Segregated Security Exchanges with Ex Ante Rights to Trade

A Market-Based Solution to Collateral-Constrained Externalities*

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Abstract

This paper studies a competitive general equilibrium model with default and endogenous collateralized contracts. The possibility of trade in spot markets creates externalities, as spot prices and the bindingness of collateral constraints interact. We propose a market based solution which overcomes the externalities problem and obviates the needs for any government policy intervention. If agents are allowed to contract ex ante on market fundamentals determining the state-contingent spot prices used to unwind collateral, over and above contracting on true underlying states of the world, then standard existence and welfare theorems apply, that is, competitive equilibria are equivalent with Pareto optima.

Keywords: Default; endogenous collateral; externalities; segregated exchanges; Walrasian equilibrium; limited commitment; financial crises

JEL Code: D52, D53, D61, D62.

1 Introduction

This paper uses a competitive general equilibrium with directly-collateralized and asset-backed securities to analyze the interaction between the endogenous valuation of collateral and corresponding default decisions. The interaction creates a "pecuniary externality", which causes a collateral-constrained equilibrium to be inefficient. The externality exists because the consumption feasibility set of an agent directly depends on the spot price and the prices of asset backed securities through the decisions of other agents in the contracting period. This impact on the feasible set in turn impacts the allocations of all agents whenever the collateral or borrowing constraints of any types of agents is binding.

The primary contribution of this paper is not the discovery of the externality, as this is now quite clear from a growing literature, (e.g., Allen and Gale, 2004; Caballero and Krishnamurthy, 2001, 2004; Farhi, Golosov, and Tsyvinski, 2009; Geanakoplos, 2003; Geanakoplos and Polemarchakis, 1986; Golosov and Tsyvinski, 2007; Greenwald and Stiglitz, 1986; Jacklin, 1987; Lorenzoni, 2008). Much of this literature is linked to modeling the recent financial crisis and emphasizes fire sales that come about from too much initial credit that has to be unwound, falls in asset prices, and a further tightening of financial constraints. This of course relates to the larger financial accelerator literature, e.g., Aghion, Banerjee, and Piketty (1999); Bernanke and Gertler (1989); Bernanke, Gertler, and Gilchrist (1996); Cooley, Marimon, and Quadrini (2004); Kiyotaki and Moore (1997); Krishnamurthy (2003); Lamont (1995); Rampini (2004); Ranciere, Tornell, and Westermann (2008). These papers use endogenous borrowing constraints as amplification mechanisms. In particular our paper which focuses on collateral constraints is closely related to Kiyotaki and Moore (1997) and Krishnamurthy (2003).

Recently Hart and Zingales (2113) have emphasized that externalities can result in too much saving, providing a model where limited ability to borrow against future human capital raises the demand for liquid assets which in turn raises the price of goods purchased. Our paper is closely related. Indeed in our class of models in which collateral is required to back promises, the over-savings phenomena is a general result. As a consequence of this, a constrained optimal system will make traders pay for the right to bring the collateral good or be compensated for not doing so, though this is relative to a market fundamental that determines the equilibrium spot prices.

The primary contribution of our paper, though, the one we emphasize, is a solution to all these pecuniary externality problems using market-based, segregated exchanges in securities. These internalize the externality by creating otherwise missing markets. The contrast with the literature is evident. We do not require portfolio restrictions, restrictions on savings, interest rate manipulation, fiscal policy, or taxes and subsidies levied by the government. We do not have to quantify any particular policy response. Rather, appropriate designed markets in rights to trade will deliver the correct prices at the Walrasian, non intervention outcomes and so deliver an efficient outcome. Lump sum taxes and subsidies can be used to compensate potential losers from the creation of there new markets and achieve any desired Pareto optimal allocation. We also do give a public finance tax/subsidy interpretation of the market based solution through the lens of the budget constraint, to make clear both the lump sum redistributive and marginal considerations that lead to a constrained optimal structure.

We now go over some of theses points in more detail. A contract or security consists of two items, a state-contingent promise and the collateral backing that promise. We take it as a primitive that default is possible or, equivalently, that collateral is required to make borrowers (or issuers of securities) repay their loans. A borrower may choose to default on a particular loan, or a particular state-contingent promise, and in doing so would lose the value of collateral backing that particular loan or security[1]. A rational borrower will base her default decision security by security on the value of the collateral backing each liability, compared to the original promise to pay. Of course the value of the collateral good at the time of repayment decisions (called the execution period) and in the market for asset backed securities (in the contract period) is an equilibrium phenomenon. Yet this market-clearing price of collateral determines whether borrowers default or not and hence the overall amount of debt and saving. In particular, the model is a general equilibrium model with endogenously determined collateral and so aggregate collateral (hence saving) is a result of the actions in the contracting period of all agents as a group. This in turns implies that the market fundamental, the price in the spot market, the price used to unwind collateral, is endogenously determined by the actions in the contracting period of all agents as a group (aggregate savings).

Contracts that promise to pay and which do not default have to be backed by a sufficient, minimum level of collateral, again depending on the promise and the value of collateral. Likewise asset-backed securities which are issued as a promise to pay have to be backed in collateral by an equivalent value of asset-backed securities acquired, the promises of others. Further, for every set of securities which actually default, handing over collateral, there is another set which would be equivalent, with the same overall payoff and no default. Adding up all such promises, over state-contingent security promises directly backed by collateral and over state-contingent securities backed by the promises of others, generates a state-contingent collateral constraint on trades which is in play in the ex ante contract market. Of course contracts which do default naturally also require collateral that is to be handed over when

[1] This paper is related to the literature on decentralization with autarky as the penalty for reneging, e.g., Alvarez and Jermann (2000); Kehoe and Levine (1993); Kocherlakota (1996). Similar to our model, they allow ex ante complete contracts, and focus on decentralization of constrained optimal allocations. On the other hand, the punishment mechanism is different from ours, as for us defaulting agents will only lose their collateral.

the borrower does not repay. That is, partially collateralized securities are still intimately associated with the exact amount of collateral which serves a backing. But rescaling these latter contracts delivers collateral constraints which are equivalent. We label such constraints collateral constraints, for brevity.

The externality problem[2] is in general a missing-market problem, as Arrow (1969), which was build on an earlier work of Meade (1952), made clear some time ago. For us, here in this paper, the markets for contracts over the "market fundamentals", those aspects of the environment which determine the market-clearing price, the valuation of collateral, are missing.

There are several key ingredients in our approach to creating these missing markets. First, we define a new object called a type's "discrepancy from the market fundamental"[3], and in equilibrium, by definition, the sum of individual discrepancies must be zero (but discrepancies are nontrivial, some types on one side of the market and some on the other). Second, we give this discrepancy a common price per unit discrepancy, determined by a market (but the

[2]This externality problem is multilateral. That is, without further restrictions, every agent will experience, as a common price ratio, the same market fundamental, regardless of the fact that other agents are also experience the same thing. In this sense the externality is a nondepletable externality. Indeed, this might suggest that one way to internalize the externality is with personalized prices, using a Lindahl equilibrium concept (Lindahl, 1958). Yet, to the contrary, here we apply a market-based solution concept with a common price per unit discrepancy. The connection is that the total paid, or received, does depend on type-specific pretrade endowment ratios and in that sense is personalized.

[3]This object is related to consumption rights in Bisin and Gottardi (2006), which internalizes the consumption externality due to adverse selection problem. The key difference is that our "discrepancy from the market fundamental" only requires own type information (endowments and savings/collateral position) and the knowledge of the equilibrium price, which is a standard Walrasian assumption, while the determination of the consumption rights for each type in an adverse selection environment utilizes information on other types (see Eq. (3.2) in Bisin and Gottardi (2010) which specifies correct conjectures of what other types are doing and the related no-envy conditions in Prescott and Townsend (1984a,b)). Another related paper is Stein (2012), who proposed a cap-and-trade approach, in which banks are granted permits (reserves) for private money creation and the interest rate is the price at which theses permits are traded in the market. The key difference from ours and Bisin and Gottardi (2006) is that his approach is a government intervention in which rights to trade are the policy instrument, so it is not entirely a decentralized, market-based approach.

quantity discrepancy depends in part on observed heterogeneity). Third, we allow agents to contract ex-ante on the market fundamental determining the state-contingent spot-market-clearing price. That is, we create security exchanges at which the value of collateral used for clearing ex post is pre-determined, for the entire range of values for collateral, including out of equilibrium values.

The particular security exchanges which emerge in equilibrium are determined by the forces of demand and supply. In any active exchange the clearing price of collateral, allowing retrade within the exchange, is indeed one that is sustained in equilibrium given the types and numbers of agents attracted to that particular exchange[4]. We then prove that the competitive equilibria with endogenous collateral constraints in this extended commodity space are equivalent with Pareto optima. Indeed, working through the optimum problem and its Lagrange multipliers, we are able to derive the price of rights to trade even for markets which will not exist in equilibrium. It is crucial that any agent thinks about the consequences of off-equilibrium deviations to different exchanges, so to speak.

One could view these results as normative, indicative of the need for a systematic but market-determined way for traders to unwind commitments. We elaborate on what actual implementation might look like in the conclusion of this paper.

Here to elaborate is the main idea of the paper: we internalize the externality by making household types pay or be paid for their influence on the spot market prices determining the value of collateral, when their pretrade endowment ratio is different from the ratio determining the market fundamental. Specifically, household types who enter into a market

[4]A price island is a language one can use to conceptualize the consistent execution of the contingencies on fundamentals. That is, a price island specifies the spot price, the value of collateral ex post, and the set of agents that end up there through their ex ante purchases or sales have to support that price. Agents can carry in goods or securities in such a way that their pretrade ratio of endowments or portfolio in a spot market deviates from the market fundamental, but the sum of the discrepancies must, by the definition of consistency, be zero so that the spot price that indexed ex ante contracts is the one which prevails in equilibrium. This is like a club constraint in other literature, e.g., Prescott and Townsend (2006). This solution concept with segregated security exchanges is also related to the assignment literature (e.g., Koopmans and Beckmann, 1957; Prescott and Townsend, 1984a,b). Mortensen and Wright (2002) internalizes a search externality using directed search into segregated submarkets that promise different expected waiting times. See also Guerrieri, Shimer, and Wright (2010).

in which the price is high for the collateral good with which they are abundantly endowed and/or have as a consequence of collateral/savings (they have a smaller pretrade endowment ratio than the market fundamental) will pay for demanding rights to trade on that market (and no other) . On the flip side, those types who are entering into a market with relatively little of the collateral good (they have a pretrade endowment ratio larger than the market fundamental) must be paid for accepting the restriction to trade on that market (and no other).

In another interpretation, ex post spot trades are replaced by ex ante trade in asset backed securities. In this interpretation, a household has to pay or be paid for the rights to trade in a particular security exchange ex ante, but these exchanges still determine the price at which asset backed securities are unwound.

The collateralization structure in this model incorporates both "tranching" and "pyramiding" (see also Geanakoplos, 1997). With "tranching", a specific piece of collateral can be used to back up several contracts as long as their promises to pay are in different states, i.e., no conflicting claims. With "pyramiding", agents are allowed to use financial assets, the contracts for promises to receive goods of others, as collateral for their own promises. This is different from the contract-specific collateralization structure as in Geanakoplos (2003), among others, where the collateral of a contract cannot be used as collateral for any other contract. On the other hand, our structure is similar to that of Chien and Lustig (2010), where several state-contingent contracts can be backed by the same collateral. However, the main results of this paper are valid under any contract-specific collateralization structure[5]; that is, the externality exists, and more importantly, our solution concept to the externality problem still works[6].

Of course agents are allowed to retrade in spot markets, and that is what delivers the

[5]Note that the need for securities with default in Geanakoplos (2003) is not a result of the contract-specific collateralization structure. As shown in Kilenthong (2011), if all promises are feasible, there will be no need for securities with default even when the collateralization structure is contract-specific. In fact, collateralized securities with default are needed in Geanakoplos (2003) because he rules out state-contingent promises ex ante, i.e., only debt-like collateralized securities are allowed.

[6]Of course, the collateral constraints are slightly different under different structures. That difference could lead to different quantitative, but not qualitative conclusions and the existence and welfare theorems.

spot-market-clearing prices. This is the easiest interpretation of what is going on in the model. However, with pyramiding, agents are indifferent between ex-ante contracting versus ex post retrading in spot markets. This is because anything which can be done in the spot market, trading one good for another, can be done in the ex ante contract market, with promises to receive one good backing promises to surrender the other. Hence agents do not need to retrade in spot markets (but they may well do so).

The remaining of the paper proceeds as follows. Section 2 describes the primitive ingredients of the model. We establish the existence of the externality in section 3. Section 4 presents our market-based solution concept with illustrative numerical examples. Section 5 formally defines the competitive equilibrium with extended commodity space using lotteries, and proves the existence and welfare theorems. Section 6 concludes the paper. Appendix A contains all proofs. Appendix B presents more related results, and Appendix C shows detailed derivations for numerical examples.

2 The Model Economy

This is a two-period economy, $t = 0, 1$. All financial (debt and insurance) contracts are traded in period $t = 0$, henceforth called the "contracting period". In addition, in period $t = 0$, both of two consumption goods can be traded and consumed. One of them can be saved. All contracts will be executed in period $t = 1$, henceforth called the "execution period". There are a finite number S of possible states of nature in this period $t = 1$, i.e., $s = 1, 2, ..., S$. This allows $S = 1$ so there is only intertemporal trade, borrowing and lending, from $t = 0$ to $t = 1$. For $S > 1$ in which contingent claims, Arrow-Debreu securities are traded, let $0 < \pi_s < 1$ be the objective and commonly assessed probability of state s occurring, where $\sum_s \pi_s = 1$. Again the two underlying goods can be traded and consumed in each state s. We refer to these $t = 1$ markets as spot markets.

Again there are two underlying goods, called good 1 and good 2. Good 1 cannot be stored (is completely perishable) from $t = 0$ to $t = 1$, while good 2 is storable. The good 2 that can be stored is collateralizable, i.e., can serve as collateral to back promises. Henceforth, good 2 and collateral good will be used interchangeably. Furthermore, good 1 will be the numeraire good in every date and state. In the concluding section, we interpret good 1 as a money and good 2 as treasuries. That is, treasuries are collateral backing all promises to pay at $t = 1$.

There is a continuum of agents of measure one. So in this paper we are not concerned with small numbers. The agents are however divided into H types, each of which is indexed by $h = 1, 2, \cdots, H$. Each type h consists of $\alpha^h \in [0, 1]$ fraction of the population such that $\sum_h \alpha^h = 1$. Each agent type h is endowed with good 1 and good 2, $\mathbf{e}_0^h = \left(e_{10}^h, e_{20}^h\right) \in \mathbb{R}_+^2$ in period $t = 0$ and $\mathbf{e}_s^h = \left(e_{1s}^h, e_{2s}^h\right) \in \mathbb{R}_+^2$, in each state $s = 1, \cdots, S$. Let $\mathbf{e}^h = \left(\mathbf{e}_0^h, \cdots, \mathbf{e}_S^h\right) \in \mathbb{R}_+^{2(1+S)}$ be the endowment profile of agent type h over period $t = 0$ and all states s in period $t = 1$. There are thus $2(1 + S)$ commodities in total. Heterogeneity of agents originates in part from the endowment profiles $\mathbf{e}^h$ (and not in preferences, but we could easily allow this extension). As a notational convention, vectors or matrices will be represented by bold letters. We also assume that the endowments in all periods and all states are publicly known. Hence, the limited commitment considered in this paper comes from a contract enforceability problem, not from an informational problem. We leave the latter for future work. In the concluding section, we interpret an agent as a trader (insurance company and/or hedge fund) and think of endowments as pretrade portfolios determined by considerations outside the model.

