(e)) = \mathbf{u}(X(e')) = H\{(0,6),(6,0)\} \cup H\{(1/5,29/5),(1,1),(29/5,1/5)\}$, $\mathbf{u}(X^e(e)) = \mathbf{u}(X^e(e')) = \mathbf{u}(X^\omega(e')) = \mathbf{u}(X^\omega(e)) = H\{(1,1),(1,5),(5,1)\}$, while $W(e) = \{(4,2)\}$ and $W(e') = \{(2,4)\}$.

4 Closing Remarks

If one were to draw a lesson from the above, what might it best be? We do not think that the best lesson to draw is really the non-existence of a Walrasian bargaining solution (except in the strict sense of our Nash-like definitions, of course), but rather the inadequacy of the classical model of bargaining theory, pursuing Nash (1950), whose idea of a “bargaining problem" and of a “bargaining solution" we employed. With a suitable model it should in fact be possible for a bargaining solution to be Walrasian, but then it must be able to utilize information which the classical bargaining problem (Nash, 1950) does not carry.

In fact, Binmore (1987, pp.240), noting that “In general, ... the competitive equilibrium yields different payoffs to the players from the ‘Nash bargaining solution’...”, offered an adaptation of the Nash axioms to bargaining about trade (rather than about welfare distributions), and obtained an adapted solution concept which solves revised bargaining problems, at least in a special Edgeworth box (with $m = n = 2$), so as to give Walrasian welfare - a Walrasian bargaining solution, in a sense. Similarly, using information about the marginal rates of substitution and choosing the bargaining set and “the bargaining powers” accordingly, Sotskov (2003) derives the Walrasian payoffs as the Nash bargaining solution to the derived bargaining set with these “bargaining powers”.[4] Of course, the derived “bargaining solution” is no longer a bargaining solution in the sense of Nash (1950), as it uses information that is not incorporated in the bargaining problem.

Our point is that, under the limited information incorporated in a bargaining problem, not just the Nash bargaining solution, but, in fact, *any* bargaining solution will generally fail to agree with Walrasian welfare, as we showed here. For a happier note from a Walrasian viewpoint, however, we refer to the recent positive results by Yildiz (2001), who shows in a sequential bargaining model where agents alternate in offering prices, that the equilibrium prices and the allocation become Walrasian as the agents become very patient.

[4] Cf. Trockel (1996), who derives the Nash bargaining solution as the Walrasian payoffs in an artificial economy in which the utilities are traded. See also Trockel (2003).

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