Let $k^h \in \mathbb{R}_+$ denote the collateral holding (equivalent to the holding of good 2) of an agent type h at the end of period $t = 0$. Note that this collateral allocation does not need to be equal to his initial endowment of good 2. In particular, since good 2 can be exchanged or acquired in the contracting period (at date $t = 0$), k^h will be equal to the net-position in the collateral good after trading in period $t = 0$. The collateral good as legal collateral backing claims is assumed to be fully registered and kept in escrow, i.e., cannot be taken away either by borrowers or lenders. However, the holding of good 2 can also include normal saving. The storage technology of good 2 whether in collateral or normal saving is linear but potentially with a random return. In some applications, it is natural to treat the returns as a constant, and focus on how collateral interacts with intertemporal trade. In other applications, the risk is in the collateral itself. Each unit of good 2 stored will become R_s units of good 2 in state $s = 1, \cdots, S$. Specifically, storing I units of good 2 at date $t = 0$ will deliver $R_s I$

units of good 2 in state s. It is noteworthy that the results in this paper are valid even if the technology R is not random. In most of the exposition, uncertainty originates in the endowment, primarily. In the concluding section, we can interpret R_s as determined outside the model, e.g., world events determine the price of treasuries, as if in a small open economy.

The preferences of agent type h are represented by the utility function $u\left(c_1^h, c_2^h\right): \mathbb{R}_+^2 \to \mathbb{R}$, where $\left(c_1^h, c_2^h\right)$ are the consumption of good 1 and good 2 of agent type h, respectively. Let $0 < \beta \leq 1$ be the common discount factor. The discounted expected utility of h is thus

$$U^h\left(\mathbf{c}^h\right) = u\left(c_{10}^h, c_{20}^h\right) + \beta \sum_{s=1}^{S} \pi_s u\left(c_{1s}^h, c_{2s}^h\right)$$

where, as with the notation for endowments, $\mathbf{c}^h = \left(\mathbf{c}_0^h, \cdots, \mathbf{c}_S^h\right) \in \mathbb{R}_+^{2(1+S)}$ is the consumption allocation with $\mathbf{c}_0^h \equiv \left(c_{10}^h, c_{20}^h\right) \in \mathbb{R}_+^2$ and $\mathbf{c}_s^h \equiv \left(c_{1s}^h, c_{2s}^h\right) \in \mathbb{R}_+^2$ for $s = 1, \ldots, S$ as the consumption of good 1 and good 2 in period $t = 0$, and in state s, respectively. In our trader interpretation in the concluding section, consumption removes securities from a portfolio, though the reason utility is derived from this is not modeled there.

Assumption 1. *For each agent type h, common utility function $u\left(c_1^h, c_2^h\right)$ is homothetic, continuous, strictly concave, strictly increasing in both arguments, and satisfies the usual Inada conditions.*

Homotheticity is special but will allow us to construct closed form solutions in the determination of spot prices. This has great expositional advantage when it comes to understanding how security markets work to correct the externality. Risk aversion with random endowments motivates trade in state-contingent securities. Heterogeneous intertemporal endowments motivates trade in bonds. We will on occasion put superscript h on the utility function for clarity, but again preference heterogeneity per se is not an essential part of what we do here.

2.1 Market Fundamentals

Each agent can trade in spot markets in each state s. In principle, the market-clearing prices in these spot markets depend on the distribution of pretrade (before ex post spot trade) endowments or the composition of agents. To be precise, let z_s be a *market fundamental* that determines the spot-market-clearing price of good 2 in state s, and accordingly $p(z_s)$ be the spot-price function.

With identical homothetic preferences, the aggregate ratio of good 1 to good 2 in state s is the market fundamental in state s; that is, $z_s = \frac{\sum_h \alpha^h e^h_{1s}}{R_s K + \sum_h \alpha^h e^h_{2s}} \in \mathbb{R}_+$, where $K = \sum_h \alpha^h k^h$ is the aggregate (endogenous) saving including collateral. Here then the spot price function can be represented by a single-valued function $p(z_s)$. In other words, the market fundamental, a ratio of goods in state s, is necessary and sufficient to pin down the spot price in state s. We summarize:

Lemma 1. *With identical homothetic preferences, the market fundamental in state s is given by*

$$z_s = \frac{\sum_h \alpha^h e^h_{1s}}{R_s \sum_h \alpha^h k^h + \sum_h \alpha^h e^h_{2s}}. \tag{1}$$

Market clearing price $p(z_s)$ is a one-to-one function, i.e. $p(z_s)$ is a single-valued, and $p(z_s) = p(z'_s)$ implies that $z_s = z'_s$. In addition, with strictly concavity of $u(\cdot)$, $p(z_s)$ is strictly monotone increasing.

Condition (1) is called a *consistency constraint*. It ensures that the market fundamental is consistently well-defined in that $p(z_s)$ is exactly the spot price that constitutes a spot market equilibrium. This is where we exploit the homotheticity assumption; ratios of the aggregate are enough to pin down equilibrium prices.

2.2 Collateralization Structure

A specific piece of collateral can be used to back up several contracts as long as their promises to pay are in different states. Thus, there is no conflict in a given state s. This is known as *tranching*. This is distinct from the contract-specific collateralization structure (in Geanakoplos, 2003, among others), in which the collateral of a given security cannot be used as collateral for any other security. For full generality here, we will consider state-contingent securities as the primitives and otherwise let the security structure be endogenous. Accordingly, we focus on securities paying in each state s with market fundamental z_s, one at a time. In words, we are dealing with an Arrow-Debreu complete security environment, but collateral will limit the securities which emerge in equilibrium.

A (contingent) security promising to pay one unit of good 1, the numeraire, in period $t = 1$ and state s with $\widehat{C}$ units of good 2 as collateral is a promise to pay a unit of good 1 if the state of nature is s and nothing otherwise. For notational convenience, we use $\widehat{}$ to distinguish securities paying in good 1, the numeraire, from securities paying in good 2. With limited commitment, that is, allowing default on the part of the person making the promise, the payoff of this security is given by

$$\widehat{D} = \begin{cases} \min\left(1, \widehat{C}R_s p(z_s)\right) & \text{if state is } s, \\ 0 & \text{otherwise,} \end{cases} \tag{2}$$

where this payoff is in units of good 1 in period $t = 1$, and $p(z_s)$ is the price of good 2 (in units of good 1) in state s converting good 2 with its potentially random return into units of good 1. That is, the person pays off as promised in good 1 or defaults surrendering collateral. The issuer or "borrower" in period $t = 0$ may not wish to honor the state-contingent obligation. This creates the limited commitment problem; that is, she will keep the promise if that promise is no larger than the value of the collateral, i.e., $1 \leq \widehat{C}R_s p(z_s)$, and will "default" otherwise, $\widehat{C}R_s p(z_s) < 1$. In case of default, the payoff of the contract in state s is equal to the value of its collateral in that state, $\widehat{C}R_s p(z_s)$ units of good 1. Note in particular that this defaulting condition depends on the spot price $p(z_s)$.

Ironically, it can be shown that there is no loss of generality in restricting attention to securities without default. Intuitively a security which would default has a known payoff structure, so we may as well start with that in the first place. But the possibility of default does restrict securities, and collateral constraints can be binding. The discussion is summarized in the following lemma.

Lemma 2. *For any state-contingent security, there exists a security with* no default *that can generate the* same total payoffs *using the* same amount of collateral.

In Appendix A, we present the result for a security paying in good 1 in state s with good 2 as collateral, and then argue that the same logic applies for all other types of securities. See Kilenthong (2011) for a similar result with contract-specific collateralization. Further, issuing securities that do default requires no less collateral than (an equivalent set of) securities that do not. In other words, and this again may seem counterintuitive, securities with default, i.e., with little collateral, do not really economize on collateral. In addition, we also show in Appendix B.1 that default cannot make collateral constraints, formally defined below, less binding.

In addition, with perfectly divisible collateral, there is no loss of generality in excluding over-collateralized securities, whose collateral value is strictly larger than the promise. More precisely, an over-collateralized security paying in good 1 in state s is a contract with a collateral $\widehat{C}$ such that $\widehat{C}R_s p(z_s) > 1$. The payoff of this security in state s is 1. This security is equivalent to a no-default security with $\frac{1}{R_s p(z_s)} < \widehat{C}$ units of good 2 as collateral, whose payoff in state s is also 1. A similar result applies to other types of securities as well.

It is worthy of emphasis, however, that own saving should not be interpreted as over-collateralization, as no securities are acquired from others; that is, each agent can save. This saving will result in the slackness of the collateral constraint (3) defined below. In particular, an agent may hold at the end of period $t = 0$ more collateral good than the (minimum) amount needed to collateralize all securities issued.

3 Collateral Constraints and Externality

There is no loss of generality in considering at most only two classes of securities[7]; (i) $\hat{\theta}^h_s$ - securities paying in good 1 in state s, (ii) θ^h_s - securities paying in good 2 in state s. Here a positive number denotes the purchaser or holder, and negative the issuer. When negative, each of the state-contingent securities must be backed by the issuer either by good 2 itself or by purchased assets (other people's promises). In other words, $\hat{\theta}^h_s$ and θ^h_s include both directly collateralized and asset-backed securities. As established in Appendix B.2, the

[7] With spot markets we actually need securities $\hat{\theta}^h_s$ paying in the numeraire only. We proceed here in more generality as what we do will not require active spot markets.

collateral constraints[8] for an agent type h take the intuitive form

$$p(z_s)R_sk^h + \hat{\theta}_s^h + p\left(z_s\right)\theta_s^h \quad \geq \quad 0, \; \forall s. \tag{3}$$

The collateral constraint (3) states that, for each state s, the net-value of all assets, including collateral good and securities, must be non-negative. If $\hat{\theta}_s^h$ and θ_s^h were negative, as promises, we could write this as $p\left(z_s\right)R_sk^h \geq -\hat{\theta}_s^h - p\left(z_s\right)\theta_s^h$. That is, there is sufficient collateral in value in state s to honor the value of all such promises. Note that all promises are converted to units of good 1 using the spot market price of the collateral good $p\left(z_s\right)$. Also as $\hat{\theta}_s^h$ and θ_s^h include both asset-backed and directly backed securities, these collateral types per se do not matter, either[9].

The collateral constraints (3) can be written in consumption space as follows. Suppose for the moment that securities are such that there is no spot trade in equilibrium (see Lemma 6 in Appendix B.4). The consumption for an agent type h in state s is given by

$$c_{1s}^h \quad = \quad e_{1s}^h + \hat{\theta}_s^h, \tag{4}$$

$$c_{2s}^h \quad = \quad e_{2s}^h + R_sk^h + \theta_s^h. \tag{5}$$

Substituting these two equations into the collateral constraint (3) yields

$$c_{1s}^h + p(z_s)c_{2s}^h \quad \geq \quad e_{1s}^h + p(z_s)e_{2s}^h. \tag{6}$$

This condition implies that, due to limited commitment and the possibility of default, the market value of consumption in a state s of an agent cannot be lower than the market value of her endowment (without collateral k^h) in the same state (related to Golosov and Tsyvinski, 2007; Kehoe and Levine, 1993, among others). Intuitively, if this constraint were to be violated, an agent type h would have promised to deliver some part of the value of her endowments, over and above her consumption, but other things equal there will be no incentive to deliver ex post. Collateral will preclude (6) from being violated.

[8] These collateral constraints are different from the ones in Geanakoplos (2003) in that they now include all state contingent collateralized securities with tranching and pyramiding. See Appendix B.2 for more details. Similar collateral constraints are discussed in liquidity literature, e.g., Caballero and Krishnamurthy (2001); Holmström and Tirole (1998); Rampini and Viswanathan (2010).

[9] In addition, we can show that the markets economize on collateral; that is, there is no gain from pooling collateral across agents type h (see Lemma 5 in Appendix B.3).

The interaction between the bindingness of collateral constraints and spot prices generates an externality. Technically, there is an externality because the consumption feasibility set of an agent type h depends on other agents' choices of saving $k^{\tilde{h}}$ through the spot price. This dependency results from the collateral constraints (6), or borrowing constraints in general. Conceivably, if there were no binding collateral constraint, the consumption feasibility set would be independent of other agents' choices (and therefore there would be no externality).

3.1 Competitive Collateral Equilibrium

Let $\widehat{P}_{as}$ and P_{as} be the prices of securities in the ex ante $t=0$ market paying in good 1 and in good 2 in state s, respectively. We allow spot price $p(z_s)$ but we do not require agents to be active in these markets since they could have all ex ante contingent commodity securities trade at $t=0$. A collateral equilibrium is defined:

Definition 1. A competitive collateral equilibrium is a specification of prices of good 2 in period $t=0$, P_{20}, the prices of securities paying in good 1, $\widehat{P}_{as}$, and the prices of securities paying in good 2, P_{as}, the spot price of good 2 in each state s, $p(z_s)$, and an allocation $\left(\mathbf{c}_0^h, k^h, \hat{\theta}^h, \theta^h\right)_h$ such that

(i) for any agent type h, $\left(\mathbf{c}_0^h, k^h, \hat{\theta}^h, \theta^h\right)$ solves

$$\max_{\left(\mathbf{c}_0^h, k^h, \hat{\theta}^h, \theta^h\right)} u\left(c_{10}^h, c_{20}^h\right) + \beta \sum_s \pi_s u\left(e_{1s}^h + \hat{\theta}_s^h, e_{2s}^h + R_s k^h + \theta_s^h\right) \tag{7}$$

subject to the collateral constraints for each state s:

$$p(z_s) R_s k^h + \hat{\theta}_s^h + p(z_s)\theta_s^h \geq 0, \ \forall s, \tag{8}$$

and the budget constraint at $t=0$:

$$c_{10}^h + P_{20}\left(c_{20}^h + k^h\right) + \sum_s \widehat{P}_{as}\hat{\theta}_s^h + \sum_s P_{as}\theta_s^h \leq e_{10}^h + P_{20}e_{20}^h, \tag{9}$$

taking prices $\left(P_{20}, \widehat{P}_{as}, P_{as}, p(z_s)\right)$ as given;

(ii) markets clear for good 1 at $t = 0$, for good 2 at $t = 0$, for $\hat{\theta}_s^h$ in state s, and for θ_s^h in state s, respectively:

$$\begin{aligned} \sum_h \alpha^h c_{10}^h &\leq \sum_h \alpha^h e_{10}^h, & (10) \\ \sum_h \alpha^h \left[c_{20}^h + k^h\right] &\leq \sum_h \alpha^h e_{20}^h, & (11) \\ \sum_h \alpha^h \hat{\theta}_s^h &= 0, \ \forall s, & (12) \\ \sum_h \alpha^h \theta_s^h &= 0, \ \forall s; & (13) \end{aligned}$$

(iii) the consistency constraint

$$z_s = \frac{\sum_h \alpha^h e_{1s}^h}{R_s \sum_h \alpha^h k^h + \sum_h \alpha^h e_{2s}^h} \tag{14}$$

holds for all s.

The necessary maximizing condition for a collateral equilibrium (ce) related to collateral allocation k^h (an interior solution to the consumer problem) is given by, for any h,

$$P_{20} = \frac{u_{20}^h}{u_{10}^h}\bigg|_{ce} = \sum_s \pi_s \beta \frac{u_{2s}^h}{u_{10}^h} R_s + \sum_s \frac{\gamma_{cc-s}^h}{u_{10}^h} p(z_s) R_s, \tag{15}$$

where $u_{i0}^h = \frac{\partial u\left(c_{10}^h, c_{20}^h\right)}{\partial c_{i0}}$, $u_{is}^h = \frac{\partial u\left(c_{1s}^h, c_{2s}^h\right)}{\partial c_{is}}$ for $i = 1, 2$, and γ_{cc-s}^h is the Lagrange multiplier for the collateral constraint (8) in state s for an agent type h.

3.2 Collateral Constrained Optimality

Attainable allocations are those that can be achieved by exchanges of securities and collateral in date $t = 0$ and exchanges of consumption goods in date $t = 1$ at state s, respecting spot prices $p(z_s)$. In other words, a planner can only reallocate goods with the same instruments as the agents. Accordingly, attainable allocations are defined using the spot-price function $p(z_s)$. Again as will be later proved (see Lemma 6 in Appendix B.4), the asset-backed securities in this model are simply substitutes for spot markets.

Definition 2. An allocation $\left(\mathbf{c}_0^h, k^h, \hat{\theta}_s^h, \theta_s^h\right)_h$ is *attainable* if

(i) it satisfies resource constraints:

$$\begin{aligned}
\sum_h \alpha^h c_{10}^h &\leq \sum_h \alpha^h e_{10}^h, && (16)\\
\sum_h \alpha^h \left[c_{20}^h + k^h\right] &\leq \sum_h \alpha^h e_{20}^h, && (17)\\
\sum_h \alpha^h \hat{\theta}_s^h &= 0, \ \forall s, && (18)\\
\sum_h \alpha^h \theta_s^h &= 0, \ \forall s; && (19)
\end{aligned}$$

(ii) for each agent type h, it satisfies the collateral constraints for each state s:

$$p(z_s)R_s k^h + \hat{\theta}_s^h + p(z_s)\theta_s^h \ \geq \ 0, \ \forall s, \tag{20}$$

(iii) the consistency constraint

$$z_s = \frac{\sum_h \alpha^h e_{1s}^h}{R_s \sum_h \alpha^h k^h + \sum_h \alpha^h e_{2s}^h} \tag{21}$$

holds for all s.

We note in passing with a non-constant, s-contingent spot-price function that the attainable set is non-convex. The main source of the non-convexity is the product of spot-price function and the sum of collateral and contract allocations, $p(z_s)\left(R_s k^h + \theta_s^h\right)$, in the collateral constraints (3). Thus there are hints already that mixed strategies or lotteries may come into play. We turn to this issue later in the paper but earmarked it here for future reference.

Lemma 3. *With identical homothetic and strictly concave preferences, the attainable set is non-convex.*

A constrained optimal allocation is characterized using the following planner's problem. Let $\overline{U}^h$ be the reservation utility level for an agent type h.

Program 1. The Pareto Program with collateral constraints:

$$\max_{\left(\mathbf{c}_0^h, k^h, \hat{\theta}^h, \theta^h\right)_h} u\left(c_{10}^1, c_{20}^1\right) + \beta \sum_s \pi_s u\left(e_{1s}^1 + \hat{\theta}_s^1, e_{2s}^1 + R_s k^1 + \theta_s^1\right) \tag{22}$$

subject to (16)-(21), and the participation constraint for each $h = 2, \cdots, H$,

$$u\big(c_{10}^h, c_{20}^h\big) + \beta \sum_s \pi_s u\big(e_{1s}^h + \hat{\theta}_s^h, e_{2s}^h + R_s k^h + \theta_s^h\big) \quad \geq \quad \overline{U}^h, \tag{23}$$

and non-negativity constraints for consumption and collateral allocations.

As is typically the case, it suffices to consider only equal-treatment-of-equals in the Pareto problem. More generally, the externalities in this class of models, if they exist, have nothing to do with the equal treatment of equals property. Let μ_{cc-s}^h, and $\mu_{\bar{u}}^h$ denote the Lagrange multipliers for the collateral constraint (3) for agent h in state s, and for the participation constraint (23) for agent h, respectively. For notational convenience, let $\mu_{\bar{u}}^1 = 1$. A necessary condition[10] for constrained optimality[11] (op) related to collateral allocation k^h is given by, for any h,

$$\frac{u_{20}^h}{u_{10}^h}\Big|_{op} = \sum_s \pi_s \beta \frac{u_{2s}^h}{u_{10}^h} R_s \quad + \quad \sum_s \frac{\mu_{cc-s}^h}{\mu_{\bar{u}}^h u_{10}^h} p(z_s) R_s - \sum_s \frac{\alpha^h}{\mu_{\bar{u}}^h u_{10}^h} \frac{p'(z_s)}{p(z_s)} \frac{\partial z_s}{\partial K} \sum_{\tilde{h}} \mu_{cc-s}^{\tilde{h}} \hat{\theta}_s^{\tilde{h}}, \tag{24}$$

where $p'(z_s) = \frac{\partial p(z_s)}{\partial z_s}$, $K = \sum_h \alpha^h k^h$, μ_{cc-s}^h is the Lagrange multiplier for the collateral constraint (20) for state s for an agent type h and $\mu_{\bar{u}}^h$ is the Lagrange multiplier for the participation constraint (23) for an agent type $h = 1, 2, \ldots, H$ with a normalization of $\mu_{\bar{u}}^1 = 1$.

Of special interest, the last term depends not only on the bindingness of collateral constraints for h but also the bindingness of other agents' collateral constraints. This implies that if an agent type's collateral constraint were binding, it would impact everyone. This is the source of the externality. Again such results are easily surmised from an extensive literature, but we quickly review in the next section.

[10]Given that the constraint set is not convex (Lemma 3), this optimality condition is necessary but may not be sufficient. Nevertheless, this does not cause any problem to our externality argument, as we simply need to show that a collateral equilibrium cannot be constrained optimal, i.e. does not satisfy the necessary optimal condition (24).

[11]For expositional simplicity and without any real loss, we consider only equal-treatment (for each type), and interior solutions (i.e., the non-negativity constraint for k^h is neglected). With homothetic and strictly concave preferences, and no non-convexity, agents of the same type will optimally choose the same allocation in an equilibrium; that is, given the same market prices in equilibrium. Thus, a collateral equilibrium allocation has equal-treatment-of-equals property.

3.3 The Externality

Note that an infinitesimal agent of type h takes a spot price, $p(z_s)$, as invariant to his or her own actions. To the contrary, the constrained planner can influence the spot prices $p(z_s)$ through collateral assignments, k^h, for the agents of type $h = 1, 2, \ldots, H$, in period $t = 0$, which affect in turn the market fundamentals z_s. This key influence is the term in $\frac{p'(z_s)}{p(z_s)} \frac{\partial z_s}{\partial K}$. The difference between the impact of the planner and that of the agents creates the externality and causes an inefficiency. If the last term in (24) were zero and we set $\gamma^h_{cc-s} = \frac{\mu^h_{cc-s}}{\mu^h_{\bar{u}}}$, then condition (15) is exactly the same as (24), and there would be no externality.

The last term in (24) could be zero if either $\mu^{\tilde{h}}_{cc-s} = 0$ for all $\tilde{h}$ or $\frac{p'(z_s)}{p(z_s)} \frac{\partial z_s}{\partial K} = 0$. With a strictly concave utility function, the spot price varies with the market fundamental[12] (is not constant), i.e. $\frac{p'(z_s)}{p(z_s)} \frac{\partial z_s}{\partial K} \neq 0$. As a result, when at least one of the collateral constraints is binding, i.e., $\mu^{\tilde{h}}_{cc-s} > 0$ for some $\tilde{h}$, the last term in (24) will be non-zero. With this non-zero term, a collateral equilibrium will not be constrained efficient. It is true that, as an exceptional case, a collateral equilibrium could be a full first-best optimum, that is, the environment could be such that despite the focus of the paper we could ignore the collateral constraint. But, otherwise, the collateral equilibrium must be constrained *sub*optimal, and that is the assumed benchmark case of interest that motivates the paper. The result is summarized in the following proposition.

Proposition 1. *Under Assumption 1, a competitive collateral equilibrium is constrained optimal if and only if all collateral constraints are not binding, i.e.* $\gamma^h_{cc-s} = \mu^h_{cc-s} = 0$ *for all* h *and all* s*. As a result, a competitive collateral equilibrium is constrained* suboptimal *if and only if it is not first-best optimal.*

In particular when the last term in (24) is not zero, we can show that it must be positive, i.e., $\sum_s \frac{\alpha^h}{\mu^h_{\bar{u}} u^h_{10}} \frac{p'(z_s)}{p(z_s)} \frac{\partial z_s}{\partial K} \sum_{\tilde{h}} \mu^{\tilde{h}}_{cc-s} \hat{\theta}^{\tilde{h}}_s > 0$. As a result, the equilibrium price of good 2 in

[12] If the utility function is linear in both goods, then the spot price is constant, i.e. $\frac{p'(z_s)}{p(z_s)} \frac{\partial z_s}{\partial K} = 0$. This clearly results in constrained efficiency. Similarly, if the amount of aggregate saving is fixed exogenously, then the market fundamental (the ratio of good 1 to good 2) is fixed. This also implies that the last term is zero, and so constrained efficiency.

period $t = 0$ will be too high relative to its shadow price from the (constrained) optimal allocation $\left.\frac{u^h_{20}}{u^h_{10}}\right|_{op}$. In addition, this implies that the competitive collateral equilibrium level of (endogenous) aggregate saving K^{ce} is too large relative to the (constrained) optimal level of aggregate saving/collateral K^{op}. Intuitively, the planner can do better by lowering the aggregate saving or collateral. This is our analogous here to the result of Hart and Zingales (2113) that it is possible for agents to be saving too much. The result is summarized in the following proposition.

Proposition 2. *Under Assumption 1, if a competitive collateral equilibrium is not first-best optimal, then*

(i) the equilibrium price of good 2 in period $t = 0$*,* P_{20}*, is too high, i.e.,* $P_{20} > \left.\frac{u^h_{20}}{u^h_{10}}\right|_{op}$*, and*

(ii) the (endogenous) aggregate saving/collateral in a competitive collateral equilibrium, K^{ce}*, is too large, i.e.,* $K^{ce} > K^{op}$*.*

4 Internalizing The Externality: The Economy with Segregated Security Exchanges

This section presents the key contribution of this paper, a market-based solution to the externality problem. A key object is the discrepancy from the market fundamental of each type, as this reflects the "marginal impact" of each type on others at an equilibrium price. This new object when priced then makes each type pay or be paid according to their marginal impact on the price used to unwind collateral, adding to or alleviating congestion, so to speak. But this object exists and is priced for out-of-equilibrium prices as well. Taken together these internalize the externality.

In particular, let $\mathbf{z} = (z_s)_{s=1}^{S}$ denotes a vector of the market fundamentals in all states s. Being in a security exchange z_s in state s means that an agent h can trade in spot markets and value collateral at spot price $p(z_s)$, as determined by the market fundamental z_s. Equivalently, even if the spot markets were shut down[13], an agent on a security exchange

[13]As proved in Lemma 6 in Appendix B.4, the spot markets are redundant in that agents are indifferent between trading in ex-ante contracts or in spot markets. Importantly, the spot markets are opened.

z_s can accomplish the same thing by trading in ex-ante securities $(\hat{\theta}_s, \theta_s)$, which has a relative price equal to the same $p(z_s)$.

Now let $\Delta_s^h(z_s) \in \mathbb{R}$ define "type h's *discrepancy from the market fundamental in state* s" a scalar[14]:

$$\Delta_s^h(z_s) = z_s \left(e_{2s}^h + R_s k^h\right) - e_{1s}^h, \forall s, \tag{25}$$

which can be rewritten as

$$\Delta_s^h(z_s) = \left(e_{2s}^h + R_s k^h\right) \left(z_s - \frac{e_{1s}^h}{e_{2s}^h + R_s k^h}\right), \forall s. \tag{26}$$

Note that if $\Delta_s^h(z_s) = 0$, then $z_s = \frac{e_{1s}^h}{e_{2s}^h + R_s k^h}$ and type h's pretrade endowment is exactly equal to the market fundamental. But typically with heterogeneity and active trade an agent type h will be on one side or the other of the market fundamental, buying or selling the collateral good 2 for good 1. Of course it takes at least two sides to make open an active market. If $\Delta_s^h(z_s) > 0$, then $z_s > \frac{e_{1s}^h}{e_{2s}^h + R_s k^h}$ and type h holds a relative low amount of good 1 and abundant amount of good 2, that is, relative to z_s. Such an agent type h brings in good two, the collateral good, in such a way that they push the valuation of collateral (the spot price of good 2 here) downward; that is, his pretrade actions have a negative marginal impact on the valuation of collateral to unwind commitments. As a result, he would need to pay for the right to trade or unwind in this market. Conversely, when $\Delta_s^h(z_s) < 0$, an agent type h has a relatively high amount of good 1 and scarce amount of good 2, relative to z_s. We know that there is oversaving in general, so this type will be compensated. Specifically, his pretrade actions have a positive marginal impact on the valuation of collateral. Therefore he would need to be paid to enter a high z_s, high $p(z_s)$ market.

Adding one unit of good 2 (via collateral k) adds to the discrepancy $\Delta_s^h(z_s)$ by exactly $z_s R_s$ (see Eq.25). This is the same for all agent types. But note also that there is a part of $\Delta_s^h(z_s)$ over which h has no control, namely her endowments. However, we assume that types, hence the endowments are publicly known. The "type h discrepancy from the fundamental" will be priced in a competitive equilibrium. The per unit price, denote $P_\Delta(z_s, s)$ below, will

[14]If we were in the underlying spaces of k and z_s, they would enter multiplicatively, hence and so we would have a non convexity problem. This is not a problem with lotteries, however.

be common. But the payment or subsidy for rights to trade in each security exchange will be proportional to the "type h discrepancy from the fundamental", that is price times quantity, and again the latter can be negative.

More formally, let an indicator function $\delta^h(z_s) \in \{0,1\}$, that is $\delta^h(z_s) = 0$ or $\delta^h(z_s) = 1$, denote an agent type h's discrete choice of security market z_s in each state $s = 1, 2, \ldots, S$. That is, each agent must choose one but only one fundamental spot market in each state s. With vector $\mathbf{z} = (z_s)_{s=1}^S$, we write this function as $\delta^h(\mathbf{z})$ such that $\sum_{\mathbf{z}} \delta^h(\mathbf{z}) = 1$. Specifically, $\delta^h(\mathbf{z}) = \Pi_{s=1}^S \delta^h(z_s) = 1$ if only if $\delta^h(z_s) = 1$ for all s. Let $P_\Delta(z_s, s)$ denote the market price of rights to trade in security exchange z_s in state s, $\Delta_s^h(z_s)$. But consistent with key notion of segregated exchanges, these choices of $\mathbf{z}$ are bundled with the consumption good, securities and collateral. Notationally, let $x^h(\mathbf{z}) = \left(\delta^h(\mathbf{z}), k^h(\mathbf{z}), c_{10}^h(\mathbf{z}), c_{20}^h(\mathbf{z}), \hat{\theta}_s^h(\mathbf{z}), \theta_s^h(\mathbf{z})\right)$ denote a typical bundle or allocation for an agent type h.

A competitive equilibrium with segregated security exchanges is defined as follows.

Definition 3. A competitive equilibrium with segregated security exchanges is a specification of allocation $\left(x^h(\mathbf{z})\right)_h$ and prices $\left(P_{20}, \hat{P}_a(z_s, s), P_a(z_s, s), P_\Delta(z_s, s), p(z_s)\right)$ such that

(i) for any agent type h, allocation $x^h(\mathbf{z}) = \left(\delta^h(\mathbf{z}), k^h(\mathbf{z}), c_{10}^h(\mathbf{z}), c_{20}^h(\mathbf{z}), \hat{\theta}_s^h(\mathbf{z}), \theta_s^h(\mathbf{z})\right)$ solves

$$\max_{x^h(\mathbf{z})} \sum_{\mathbf{z}} \delta^h(\mathbf{z}) \left[u\left(c_{10}^h(\mathbf{z}), c_{20}^h(\mathbf{z})\right) + \sum_s \pi_s u\left(e_{1s}^h + \hat{\theta}_s^h(\mathbf{z}), e_{2s}^h + R_s k^h(\mathbf{z}) + \theta_s^h(\mathbf{z})\right)\right] \tag{27}$$

subject to collateral constraints

$$\sum_{\mathbf{z}} \delta^h(\mathbf{z}) \left[p(z_s)\left[R_s k^h(\mathbf{z}) + \theta_s^h(\mathbf{z})\right] + \hat{\theta}_s^h(\mathbf{z})\right] \;\; \geq \;\; 0, \forall s, \tag{28}$$

and budget constraints

$$\begin{aligned} &\sum_{\mathbf{z}} \delta^h(\mathbf{z}) \Big\{ c_{10}^h(\mathbf{z}) + P_{20}\left[c_{20}^h(\mathbf{z}) + k^h(\mathbf{z})\right] + \sum_s \hat{P}_a(z_s, s)\, \hat{\theta}_s^h(\mathbf{z}) \\ &+ \sum_s P_a(z_s, s)\, \theta_s^h(\mathbf{z}) + \sum_s P_\Delta(z_s, s)\, \Delta_s^h(z_s) \Big\} \leq e_{10}^h + P_{20} e_{20}^h, \end{aligned} \tag{29}$$

taking prices prices $\left(P_{20}, \hat{P}_a(z_s, s), P_a(z_s, s), P_\Delta(z_s, s), p(z_s)\right)$ as given;

(ii) markets clear for good 1 in period $t = 0$, for good 2 in period $t = 0$, for securities paying good 1, for securities paying good 2, and for rights to trade, respectively,

$$
\begin{aligned}
\sum_h \sum_{\mathbf{z}} \delta^h(\mathbf{z}) \alpha^h c_{10}^h(\mathbf{z}) &= \sum_h \alpha^h e_{10}^h, && (30)\\
\sum_h \sum_{\mathbf{z}} \delta^h(\mathbf{z}) \alpha^h \left[c_{20}^h(\mathbf{z}) + k^h(\mathbf{z})\right] &= \sum_h \alpha^h e_{20}^h, && (31)\\
\sum_h \sum_{\mathbf{z}_{-s}} \delta^h(\mathbf{z}) \alpha^h \hat{\theta}_s^h(\mathbf{z}) &= 0, \forall s; z_s, && (32)\\
\sum_h \sum_{\mathbf{z}_{-s}} \delta^h(\mathbf{z}) \alpha^h \theta_s^h(\mathbf{z}) &= 0, \forall s; z_s, && (33)\\
\sum_h \sum_{\mathbf{z}_{-s}} \delta^h(\mathbf{z}) \alpha^h \Delta_s^h(z_s) &= 0, \forall s; z_s, && (34)
\end{aligned}
$$

where $\mathbf{z}_{-s} = (z_1, \ldots, z_{s-1}, z_{s+1}, \ldots, z_S)$ is a vector of market fundamentals in all states but state s.

4.1 Public Finance Interpretation

This section presents an alternative interpretation of the market based solution. The budget constraint with its prices for discrepancies from the market fundamentals has a public finance interpretation, as if we were to try to implement the optimum solution by taxes and subsidies, both lump sum and marginal.

Specifically, substituting (25), $\Delta_s^h(z_s) = z_s\left(e_{2s}^h + R_s k^h\right) - e_{1s}^h$, into the budget constraint for an agent type h (29) gives

$$
\begin{aligned}
&\sum_{\mathbf{z}} \delta^h(\mathbf{z}) \Big\{ c_{10}^h(\mathbf{z}) + P_{20}\left[c_{20}^h(\mathbf{z}) + k^h(\mathbf{z})\right] + \sum_s \hat{P}_a(z_s, s)\, \hat{\theta}_s^h(\mathbf{z}) + \sum_s P_a(z_s, s)\, \theta_s^h(\mathbf{z}) \\
&\qquad + \sum_s P_\Delta(z_s, s)\left(z_s\left(e_{2s}^h + R_s k^h\right) - e_{1s}^h\right) \Big\} \leq e_{10}^h + P_{20} e_{20}^h, \qquad (35)
\end{aligned}
$$

which can be rewritten as

$$
\begin{aligned}
&\sum_{\mathbf{z}} \delta^h(\mathbf{z}) \Big\{ c_{10}^h(\mathbf{z}) + P_{20}\left[c_{20}^h(\mathbf{z}) + k^h(\mathbf{z})\right] + \sum_s \hat{P}_a(z_s, s)\, \hat{\theta}_s^h(\mathbf{z}) + \sum_s P_a(z_s, s)\, \theta_s^h(\mathbf{z}) \Big\} \leq \\
&\sum_{\mathbf{z}} \delta^h(\mathbf{z}) \Big\{ \big[1 + \sum_s P_\Delta(z_s, s)\big] e_{10}^h + \big[P_{20} - \sum_s P_\Delta(z_s, s)\, z_s\big] e_{20}^h - \big[\sum_s P_\Delta(z_s, s)\, z_s R_s\big] k^h(\mathbf{z}) \Big\}. (36)
\end{aligned}
$$

We can now see that we need to have three types of taxes/subsidies, (i) saving/collateral tax of $\sum_s P_\Delta(z_s, s) z_s R_s$ per unit of saving/collateral, (ii) capital good endowment tax of $\sum_s P_\Delta(z_s, s) z_s$ per unit of collateral good endowment, and (iii) income or consumption good endowment (negative) tax of $-\sum_s P_\Delta(z_s, s)$ per unit of consumption good endowment. The endowment/income taxes are easy to see and since endowments are fixed, they can be viewed as lump sum taxes/subsidies or budget shifters. The saving/collateral tax enters the budget too but it is also endogenous as saving is; hence we can think of this as a marginal tax rate the agent type is facing for a given, fixed security exchanges z_s that the agent chooses. However, the security exchange z_s itself is a choice as far as the household is concerned, so in some sense even the fixed budget shifters are a choice for the agent. This is like looking up fixed and marginal rates in a big tax book and settling on which page (or pages) to use, indexed by active security exchanges z_s that the agent chooses.

4.2 Example Economies with Segregated Security Exchanges

The following three examples illustrate the competitive equilibrium with segregated security exchanges.

Environment 1 (Intertemporal Smoothing)**.** There are two periods, $t = 0, 1$, and a single state, $S = 1$ in period $t = 1$. So this is a pure intertemporal economy. Henceforth we drop all subscript s from the notation. There are two types of agents, $H = 2$, both of which have an identical constant relative risk aversion (CRRA) utility function

$$u(c_1, c_2) = -\frac{1}{c_1} - \frac{1}{c_2}, \forall h. \tag{37}$$

Each type consists of $\frac{1}{2}$ fraction of the population, i.e. $\alpha^h = \frac{1}{2}$. In addition, the discount factor $\beta = 1$. The storage technology is given by $R = 1$. The endowment profiles of the agents are shown in Table 1 below. Recall that e^h_{it} is an agent h's endowment of good i in period t. Note that endowments for both agents are symmetric. In particular, an agent type 1 is well endowed with both goods in period $t = 0$ and vice versa for type 2.

The symmetry of endowments and preferences together with $\beta = 1$ implies that the first-best aggregate saving is zero, and each agent gets the average 2 units of each good in each

Table 1: Endowment profiles of the agents.

	endowments				first-best allocations				
	e^h_{10}	e^h_{20}	e^h_{11}	e^h_{21}	k^h	c^h_{10}	c^h_{20}	c^h_{11}	c^h_{21}
$h = 1$	3	3	1	1	0	2	2	2	2
$h = 2$	1	1	3	3	0	2	2	2	2

period (see Table 1). Accordingly, the first-best price of good 2 in period $t = 0$ is $P^{fb}_{20} = 1$ ("fb" stands for first-best), and the market fundamental is $z = 1$. Unfortunately, but to make our point, the first-best allocation is not attainable; that is, it violates the collateral constraints.

We now consider the economy with default and collateral (with an externality). The endowment profile and the first-best allocation suggest that agent 2 would like to move resources backwards in time from $t = 1$ to $t = 0$, i.e., borrow and therefore will be constrained. Hence, we will assume that agents type 2 hold no collateral, i.e. $k^1 = k^{ex}$ ("ex" stands for externality) and $k^2 = 0$. We will then solve for an equilibrium k^{ex}. Note that if agent type 1 wants to save and if there is no lending to another agent, this saving must be in good 2.

As shown in Appendix C.1, the equilibrium collateral is $k^{ex} = 1.3595$. In addition, there is no loss of generality to consider a solution with no security trading, i.e., $\theta^h = \hat{\theta}^h = 0$ for all h. As a result, agents actively trade in spot markets, which we denote by trades τ^h below. Despite this seeming triviality, this is a good starting point, a good example economy as there remains the issue of the value for unwinding collateral. The zero debt aspect resembles Hart and Zingales (2113) in which an entire market in human-capital-backed trade is shut down. It is worthy of emphasis, however, that all spot trades can be done using ex-ante security trades and we return to that interpretation below. The price of good 2 in period $t = 0$ is $P^{ex}_{20} = \left(\frac{4}{4-k^{ex}}\right)^2 = 2.2948$, and the market fundamental in period $t = 1$ is $z^{ex} = \frac{4}{4+k^{ex}} = 0.7463$, which implies that the spot price is $p(z^{ex}) = 0.5570$. Note that the collateral price at $t = 0$ is higher in the equilibrium with an externality, i.e., $P^{fb}_{20} = 1 < P^{ex}_{20} = 2.2948$. On the other hand, the spot price of good 2 in period $t = 1$ is lower in the equilibrium with the externality, i.e., $p(z^{fb}) = 1 > p(z^{ex}) = 0.5570$. In words, the collateral distortion makes the price of good 2 higher in the first period and lower in the second period relative to the first-best.

Figure 1 illustrates the competitive collateral equilibrium allocation (with externality) in period $t = 0$. It shows that an agent type 1, the unconstrained agent, sells good 1 and buys good 2, and vice versa for agent 2. In addition, the allocation is on the budget line of constrained agent 2, which is the line passing through $\mathbf{e}_0^2$. An unconstrained agent effectively does all the saving, $k^1 = k^{ex} = 1.3595$ and $k^2 = 0$. In the spot market the direction of trades is reversed. That is, an agent type 1 buys $\hat{\tau}_{ex}^1 = 0.3252$ units of good 1 and sells $\tau_{ex}^1 = -0.5839$ units of good 2 at price $p(z^{ex}) = 0.5570$ in spot markets, and vice versa. See figure 2a below. In addition, the expected utility of an agent type 1 and type 2 are $U_{ex}^1 = -2.2527$ and $U_{ex}^2 = -2.5724$, respectively.

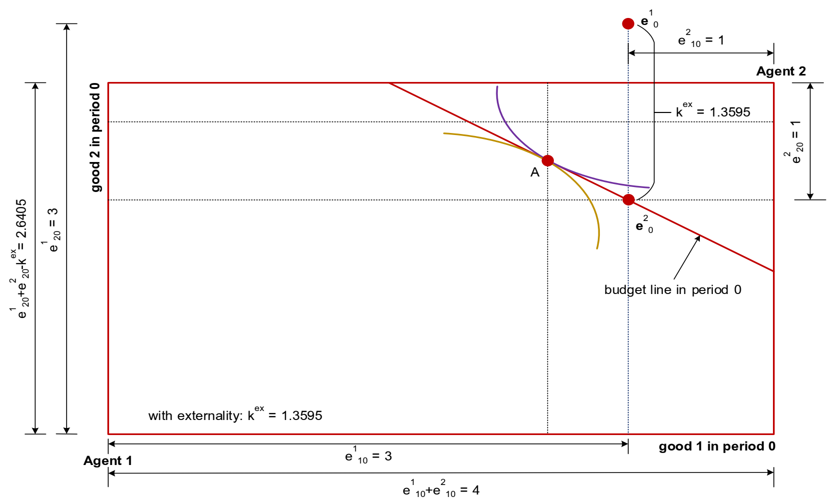


Figure 1: Point A is the competitive collateral equilibrium allocation (with externality) in period $t = 0$ with $\mathbf{c}_0^1 = (2.6899, 1.7756)$ and $\mathbf{c}_0^2 = (1.3101, 0.8649)$. An agent type 1 sells 0.3101 units of good 1, buys 0.1128 units of good 2, and $k^{ex} = 1.3595$ saves units of good 2, and vice versa.

We will now turn to a corresponding competitive equilibrium with segregated security exchanges (without an externality). There is *one active security exchange*, $z^{op} = 0.7729$ ("op" stands for optimality), even though all security exchanges are available for trade. That is, in equilibrium, both types optimally choose to trade in the same security exchange with specified market fundamental $z^{op} = 0.7729$. More formally, $\delta^1 (z^{op} = 0.7729) =$

$\delta^2 (z^{op} = 0.7729) = 1$. The collateral allocation in the competitive equilibrium with segregated security exchanges is $k^1 (z^{op} = 0.7729) = k^{op} = 1.1753$ and $k^2 = 0$. The equilibrium average or per capita saving (without externality) is $\sum_h \sum_z \alpha^h \delta^h (z) k^h (z) = \alpha^1 k^{op} = 0.5877$, which clearly smaller than the aggregate saving in the equilibrium with the externality $\sum_h \alpha^h k^h = \alpha^1 k^{ex} = 0.6798$, though more than the first-best which is zero. In this sense the distortion is corrected, and we shall see below the contracted second best allocation. In addition, an agent type 1 buys $\hat{\tau}^1_{op} = 0.2970$ units of good 1 and sells $\tau^1_{op} = -0.4972$ units of good 2 at price $p(z^{op}) = 0.5974$ in spot markets, and vice versa for agent 2. See figure 2b below.

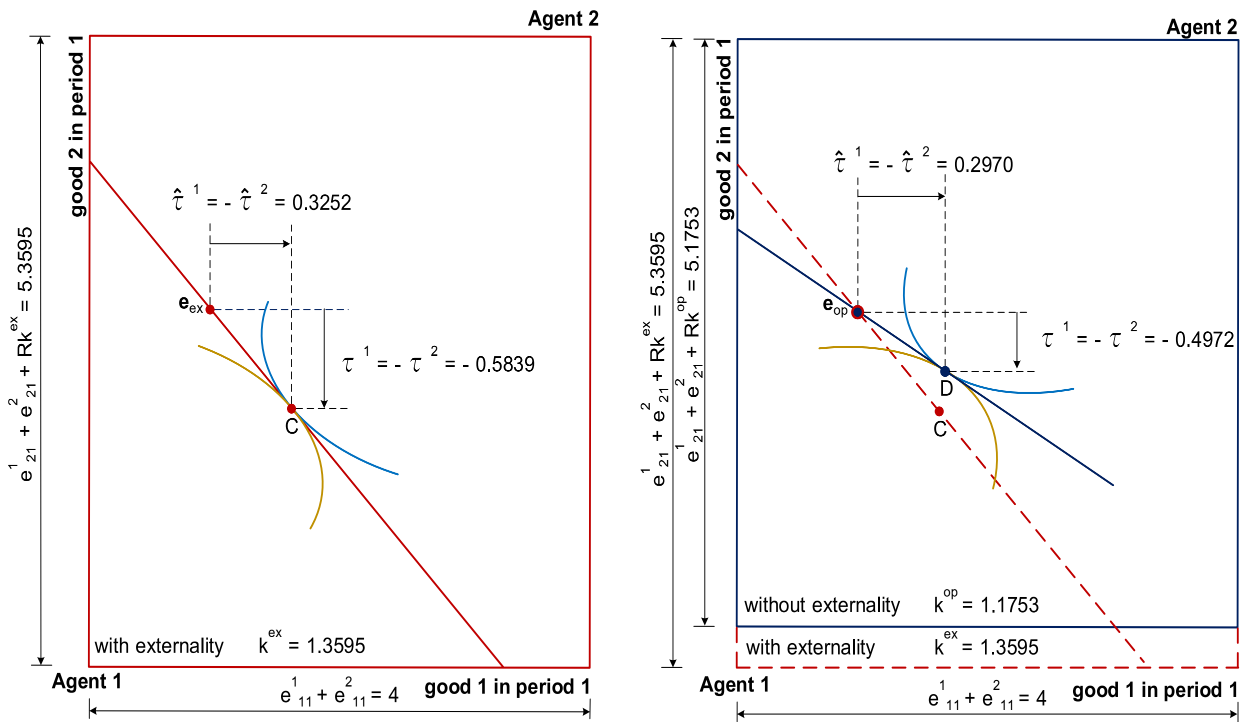
Table 2 presents equilibrium prices/fees of rights to trade in security exchanges $P_\Delta (z)$ for different market fundamental levels. Notice that as anticipated, the prices/fees of rights to trade in security exchanges are increasing with the market fundamentals z; that is, the larger the specified market fundamental of a security exchanges, the higher the fee of the security exchanges will be. Note also that the prices/fees of out-of-equilibrium (non-active) security exchanges are available, but at such prices agents do not want to trade them.

Table 2: Equilibrium prices of rights to trade in security exchanges $P_\Delta (z)$. Bold numbers are equilibrium prices for actively traded security exchanges.

	$z = 0.7479$	$\mathbf{z = 0.7729}$	$z = 0.7979$
$P_\Delta (z)$	0.4639	**0.5375**	0.6118

An agent type 1 is coming in with good 2 in storage, and therefore his discrepancy is positive. On the other hand, an agent type 2's discrepancy is negative. Thus, with a positive equilibrium fee $P_\Delta(z^{op}) = 0.5375$, an agent type 2 whose discrepancy from the fundamental is negative must get paid for the access to the security exchange. In particular, a constrained agent $(h = 2)$ with $\Delta^2(z^{op}) = -0.6813$, is receiving a transfer of $-P_\Delta(z^{op})\Delta^2(z^{op}) = 0.3662$ in period $t = 0$ for being in the security exchange $z^{op} = 0.7729$. Graphically, this shifts her budget line outward by $T = 0.3662$ as shown in figure 3. This illustrates how trading in rights to trade in security exchanges generates the redistribution of wealth in general equilibrium.

In addition, the expected utility of an agent type 1 and type 2 in this competitive equi-



(a) (b)

Figure 2: (a) Point C is the competitive collateral equilibrium allocation (with externality) in period $t = 1$ with $\mathbf{c}_1^1 = (1.3252, 1.7756)$ and $\mathbf{c}_1^2 = (2.6748, 3.5839)$. An agent type 1 buys 0.3252 units of good 1, sells 0.5839 units of good 2 at price $p(z^{ex}) = 0.5570$ in spot markets, and vice versa. Note that $\mathbf{e}_{ex}$ is the pre-trade allocation (with externality) in period $t = 1$. (b) Point D is the competitive equilibrium with segregated security exchanges allocation (without externality) in period $t = 1$ with $\mathbf{c}_1^1 = (1.2970, 1.6781)$ and $\mathbf{c}_1^2 = (2.7030, 3.4972)$. An agent type 1 buys 0.2970 units of good 1, sells 0.4972 units of good 2 at price $p(z^{op}) = 0.5974$ in spot markets, and vice versa. Note that $\mathbf{e}_{op}$ is the pre-trade allocation (without externality) in period $t = 1$.

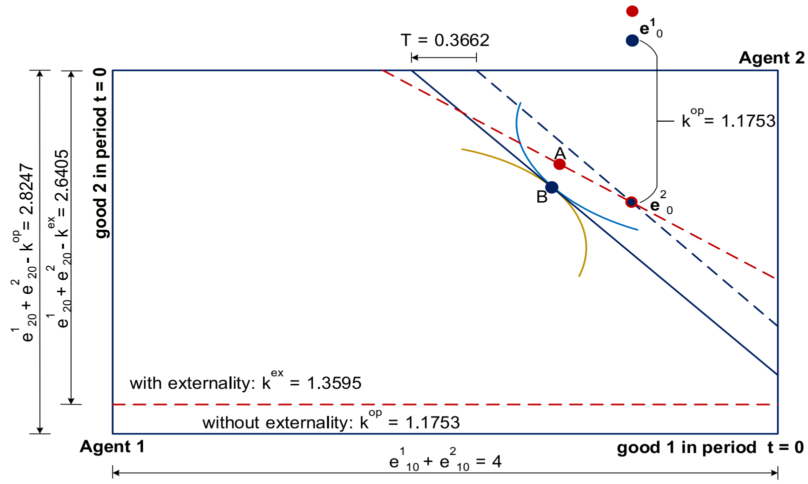


Figure 3: Point B is the competitive collateral equilibrium allocation (without externality) in period $t = 0$ with $\mathbf{c}_0^1 = (2.6073, 1.8410)$ and $\mathbf{c}_0^2 = (1.3927, 0.9837)$. where an agent type 1 sells 0.3927 units of good 1, buys 0.0163 units of good 2, and $k^{op} = 1.1753$ saves units of good 2, and vice versa.

librium with segregated security exchanges (without externality) are $U_{op}^1 = -2.2936, U_{op}^2 = -2.3905$, respectively. Recall that the expected utility of an agent type 1 and type 2 in the competitive collateral equilibrium allocation (with externality) are $U_{ex}^1 = -2.2527$ and $U_{ex}^2 = -2.5724$, respectively. This shows that internalizing the externality is beneficial to an agent type 2 (constrained agent) but may be harmful for an agent type 1 if there is no other remedy is employed. This is a (distributional) general equilibrium effect. Internalizing the externality improves efficiency of the economy, but also redistributes wealth. All agents can benefit from the efficiency effect, which shifts the Pareto frontier outward, but some agents may be harmed by the distributional effect. To induce welfare gain for all of agents, there must be lump sum transfers, as in the second welfare theorem which we establish below.

Note that with lower aggregate saving, the price of good 2 in period $t = 0$ in this competitive equilibrium with segregated security exchanges (without externality) is lower ($P_{20}^{op} = 2.0073 < P_{20}^{ex} = 2.2948$) but the spot price of good 2 is higher ($p(z^{op}) = 0.5974 > p(z^{ex}) = 0.5570$), relative to the one in the competitive collateral equilibrium allocation (with externality). That is, the collateral distortion makes the price of good 2 higher in the first period and lower in the second period relative to the constrained optimality, as proved in Proposition 2. In other words, the price of good 2 varies less over time when the externality is internalized. In this sense we mitigate fluctuations.

If we shut down all active spot markets in this example environment, we can still achieve a second best allocation by allowing agents to make date $t = 1$ promises to deliver good 1 or good 2. A promise to deliver must be backed by a promise issued by another type. As we establish more generally in Appendix B.2, there are collateral constraints for asset-backed promises and a need for valuation. Intuitively, the valuation can be done using $P_a(z_s, s) = p(z_s)\widehat{P}_a(z_s, s)$, and the collateral constraints for an agent type h become

$$P_a(z_s, s) R_s k^h + \widehat{P}_a(z_s, s)\hat{\theta}_s^h + P_a(z_s, s)\theta_s^h \geq 0, \forall s. \tag{38}$$

The next economy illustrates an economy with uncertainty where collateralized securities, $\hat{\theta}$, are actively traded (cannot be substituted by spot trades). All agents are constrained, but at different states. In particular, an agent will be binding in a state where her endowment is large. This is because she would like to transfer a part of such a large amount of wealth backwards in time from $t = 1$ to $t = 0$ but cannot do so because of the collateral constraints.

Environment 2 (State Contingent Securities)**.** The economy in this example is similar to the one in example 1 with two periods, but there are two states, $S = 2$. There are two types of agents, $H = 2$, both of which have an identical constant relative risk aversion (CRRA) utility function as in (37). Each type consists of $\frac{1}{2}$ fraction of the population, i.e. $\alpha^h = \frac{1}{2}$. In addition, the discount factor $\beta = 1$. The storage technology is constant and given by $R_s = 1$ for $s = 1, 2$. The endowment profile is presented in Table 3. Note unlike the first example that the agents are ex-ante identical in endowments. But agent type 1 has relatively more of both goods in state $s = 1$ than in state $s = 2$ and vice versa for agent type 2.

Table 3: Endowment profiles of the agents.

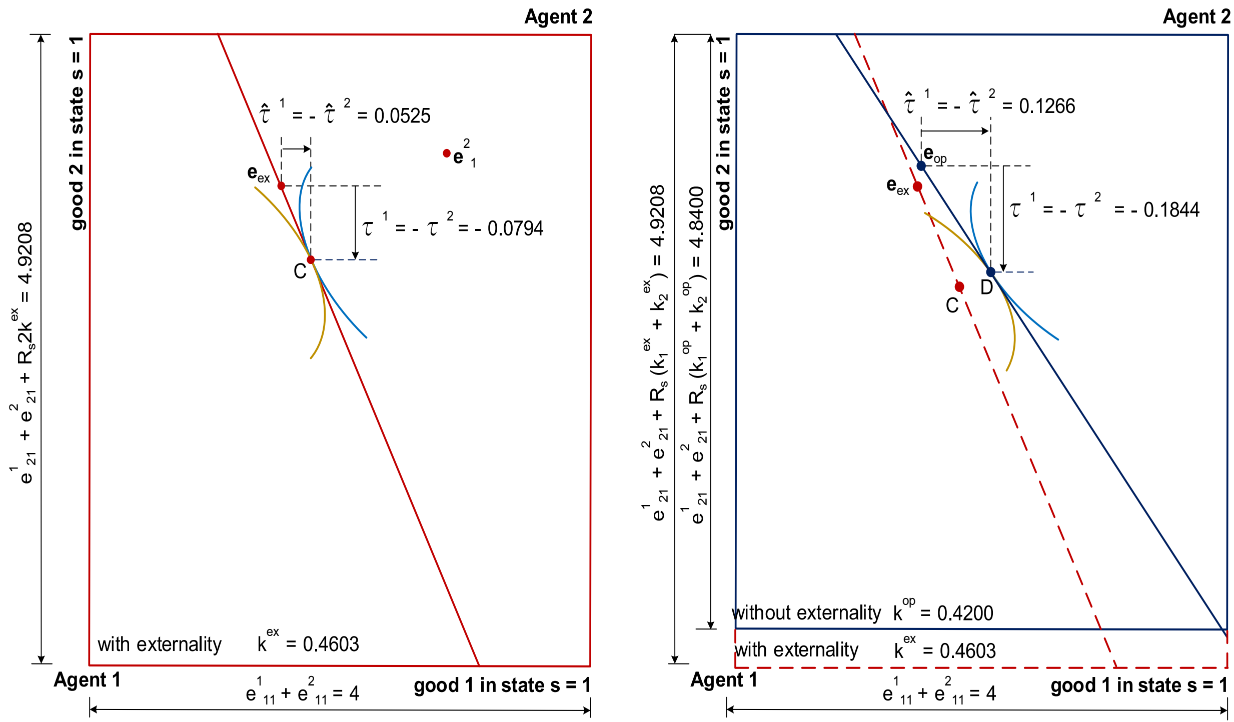
	e_{10}^h	e_{20}^h	e_{11}^h	e_{21}^h	e_{12}^h	e_{22}^h
$h = 1$	2	2	3	3	1	1
$h = 2$	2	2	1	1	3	3

First, the symmetry of the endowments and preferences implies that an equilibrium allocation in period $t = 0$ should be the same for all agents; that is, $c_{10}^h = c_{10}$ and $c_{20}^h = c_{20}$, for all h. Further, the indeterminacy between k^h and $\theta^h(s)$ implies that there is no loss of generality in considering the case with symmetric collateral allocation, i.e. $k^h = k$, for all h. The first-best is to assign 2 units of each goods to each agent in every period and every state. This involves making promises which would have required collateral in our limited commitment world.

The competitive collateral equilibrium (with externality) is as follows. The detailed derivation is again omitted and presented in Appendix C.3. The unique competitive collateral equilibrium (with externality) of this economy has $k^h = k^{ex} \approx 0.4603$, for all $h = 1, 2$. Accordingly, the market fundamental and spot price in state s are $z_s^{ex} = 0.8129$ and $p(z_s^{ex}) = 0.6608$, respectively, for all $s = 1, 2$. The price of good 2 in period $t = 0$ is $P_{20}^{ex} = 1.6872$, the prices of collateralized securities paying in units of good 1 and good 2 in state s are $\widehat{P}_s^{ex} = 1.2766, P_s^{ex} = 0.8436$, respectively, for all $s = 1, 2$. Note that the symmetry also makes the prices of collateralized securities the same across states.

Interestingly, there are security trades in this economy. In particular, an agent type 1 issues a security paying in units of good 1 in state $s = 1$, $\hat{\theta}_1^1 = -0.3042$ units backed by all of his collateral $k^1 = k^{ex} = 0.4603$, while an agent type 2 buys the same amount of this security. That is, an agent type 1's collateral constraint in state $s = 1$ is binding again here with positive collateral. In addition, an agent type 1 reverse the transaction in the spot market, buys $\hat{\tau}_{ex}^1 = 0.0525$ units of good 1 and sells $\tau_{ex}^1 = -0.0794$ units of good 2 at price $p(z_s^{ex}) = 0.6608$ in spot markets in state $s = 1$, and vice versa. The positions are reversed at state $s = 2$. In addition, the expected utility of an agent type 1 and type 2 are $U_{ex}^1 = U_{ex}^2 = -2.2035$. Figure 4a illustrates the competitive collateral equilibrium allocation (with externality) in state $s = 1$.

We now turn to the competitive equilibrium with segregated security exchanges (without externality). There is *one active security exchange* in each state, $z_s^{op} = 0.8285$ for all $s = 1, 2$, even though again all security exchanges are available for trade. That is, in equilibrium, both types optimally choose to trade in the same security exchange with specified market fundamental $z_s^{op} = 0.8285$ in both states. More formally, $\delta^1(\mathbf{z}^{op}) = \delta^2(\mathbf{z}^{op}) = 1$ where $\mathbf{z}^{op} =$



(a) (b)

Figure 4: (a) Point C is the competitive collateral equilibrium allocation (with externality) in state $s = 1$ with $\mathbf{c}_1^1 = (2.7483, 3.3808)$ and $\mathbf{c}_1^2 = (1.2517, 1.5397)$. An agent type 1 buys 0.0525 units of good 1, sells 0.0794 units of good 2 at price $p(z_1^{ex}) = 0.6608$ in spot markets, and vice versa. The positions are reversed in state $s = 2$. Note that $\mathbf{e}_{ex}$ is the pre-trade allocation (with externality) in state $s = 1$, and $\mathbf{e}_1^2$ is the endowment of agent type $h = 2$ in state $s = 1$ (excluding savings and securities trades). (b) Point D is the competitive equilibrium with segregated security exchanges allocation (without externality) in state $s = 1$ with $\mathbf{c}_1^1 = (2.7644, 3.3449)$ and $\mathbf{c}_1^2 = (1.2356, 1.4951)$. An agent type 1 buys 0.1266 units of good 1, sells 0.1844 units of good 2 at price $p(z^{op}) = 0.6864$ in spot markets, and vice versa. The positions are reversed in state $s = 2$. Note that $\mathbf{e}_{op}$ is the pre-trade allocation (without externality) in state $s = 1$.

$[0.8285, 0.8285]$ is the vector of active exchanges in both states. The collateral allocation in the competitive equilibrium with segregated security exchanges is $k^1(\mathbf{z}^{op}) = k^2(\mathbf{z}^{op}) = k^{op} = 0.4200$. That is, each type holds the same amount of collateral at $t = 0$. The equilibrium average or per capita saving without externality is $\sum_h \sum_{\mathbf{z}} \alpha^h \delta^h(\mathbf{z}) k^h(\mathbf{z}) = k^{op} = 0.4200$, which clearly smaller than the aggregate saving in the equilibrium with externality $\sum_h \alpha^h k^h = k^{ex} = 0.4603$, though still more than the first-best which is zero. As a result, the market fundamental and the spot price of good 2 in each state s are higher in the competitive equilibrium with segregated security exchanges (without externality), i.e., $z_s^{op} = 0.8285 > z_s^{ex} = 0.8129$ and $p(z_s^{op}) = 0.6864 > p(z_s^{ex}) = 0.6608$. In addition, the price of good 2 in period 0 in the competitive equilibrium with segregated security exchanges (without externality) is $P_{20}^{op} = 1.5903 < P_{20}^{ex} = 1.6872$, which is again lower than the one in the competitive equilibrium with the externality.

We can recover the positions of each security and spot trades using the same approach as in the competitive collateral equilibrium with the externality. An agent type 1 issues a security paying in units of good 1 in state $s = 1$, $\hat{\theta}_1^1 = -0.2872$ units backed by all of his collateral $k^1 = k^{op} = 0.4200$, while an agent type 2 buys the same amount of this security. That is, an agent type 1's collateral constraint in state $s = 1$ is binding. In addition, an agent type 1 buys $\hat{\tau}_{op}^1 = 0.1266$ units of good 1 and sells $\tau_{op}^1 = -0.1844$ units of good 2 at the larger price $p(z_s^{op}) = 0.6864$ in spot markets in state $s = 1$, and vice versa for an agent type 2. The positions are reversed at state $s = 2$. The symmetry of the equilibrium solution implies that net borrowing of each agent is zero. In addition, the expected utility of an agent type 1 and type 2 are $U_{op}^1 = U_{op}^2 = -2.2024$. Figure 4b illustrates the competitive equilibrium with segregated security exchanges allocation (without externality) in state $s = 1$. Note that agents trade less securities relative to the equilibrium with the externality. This is because the agents save less, and are issuing fewer securities. That is, the externality generates too much saving/collateral.

Table 4 presents equilibrium prices/fees of rights to trade in security exchanges in each state $P_\Delta(z_s, s)$ for different market fundamental levels. Notice that the prices/fees of rights to trade in security exchanges are again increasing with the market fundamentals z_s in both states. Recall that the net transfer in period $t = 0$ for an agent type h is the sum of all payments/subsidies for rights to trade in all active security exchanges in all states, $\sum_{\mathbf{z}} \delta^h(\mathbf{z}) \sum_s P_\Delta(z_s, s) \Delta_s^h(\mathbf{z})$. The symmetry of the equilibrium solution implies that the equilibrium rights to trade for an agent type h in state $s = 1$ is exactly the opposite of his rights to trade in state $s = 2$, and the prices of the rights to trade in both exchanges are identical. As a result, payments/subsidies for an agent type h from trading in those security exchanges are canceling out each other completely, and therefore each agent type h receives zero net transfer in period $t = 0$. That is, there is no distributional general equilibrium effect in this case[15].

Table 4: Equilibrium prices of rights to trade in security exchanges $P_\Delta(z_s, s)$ which are the same in both states due to symmetry. Bold numbers are equilibrium prices for actively traded security exchanges.

	$z_s = 0.8035$	$\mathbf{z_s = 0.8285}$	$z_s = 0.8535$
$P_\Delta(z_s, s)$	0.0900	**0.1142**	0.1400

The following example presents an economy where it is possible to assign agents to different security exchanges and have multiple segregated security exchanges. A formal formulation with randomization is presented in the next section.

Environment 3 (Heterogeneous Borrowers and the Role of Randomization)**.** There are two periods, $t = 0, 1$, and a single state, $S = 1$, in period $t = 1$ (no uncertainty). There are three types of agents, two borrower types and one lender. Each agent is given the same utility function as in (37). Each type consists of $\frac{1}{3}$ fraction of the population, i.e. $\alpha^h = \frac{1}{3}$. Similar to the previous example, $\beta = 1$, and $R = 1$. The endowment profile is given in Table 5 below. Note that to conserve on space we do not present the first-best allocation of this economy.

We first consider the competitive collateral equilibrium (with externality). The detailed derivation is presented in Appendix C.3. With relatively large endowments in $t = 1$, an agent type 2 and an agent type 3 want to move resources backwards in time from $t = 1$ to $t = 0$, i.e., borrow. The scarcity of collateral then implies that both of them will be collateral constrained. Similar to Environment 1, we assume that only an agent type 1 hold

[15]This is special to the example and would not happen with more heterogeneity.

Table 5: Endowment profiles of the agents.

Type of Agents	e_{10}^h	e_{20}^h	e_{11}^h	e_{21}^h
$h=1$	4.26	11.5	0.5	0.5
$h=2$	3.92	0.5	7	5
$h=3$	4.32	0.5	5	7

collateral/storage, $k^1 = k^{ex}$ while an agent type 2 and an agent type 3 hold no collateral, i.e., $k^2 = k^2 = 0$. The unique competitive collateral equilibrium (with externality) has $k^1 = k^{ex} = 7.2836$, and there is no security trading, i.e., $\hat{\theta}^h = \theta^h = 0$ for all h. The price of good 2 in period $t = 0$ is $P_{20}^{ex} = 5.7422$, and the market fundamental in period $t = 1$ is $z^{ex} = \frac{12.5}{12.5+k^{ex}} = 0.6318$, which implies that the spot price is $p\left(z^{ex}\right) = 0.3992$. The competitive equilibrium allocation is shown in Table 6 below.

Table 6: The competitive collateral equilibrium allocation for Environment 3.

Agent Type	k^h	$\hat{\tau}^h$	τ^h	c_{10}^h	c_{20}^h	c_{11}^h	c_{21}^h	U^h
$h=1$	7.284	1.711	-4.285	8.384	3.499	2.211	3.499	-1.1433
$h=2$	0.000	-1.487	3.725	1.999	0.834	5.513	8.725	-1.9950
$h=3$	0.000	-0.223	0.560	2.117	0.883	4.777	7.560	-1.9460

We now turn to the competitive equilibrium with segregated security exchanges[16] (without externality), shown in Table 7 below. Interestingly, there are *two active security exchanges*, $z = 0.6113$ and $z = 0.8132$. The security exchange $z = 0.6113$ consists of some fraction of agents type 1 (19.69 percent), and all of agents type 3 (a constrained type). On the other hand, the security exchange $z = 0.8132$ consists of some residual fraction of agents type 1 (80.31 percent), and all of agents type 2 (a constrained type). In addition, an agent type 1 holds collateral which is a weighted average of collateral for each segregated market: $k^1 = 0.1969 \times 6.3082 + 0.8031 \times 4.6072 = 4.9421$. Note that there are more agents type 1 in security exchange $z = 0.8132$ than in security exchange $z = 0.6113$, and agent 1's spot trading varies across security exchanges due in part to a difference in spot prices.

[16]The competitive equilibrium with segregated security exchanges here corresponds to a Pareto optimal allocation (a solution to Program 2) with Pareto weights $\lambda^1 = 0.8, \lambda^2 = \lambda^3 = 0.1$.

Table 7: Equilibrium allocation of (non-zero-mass) lotteries. There are multiple active security exchanges; $z = 0.6113$ and $z = 0.8132$.

	$h=1$		$h=2$	$h=3$
k	6.3082	4.6072	0.0000	0.0000
$\hat{\tau}$	1.3892	1.6384	-1.3159	-0.2735
τ	-3.7176	-2.4776	1.9898	0.7319
c_{10}	5.6204	5.6204	4.4835	2.3961
c_{20}	3.3982	3.3982	2.7106	1.4491
c_{11}	1.8892	2.1384	5.6841	4.7265
c_{21}	3.0905	2.6296	6.9898	7.7319
z	**0.6113**	**0.8132**	**0.8132**	**0.6113**
Δ	3.6618	3.6532	-2.9340	-0.7209
x^h	0.1969	0.8031	1.0000	1.0000
U^h	-1.3211		-0.9110	-1.4483

The key reason for the existence of multiple segregated exchanges in equilibrium is that it is socially optimal to compensate constrained agents with positive transfers at period $t = 0$, to try to move back toward the first best, i.e., alleviate borrowing constraints, in this case. As mentioned earlier, agents type 2 and agents type 3 are collateral constrained, and therefore have relatively high marginal utility at period $t = 0$. As a result, it is socially optimal to give them more resources in period $t = 0$. This is the case in the competitive equilibrium with segregated security exchanges (without externality), where the discrepancy from the market fundamental of both types are negative (the 8^{th} row of Table 7), i.e., $\Delta^2 = -2.9340$ and $\Delta^3 = -0.7209$. With positive equilibrium price of the discrepancy, agents type 2 and agents type 3 receive transfers from rights to trade fees $P_\Delta(z)\,\Delta^h(z)$ of 6.6122 units of good 1 in period $t = 0$ and 0.6739 units of good 1 in period $t = 0$, respectively. Of course, agents type 1 pay all these fees paid in proportion to the relative number of types assigned to each exchange. Equilibrium fees of security exchanges, including the fees of inactive (out-of equilibrium) security exchanges are summarized in Table 8 below.

Intuitively, agents type 1 would like to buy into the higher z security exchange, which

Table 8: Equilibrium fees of security exchanges. The bold numbers are (actively traded) equilibrium prices.

	$z = 0.6088$	$\mathbf{z = 0.6113}$	$z = 0.6138$	$z = 0.8088$	$\mathbf{z = 0.8132}$	$z = 0.8138$
$P_\Delta(z)$	0.9119	**0.9348**	0.9589	2.2339	**2.2537**	2.2564

is $z = 0.8132$ in this case, where good 2 is more valuable because with (endogenous) saving she will end up with more of good 2 than good 1 in period $t = 1$. The question is then why did not all of type 1 choose that exchange with certainty in equilibrium. The answer is that such deterministic choice is not affordable. The total expenditure for such deterministic allocation would be of 35.7523 units of the numeraire while his income is only 35.7214 units of the numeraire, which is not enough to cover his expenditure. On the other hand, his total expenditure for the randomized equilibrium allocation is 35.7214 units of the numeraire, which is exactly equal to his income.

In addition, it is also natural to ask if it would be socially optimal to move everyone to a unique security exchange instead of segregating them as in the actual equilibrium. Contrary to the equilibrium outcome, if an agent type 3 were to enter into a security exchange with $z > 0.7143$, he would have had to pay for the right to trade in that security exchange given that her ratio of good 1 to good 2 in period $t = 1$ is $\frac{5}{7} = 0.7143$. In particular, if all agents were in one exchange but keeping the same allocation including storage, the market fundamental in that exchange would have been $z = \frac{12.5}{12.5+4.9421} = 0.7167 > 0.7143$. As a result, the discrepancy from the market fundamental of an agent type 3 would be $\Delta^3 = 0.0166$. This positive discrepancy then would imply that agent type 3 would have to pay for the right to trade in that security exchange, given that the corresponding fee is strictly positive. As with discussion earlier, charging fees in period t = 0 to this constrained agent type 3 would move against the socially optimal direction. That is, putting all agents in only one security exchange is not socially optimal in this case. It is socially optimal to allocate agents to multiple segregated exchanges, though this is at the cost of divergent marginal rates of substitution. The more general point is that mixing with lotteries can be useful, and we incorporate that into the more notation and proofs below.

This example also suggests that the number of active segregated exchanges is equal to the number of constrained types. Consistent with this idea, Appendix C.6, an economy with four agent types, three of which are constrained, has three active segregated exchanges. Intuitively, it is socially optimal to put different types in different exchanges to face distinct prices, as in our taxes/subsidy, budget-shifter impact from the choice of the fundamental z in section 4.1.

5 Existence and Welfare Theorems of Competitive Equilibrium with Segregated Security Exchanges

This section presents a more formal representation of our competitive equilibrium with segregated security exchanges. To deal with the non-convexity problem generated by the collateral constraints, we now use a probability measure or a lottery as the commodity. That is, we now suppose it is possible to assign agents to different security exchanges even in state s as if by a lottery as in example 3. Security trades are also bundled into this potentially random assignment. Security exchange assignments, by lottery or not, are still state-contingent. Importantly, a member of a security exchange z_s can trade securities with other members in the same security exchange only.

More formally, for each agent type h, let $x^h(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta) \geq 0$ denote a probability measure on $\left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right)$, where Δ_s satisfies (25) for all s. In other words, $x^h\left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right)$ is the probability of receiving period $t = 0$ consumption, $\mathbf{c}_0 \equiv (c_{10}, c_{20})$, collateral, k, securities paying in good 1, $\hat{\theta}_s$, securities paying in good 2, θ_s, and being in security exchange z_s in state s where all securities are executed and all spot trades take place also. Recall that a positive (negative) amount of trade means receiving (transferring out) the specified good.

As a probability measure, a lottery of an agent type h satisfies

$$\sum_{\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta} x^h\left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right) = 1, \tag{39}$$

but unlike discrete choice notation δ^h, this lottery may be non degenerate for some bundles, i.e., $0 < x^h < 1$. With a continuum of agents, $x^h\left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right)$ can be interpreted as the fraction of agents type h assigned to a bundle $\left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right)$. More formally, with all choice objects gridded up as an approximation, the commodity space L is assumed to be a finite n-dimensional linear space[17].

For notational purposes, let $b = \left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right)$ be a typical commodity, called a bundle. We will use b and $\left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right)$ interchangeably. Accordingly, we can write $\mathbf{x}^h \equiv \left[x^h(b)\right]_b \in \mathbb{R}^n_+$ as a typical lottery for an agent type h.

A holder of a bundle $b = \left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right)$ will receive k units of collateral and hold portfolio of securities $\left(\hat{\theta}, \theta\right)$. With limited commitment, each bundle b will be feasible only if the collateral and security assignments satisfy the collateral constraints (3) which we repeat here:

$$p\left(z_s\right) R_s k + \hat{\theta}_s + p\left(z_s\right)\theta_s \geq 0,\ \forall s. \tag{40}$$

Accordingly, we impose the following condition on a probability measure $x^h\left(b\right)$.

$$\begin{aligned} x^h\left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right) &\geq 0 \text{ if } \left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right) \text{ satisfies (25) and (40)}, \\ &= 0 \text{ if otherwise.} \end{aligned} \tag{41}$$

In words, a positive measure can be defined only on feasible bundles, which have to satisfy conditions (25) and (40).

More formally, the consumption possibility set of an agent type h is defined by

$$X^h = \left\{\mathbf{x}^h \in \mathbb{R}^n_+ : \sum_b x^h\left(b\right) = 1, \text{ and for any } b, x^h\left(b\right) \text{ satisfies (41)}\right\}. \tag{42}$$

Let $\mathbf{x}^h$ be a typical element of X^h. Note that $X^h \subset L$ is compact and convex. In addition, the non-emptiness of X^h is guaranteed by assigning mass one on each agent's endowment.

5.1 Competitive Equilibrium with Segregated Security Exchanges

Let P_{20} be the price of good 2 in period $t = 0$, and $P\left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right)$ be the price of a bundle $\left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right)$. Note that the price of good 1 in period $t = 0$ is $P_{10} = 1$ as good 1 is the numeraire good. Each agent is infinitesimally small relative to the entire economy and will take all prices as given. The broker-dealers introduced below will also act competitively. Note as well that Δ is also competitively priced.

[17] The limiting arguments under weak-topology used in Prescott and Townsend (1984a) can be applied to establish the results if L is not finite.

Consumers: Each agent h, taking prices, $P_{20}, P\left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right)$, as given, chooses $\mathbf{x}^h$ in period $t = 0$ to maximize its expected utility:

$$\max_{\mathbf{x}^h} \sum_{\left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right)} x^h\left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right)\left\{u\left(c_{10}^h, c_{20}^h\right) + \beta \sum_s \pi_s u\left(e_{1s}^h + \hat{\theta}_s, e_{2s}^h + R_s k + \theta_s\right)\right\} \tag{43}$$

subject to $\mathbf{x}^h \in X^h$, and period $t = 0$ budget constraint

$$\sum_{\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta} P\left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right) x^h\left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right) \leq e_{10}^h + P_{20} e_{20}^h, \tag{44}$$

which states that the agent sells all her endowments[18] including good 2 at price P_{20} and uses this income to buy lotteries $\mathbf{x}^h$, which includes consumption in period $t = 0$, $\left(c_{10}^h, c_{20}^h\right)$.

In state s, a holder of bundle $\left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right)$ receives, in addition to her endowments of good 1 and good 2 $\left(e_{1s}^h, e_{2s}^h\right)$, $\hat{\theta}_s$ units of good 1 as the net-payment of portfolio $\hat{\theta}$, $R_s k$ units of good 2 from the collateral/saving good, θ_s units of good 2 as the net-payment of portfolio θ. Of course, if $\hat{\theta}_s$ and θ_s are negative, these are promises to pay and require collateral. It is worthy of emphasis that the agent will trade in security exchange z_s, where she can in principle trade good 1 and good 2 at price $p(z_s)$ in spot markets. (Again in the equilibrium under consideration it will not be necessary to trade in spot markets even though they believe they could.)

Broker-Dealers: Broker-dealers are agents who try to put together deals, much like brokers on Wall street who put buyers and sellers of securities together. When ask and bid prices are not matched, brokers will try to "clear the market" by communicating with both sides of the markets (trying to change quantities and/or prices), so that demand equals supply at a given price at the end.

Formally, the broker-dealer issues (sells) $y\left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right) \in \mathbb{R}_+$ units of each bundle $\left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right)$, at the unit price $P\left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right)$. Note that the broker-dealer can issue any non-negative number of a bundle $\left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right)$; that is, the number of bundles issued

[18] It is worthy of emphasis that we can write an equivalent problem specifying consumption transfers in period $t = 0$, instead of consumption allocation. By doing so, agents do not need to sell their entire endowments but simply buy and sell consumption transfers. In other words, it is not restrictive that we make agents sell their entire endowments and buy consumption allocation through lotteries.

does not have to be between zero and one and is not a lottery. It is simply the number of bundles, a real number. Let $\mathbf{y} \in L$ be the vector of the number of bundles issued as one move across $\left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right)$. With constant returns to scale, the profit of a broker-dealer must be zero and the number of broker-dealers becomes irrelevant. Therefore, without loss of generality, we assume there is one representative broker-dealer, which takes prices as given.

The objective of the broker-dealer is to maximize its profit by supplying $\mathbf{y} \in L$ as follows:

$$\max_{\mathbf{y}} \sum_{\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta} y\left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right)\left[P\left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right) - c_{10} - P_{20}c_{20} - P_{20}k\right] \tag{45}$$

subject to technology constraints:

$$\sum_{\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}_{-s}, \Delta} y\left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}_{-s}, z_s, \Delta\right)\hat{\theta}_s = 0, \ \forall s; z_s, \tag{46}$$

$$\sum_{\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}_{-s}, \Delta} y\left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}_{-s}, z_s, \Delta\right)\theta_s = 0, \ \forall s; z_s, \tag{47}$$

$$\sum_{\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}_{-s}, \Delta} y\left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}_{-s}, z_s, \Delta\right)\Delta_s = 0 \ , \forall s; z_s, \tag{48}$$

taking prices $P_{20}, P\left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right)$ as given. See our working paper Kilenthong and Townsend (2011b) for more details.

The existence of an optimum to the broker-dealer's problem requires, that for any bundle $\left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right)$,

$$P\left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right) \leq c_{10} + P_{20}c_{20} + P_{20}k + \sum_s \widehat{P}_a\left(z_s, s\right)\hat{\theta}_s + \sum_s P_a\left(z_s, s\right)\theta_s + \sum_s P_\Delta\left(z_s, s\right)\Delta_s \tag{49}$$

where $\widehat{P}_a(z_s, s)$, $P_a(z_s, s)$, and $P_\Delta(z_s, s)$ are the Lagrange multipliers for the zero-net-supply constraints for securities paying in good 1 (46), for the zero-net-supply constraints for securities paying in good 2 (47), and for consistency constraints (48), respectively. In particular, for a security exchange z_s in state s, $\widehat{P}_a(z_s, s)$, $P_a(z_s, s)$, and $P_\Delta(z_s, s)$ are the shadow prices of a securities paying in good 1 and good 2, and the shadow price of "type h discrepancies from the fundamental" in the security exchange z_s, respectively. Condition (49) holds with equality if $y\left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right) > 0$. Here $P\left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right)$ is the revenue from the sale of one unit of bundle $\left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right)$. This condition is in fact the necessary and sufficient condition for the saddle-point profit maximization problem.

Market Clearing: The market clearing condition for good 1 in period $t = 0$ is

$$\sum_{\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta} y\left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right) c_{10} = \sum_h \alpha^h e_{10}^h \tag{50}$$

Similarly, the market clearing condition for good 2 in period $t = 0$ is

$$\sum_{\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta} y\left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right) [c_{20} + k] = \sum_h \alpha^h e_{20}^h \tag{51}$$

The market clearing conditions for lotteries in period $t = 0$ are

$$\sum_h \alpha^h x^h \left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right) = y\left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right), \ \forall \left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right) \tag{52}$$

Definition 4. A competitive equilibrium with segregated security exchanges (with lottery) is a specification of allocation $(\mathbf{x}, \mathbf{y})$, and prices $P_{20}, P\left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right)$ such that

(i) for each h, $\mathbf{x}^h \in X^h$ solves utility maximization problem (43) subject to period $t = 0$ budget constraint (44), taking prices $P_{20}, P\left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right)$ as given;

(ii) for the broker-dealer, $\left\{\mathbf{y}, \widehat{P}_a(z_s, s), P_a(z_s, s), P_\Delta(z_s, s)\right\}$ solves profit maximization problem (45) subject to technology constraints (46), (47) and (48) taking prices P_{20}, $P\left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right)$ as given;

(iii) markets for good 1, for good 2, and for lotteries in period $t = 0$ clear, i.e., (50), (51) and (52) hold.

5.2 Constrained Optimal Allocations

An allocation $\mathbf{x} \equiv \left(\mathbf{x}^h\right)_h$ is attainable if $\mathbf{x}^h \in X^h$ for all h, and it satisfies the following feasibility constraints.

Recall that good 1 cannot be stored; only good 2 is storable. The aggregate endowment of good 1 in period $t = 0$ is $\sum_h \alpha^h e_{10}^h$. Therefore, the resource constraint for good 1 in period $t = 0$ is given by

$$\sum_h \sum_{\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta} \alpha^h x^h \left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right) c_{10} \leq \sum_h \alpha^h e_{10}^h. \tag{53}$$

Similarly, the resource constraint for good 2 in period $t = 0$ is given by

$$\sum_h \sum_{\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta} \alpha^h x^h \left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right) [c_{20} + k] \leq \sum_h \alpha^h e^h_{20}. \tag{54}$$

Note that the non-negativity constraint on k guarantees that the aggregate saving is non-negative.

Recall that all securities are executed within each assigned security exchange only. In particular, for a security exchange z_s in state s, the net supply of a security paying in good 1 in state s, $\hat{\theta}_s$ must be zero:

$$\sum_h \sum_{\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}_{-s}, \Delta} \alpha^h x^h \left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}_{-s}, z_s, \Delta\right) \hat{\theta}_s = 0 \ , \forall s; z_s, \tag{55}$$

where $\mathbf{z}_{-s} = (z_1, \ldots, z_{s-1}, z_{s+1}, \ldots, z_S)$ is a vector of market fundamentals in all states but state s. This feasibility condition holds for every state s and every security exchange z_s. Similarly, the feasibility or market clearing constraints for securities paying in good 2 are as follows:

$$\sum_h \sum_{\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}_{-s}, \Delta} \alpha^h x^h \left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}_{-s}, z_s, \Delta\right) \theta_s = 0 \ , \forall s; z_s. \tag{56}$$

The market fundamental in each security exchange must be consistent. The planner must choose the composition of agents to set the market fundamental for each security exchange to its specified level. With identical homothetic preferences, the consistency constraint for a security exchange z_s is that the aggregate ratio of good 1 to good 2 within the security exchange z_s must be exactly z_s:

$$z_s = \frac{\sum_h \sum_{\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}_{-s}, \Delta} \alpha^h x^h \left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}_{-s}, z_s, \Delta\right) \left(e^h_{1s} + \hat{\theta}_s\right)}{\sum_h \sum_{\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}_{-s}, \Delta} \alpha^h x^h \left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}_{-s}, z_s, \Delta\right) \left(e^h_{2s} + R_s k + \theta_s\right)}. \tag{57}$$

Using the feasibility conditions for securities within each security exchange, (55)-(56), and the definition of "type h discrepancy from the fundamental" (25), these consistency constraints can be rewritten as

$$\sum_h \sum_{\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}_{-s}, \Delta} \alpha^h x^h \left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}_{-s}, z_s, \Delta\right) \Delta_s \ = \ 0, \ \forall s, z_s. \tag{58}$$

This consistency constraint reflects the depletability of the externality.

Definition 5. An allocation $\mathbf{x} \equiv \left(\mathbf{x}^h\right)_{h=1}^H \in X^1 \times \ldots \times X^H$ is said to be *attainable* if $\mathbf{x}^h \in X^h$ for every h, and it satisfies (53)-(56) and (58).

Let X denote the set of all attainable allocations. With finite linear weak-inequality constraints, the attainable set X is compact and convex. In addition, the assumption that the endowment is on the grids also ensures that X is nonempty.

A constrained optimal allocation is an attainable allocation such that there is no other attainable allocation that can make at least one agent type strictly better off without making any other agent type worse off. To be precise, the expected utility of an agent type h, holding a lottery $\mathbf{x}^h$, is given by

$$U^h\left(\mathbf{x}^h\right) = \sum_{\mathbf{c}_0,k,\hat{\theta},\theta,\mathbf{z},\Delta} x^h\left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right)\left\{u\left(c_{10}^h, c_{20}^h\right) + \beta\sum_s \pi_s u\left(e_{1s}^h + \hat{\theta}_s, e_{2s}^h + R_s k + \theta_s\right)\right\}$$

Definition 6. An attainable allocation $\mathbf{x}^* \in X$ is said to be a *constrained optimal* allocation if there is no another attainable allocation $\mathbf{x} \in X$ such that

$$U^h\left(\mathbf{x}^h\right) \geq U^h\left(\mathbf{x}^{*h}\right) \text{ for every } h\text{, and } U^{\bar{h}}\left(\mathbf{x}^{\bar{h}}\right) > U^{\bar{h}}\left(\mathbf{x}^{*\bar{h}}\right) \text{ for some } \bar{h}$$

We characterize constrained optimality using the following Pareto program. Let $\lambda^h \geq 0$ be the Pareto weight of agent type h. There is no loss of generality to normalize the weights such that $\sum_h \lambda^h = 1$. A constrained Pareto optimal allocation $\mathbf{x}^*$ solves the following Pareto program.

Program 2. The Pareto Program with endogenous segregated security exchanges is defined as follows:

$$\max_{\left(\mathbf{x}^h \in X^h\right)_h} \sum_h \lambda^h \alpha^h \sum_{\mathbf{c}_0,k,\hat{\theta},\theta,\mathbf{z},\Delta} x^h\left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right)\left\{u\left(c_{10}^h, c_{20}^h\right) + \beta\sum_s \pi_s u\left(e_{1s}^h + \hat{\theta}_s, e_{2s}^h + R_s k + \theta_s\right)\right\} \tag{59}$$

subject to

$$\sum_h \sum_{\mathbf{c}_0,k,\hat{\theta},\theta,\mathbf{z},\Delta} \alpha^h x^h \left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right) c_{10} \leq \sum_h \alpha^h e_{10}^h, \tag{60}$$

$$\sum_h \sum_{\mathbf{c}_0,k,\hat{\theta},\theta,\mathbf{z},\Delta} \alpha^h x^h \left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right) [c_{20} + k] \leq \sum_h \alpha^h e_{20}^h, \tag{61}$$

$$\sum_h \sum_{\mathbf{c}_0,k,\hat{\theta},\theta,\mathbf{z}_{-s},\Delta} \alpha^h x^h \left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}_{-s}, z_s, \Delta\right) \hat{\theta}_s = 0 \ , \forall s; z_s, \tag{62}$$

$$\sum_h \sum_{\mathbf{c}_0,k,\hat{\theta},\theta,\mathbf{z}_{-s},\Delta} \alpha^h x^h \left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}_{-s}, z_s, \Delta\right) \theta_s = 0 \ , \forall s; z_s, \tag{63}$$

$$\sum_h \sum_{\mathbf{c}_0,k,\hat{\theta},\theta,\mathbf{z}_{-s},\Delta} \alpha^h x^h \left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}_{-s}, z_s, \Delta\right) \Delta_s = 0, \ \forall s, z_s. \tag{64}$$

Note again that we already embedded the "individual discrepancies from the fundamental" (25), the probability constraints (39) and the collateral constraints (40) into the consumption possibility sets X^h.

It is clear that the objective function now is linear in x^h. Thereby it is continuous and weakly concave. As discussed earlier, the feasible set X is non-empty, compact, and convex. Therefore, a solution to the Pareto program for given positive Pareto weights exists and is a global maximum. The proof of the equivalence between Pareto optimal allocations and the solutions to the program is omitted for brevity (see Prescott and Townsend, 1984b, for a similar proof).

As in the classical general equilibrium model, the economy is a well-defined convex economy, i.e., the commodity space is Euclidean, the consumption set is compact and convex, the utility function is linear. As a result, the first and second welfare theorems hold, and a competitive equilibrium exists. In particular, this section proves that the competitive equilibrium is constrained optimal and any constrained optimal allocation can be supported by a competitive equilibrium with transfers. Then, we use Negishi's method to prove the existence of a competitive equilibrium. For brevity, all related definitions and proofs are omitted here but are available in Kilenthong and Townsend (2011b), and similar proofs are also available in Prescott and Townsend (2006).

The standard contradiction argument is used to prove the first welfare theorem below. We assume that there is no local satiation point in the consumption set.

Assumption 2. *For any $\mathbf{x}^h \in X^h$, there exists $\tilde{\mathbf{x}}^h \in X^h$ such that*

$$U^h\left(\tilde{\mathbf{x}}^h\right) > U^h\left(\mathbf{x}^h\right) \tag{65}$$

where $U^h\left(\mathbf{x}^h\right)$ is the expected utility of agent h derived from allocation $\mathbf{x}^h$.

This assumption is easily satisfied using reasonable specifications of the grid of consumption allocation in period $t = 0$. For example, with a strictly increasing utility function, if we include a very large consumption allocation in period $t = 0$ into the grid (larger than what can be attained with endowments and storage), then the local nonsatiation assumption will be satisfied.

Theorem 1. *With local nonsatiation of preferences (Assumption 2), a competitive equilibrium with segregated security exchanges allocation is constrained optimal.*

The Second Welfare theorem states that any constrained optimal allocation, corresponding to strictly positive Pareto weights, can be supported as a competitive equilibrium with segregated exchanges with transfers. The standard approach applies here. In particular, we first prove that any constrained optimal allocation can be decentralized as a compensated equilibrium with segregated exchanges. Then, we use a standard cheaper-point argument (see Debreu, 1954) to show that any compensated equilibrium with segregated exchanges is a competitive equilibrium with segregated exchanges with transfers.

Theorem 2. *Any constrained optimal allocation corresponding with strictly positive Pareto weights $\lambda^h > 0, \forall h$ can be supported as a competitive equilibrium with segregated security exchanges with transfers.*

We use Negishi's mapping method (Negishi, 1960) to prove the existence of competitive equilibrium with segregated exchanges. The proof benefits from the second welfare theorem. Specifically, a part of the mapping applies the theorem in that the solution to the Pareto program is a competitive equilibrium with segregated exchanges with transfers. We then show that a fixed-point of the mapping exists and it represents a competitive equilibrium with segregated exchanges (without transfers).

Theorem 3. *For any positive endowments, a competitive equilibrium with segregated security exchanges exists.*

6 Concluding Remarks on Implementation

Our solution to the externality problem is intuitive: create a market that allows agents to contract on the state contingent price under which they will unwind their contract commitments, over and above contracting on intertemporal or state-contingent security exchanges. Of course that unwind price is still endogenous, and the contracted price must equal the market clearing price at which supply equals demand[19], taking into account exogenous endowments, saving, contract positions and who is in the market. So when agents contract on the unwind price at which collateral is valued for clearing, they essentially are counting on having the requisite number and types of traders around to support that contracted price. As is usual in a Walrasian equilibrium, and in rational expectations, this presumption is validated, there is a decentralization, and agents need only pay attention to prices, making their own decision independently. No agent cares specifically about the identity or name of other traders. They do care but only implicitly about the composition of traders (or in our set up with homotheticity, the ratio of pretrade endowments) in the sense they are counting on a promised fundamental, the contracted price. So the new market mechanism does require knowledge of which side of a market a trader will be on, contingent on the state of the world, and hence what commitments they have made previously, in a certain well defined sense.

Practically, the markets for the rights to trade can be implemented using markets for certificates, each of which specifies the security exchange a trader wants to be in, at a price paid, or is willing to be in, at a price received, and the amount of the discrepancy from the fundamental that the trader will be holding. These are like market participation rights with market access fees, but for us rather than a fixed fee independent of volume as in contemporary markets, that volume here for us is implicit in the pretrade position of the trader and the market fundamental. These markets for rights to trade will be opened in

[19] We are abstracting away from broker dealers who absorb trades on their own account, to make a market so to speak. We have jumped to the standard Walrasian limit with a large (continuum) number of traders of each type, in which markets clear at an anticipated equilibrium price. However, the broker-deal firms in this paper would be the outcome of completion among those trying to set up exchanges and attract customers, as in Townsend (1983).

the contracting period, and traders can buy any certificate they want and can afford, or be compensated if this puts them in a disadvantageous position. When there is more than one active exchange for a given state contingent contracted price, then we allow for queuing with randomized execution of trade. That is traders will buy an actually fair lottery over certificates at the beginning of the contracting period, and then a platform/utility exchange will draw the outcome of the lottery and assign the certificate accordingly, to get the fractions of traders right, at the end of the contracting period[20]. In the execution period, markets are segregated or restricted in the sense that a trader with a certificate can trade in the specified security exchange only so long as its discrepancy from the fundamental on its certificate is the same as the true one or at least the one agreed to in the contracting period. If a household or trader comes to a wrong security exchange or holds an inconsistent discrepancy from the fundamental, its right to trade will be forfeited. This mechanism requires a technology that can verify ex post, in the execution period, a household's collateral/saving and its endowment profiles. On this we elaborate below.

In order to visualize more clearly the mechanics of our proposed market structure, we try to place it in a contemporary setting. We imagine that there are two commodities. One is a money (good 1), namely deposits or accounts at the Federal Reserve used to secure payments, as in Fed wire. The second is a treasury obligation (good 2, the collateral good). All promises to pay money in the future, whether a simple loan or a state contingent promise as in an insurance indemnity, required collateral, the treasuries. In practice both treasuries and money can clear and settle obligations ex post, but the rate of exchange between the two uses the ex post spot market price or collateral valuation. Even though money and treasuries can not be consumed directly, as can the commodities of our model, each participant (financial institutions, e.g., banks, insurance companies, hedge funds) derives an indirect utility from

[20]Trades are executed in some dark pool equity markets in a manner which can best be thought of as random. In a dark pool, demand is often not equal to supply, and market typically does not clear. In dark pools that simply take the exchange price as given, allocation rules on the heavy side have two main forms: pro-rata and time priority. An example of pro-rata: if there are 300 shares to buy and 200 shares to sell, then 2/3 of each buy order is filled. (If the dark pool runs continuously, then time priority is the most common, but that is not the one we want to emphasize here.) The point is that by increasing her bid, a buyer can make it more likely to sell more, but the outcome is stochastic (e.g., Zhu, 2013).

holding them in their portfolio at the end of today, this period, and also from holding them given a certain state of the world tomorrow, next period (due to reasons that we do not model here). But the utility is less from treasuries when they are used as collateral backing promises to pay. It is as if they were subtracted from end of period portfolio holdings, that is, not used for consumption/utility now. In the initial date these financial players borrow and lend in the securities markets and buy and sell insurance obligations, again with loans and insurance contracts dominated in money. The market fundamental in future spot markets under a given state of the world is determined by the relative ratio of money to treasuries at that date and state, equivalently the interest rate at that time. (Obviously, the example requires a more generous interpretation of the model, which actually ends after the second period.)

Some market participants buy for cash a vector of market exchange certificates, designating the future state contingent spot price of treasures for each state of the world. Other participants are paid to hold each item in a vector of market exchange certificates. There can of course be active trade in the sense that traders can be long or short on treasuries, even conditioned on a given state of the world. The arrangement we envision essentially offers a guarantee of the spot price of treasuries which will be used to settle obligations, hence not subject to market fluctuations beyond the usual state-of-the-world contingencies. But the market in the certificates in effect restricts the set of traders with whom there is unwinding of positions tomorrow in such a way that the contracted, insured price is the market clearing spot price. Broker-dealers will clear all the markets for securities and markets of certificates. Of course these institutional arrangements will require a registration system, to keep track of which exchange market traders are allowed to use (and hence the securities which are held). It is important to ensure that agents cannot participate across markets where they do not have the right to buy and sell and unwind trades, to forestall the obvious arbitrage when multiple exchanges emerge in equilibrium.

Registration and exclusivity might seem at first blush to be demanding requirements, but these have become standard in the operation of US financial markets, as we now argue. In contemporary financial markets, traders do not take physical possession of securities. The US has moved from a system in which securities and money (checks) were use to complete trades bilaterally, which one or the other in currier black bags being raced around downtown NY via bicycle, subject to a deadline, to a system in which securities are registered and fixed in place and do not change hands physically. A primary institution is DTCC , Depository Trust and Clearing Corp. Essentially all issuance and ownership is now electronic. Older securities are in a vault.

Ownership changes by trading on financial markets. There is a Trade Reporting Facility (TRF). One of the most obvious exchanges is New York Stock Exchange. An order to buy comes with the name of the trader, typically an identification number, and desired trades (limit order). Of course much of this information is not revealed to the public, but the exchanges know, regulators can know in principal, and records are kept. That is, trades are reported in practice such that the trading venues are disclosed to the public, whereas the trader identities are only known to the venues and regulators. In contrast, over-the-counter trades might seem to be bilateral or among dealers and unobserved, but at least the trade in some derivatives (credit default insurance) is now regulated under Dodd Frank legislation, and the point here, the collateral is recorded in a CCP, central clearing party. Further, the responsibility to finalize trade, to transfer securities and money now lies after Trade-plus-2 days with that CCP utility. Evidently many new registration and clearing platforms are being created. These private entities are a bit like our intermediaries, i.e., trades are netted and cleared though them. Our more general point is that these kinds of reforms have been implemented and in that sense our proposal would not seem to require more technology.

Exclusivity is also not uncommon. For example, some of the dark pools do not want to deal with hedge funds or high frequency (computer) traders, so they just prohibit them from entering the platform.

Asset-backed securities are allowed in our set up and do not cause a problem. Neither are they essential in that various combinations of securities and markets are equivalent. Asset-backed security trades mimic spot market trade, and become an essential part of the set up if and only if spot market exchange is for some reason more limited. As a result, all arguments and institution stated in terms of spot markets can be restated using the language of asset-backed securities. In particular, we can solve the externality problem by creating segregated security exchanges where agents can trade ex ante collateralized and asset backed securities indexed for clearing at a posted price, that is by the market fundamental. Methodologically, we do not both allow spot trade ex post then restrict spot trade with segregated markets. That is, even without spot markets, there is an externalities problem in the valuation of asset backed securities which are used to underwrite promises, and this requires some kind of market clearing valuation.

Our methods extend to other set ups in which spot market exchange is desirable or cannot be limited a priori. First, the model can be readily extended to incorporate the contract-specific collateralization without pyramiding and tranching as in Geanakoplos (2003), among others. In this case, spot trades will be necessary and cannot be substituted by ex-ante contracting. Second, this model can also be extended to general preferences and dynamic environments. This extended version will be used to study equilibrium cascades. This is again closely related to Geanakoplos (2003). Third, we can use our approach to study re-trading or anonymous trading in spot markets in incomplete market settings as in Greenwald and Stiglitz (1986); retrade under moral hazard environments with unobserved actions as in Acemoglu and Simsek (2008); Kilenthong and Townsend (2011a); and retrade in a Diamond and Dybvig (1983) preference shocks bank runs environment as in Jacklin (1987). Kilenthong and Townsend (2014) create the requisite notation and embed all these environments, including the collateral environment of this paper, into a common general framework.

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A Proofs

Lemma 2: For any state-contingent security, there exists a security with *no default* that can generate the *same total payoffs* using the *same amount of collateral.*

Proof of Lemma 2: Default is Irrelevant under Complete Contracts. Consider a contingent security *that will be in default* in state s, with collateral $\widehat{C} < \frac{1}{R_s p(z_s)}$. That is, an issuer of this security will "default" in state s. Hence, according to condition (2), the payoff of this security (in units of good 1) in state s is

$$\min\left(1, \widehat{C} R_s p(z_s)\right) = \widehat{C} R_s p(z_s) < 1. \tag{66}$$

We now argue that there is an alternative security that does not default but generates exactly the same total payoffs using the same amount of collateral overall. Consider a state-s contingent security with collateral amount $\widetilde{C} = \frac{1}{R_s p(z_s)}$. This security will not default. It is straightforward to show that the payoff of this security is one unit of good 1 in state s. Now consider $\widehat{C} R_s p(z_s)$ units of the *alternative* security. That collection of securities pays in state s one per unit or $\widehat{C} R_s p(z_s)$ in total. This is exactly the same as the payoff of the original security with default: see (66). In addition, the total collateral for $\widehat{C} R_s p(z_s)$ units of the alternative security with $\frac{1}{R_s p(z_s)}$ collateral per unit is $\widehat{C}$, which is exactly the same as the collateral level of the original security. Therefore, the alternative security can generate the same payoffs using the same total amount of collateral but without default. A similar argument also applies to all other types of securities. □

Lemma 3: With identical homothetic and strictly concave preferences, the attainable set is non-convex.

Proof of Lemma 3. For simplicity, both state and agent indices will be kept implicit here. Without loss of generality, let $R_s = 1$ for all s. Consider two different allocations $\left(k, \hat{\theta}, \theta\right)$ and $\left(k', \hat{\theta}', \theta'\right)$ with two different market fundamentals z, z', respectively. The collateral constraints (3) for an agent h in state s with these two allocations be binding:

$$p(z)k + \hat{\theta} + p(z)\theta = 0 \Longrightarrow \hat{\theta} = -p(z)\left(k + \theta\right). \tag{67}$$

$$p(z')k' + \hat{\theta}' + p(z')\theta' = 0 \Longrightarrow \hat{\theta}' = -p(z')\left(k' + \theta'\right). \tag{68}$$

Since we are looking for a counter example, we can pick these two allocations to satisfy

$$\hat{\theta} = \hat{\theta}' < 0 \Longrightarrow p(z)\,(k+\theta) = p(z')\,(k'+\theta') > 0. \tag{69}$$

The positivity of the prices implies that $k+\theta > 0$ and $k'+\theta' > 0$.

Now consider a convex combination allocation: $k^\lambda = \lambda k + (1-\lambda)\,k'$, $\hat{\theta}^\lambda = \lambda\hat{\theta} + (1-\lambda)\,\hat{\theta}'$, $\theta^\lambda = \lambda\theta + (1-\lambda)\,\theta'$, $\mathbf{c}^\lambda = \lambda\mathbf{c} + (1-\lambda)\,\mathbf{c}'$, and $z^\lambda = z(\mathbf{c}^\lambda)$, where $0 < \lambda < 1$. Using equations (67)-(68), we can write

$$p(z^\lambda)k^\lambda + \hat{\theta}^\lambda + p(z^\lambda)\theta^\lambda \;=\; \left(\frac{k+\theta}{p(z')}\right)\left[\lambda p(z^\lambda)\,(p(z') - p(z)) + p(z)\left(p(z^\lambda) - p(z')\right)\right].$$

There is no loss of generality to assume that $p(z) < p(z^\lambda) < p(z')$. Then, pick λ that is smaller than λ^*:

$$\lambda^* = \left(\frac{p(z') - p(z^\lambda)}{p(z') - p(z)}\right)\left(\frac{p(z)}{p(z^\lambda)}\right). \tag{70}$$

Using the condition that $p(z) < p(z^\lambda) < p(z')$, we can show that $0 < \lambda^* < 1$. This condition implies that we can pick $0 < \lambda < \lambda^* < 1$ such that

$$\lambda p(z^\lambda)\,(p(z') - p(z)) + p(z)\left(p(z^\lambda) - p(z')\right) < 0. \tag{71}$$

Using $k+\theta > 0$, this clearly violates the collateral constraint (3). Therefore, the attainable set is non-convex. □

Proposition 1: Under Assumption 1, a competitive collateral equilibrium is constrained optimal if and only if all collateral constraints are not binding, i.e. $\gamma^h_{cc-s} = \mu^h_{cc-s} = 0$ for all h and all s. As a result, a competitive collateral equilibrium is constrained *suboptimal* if and only if it is not first-best optimal.

Proof of Proposition 1. We first prove that a competitive collateral equilibrium is constrained optimal if and only if all collateral constraints are not binding, i.e. $\gamma^h_{cc-s} = \mu^h_{cc-s} = 0$ for all h and all s. The proof is based on the first-order conditions for Pareto program (22) and the first-order conditions for a competitive collateral equilibrium. Note that the resource constraints in the Pareto program (22) and the market-clearing constraints in the competitive collateral equilibrium are clearly equivalent. In addition, the collateral constraints are the same in both problems as well. Hence, we only need to match all first-order conditions from both problems. In addition, with limited space, we will focus only on the term that generates an externality.

Optimal Conditions for the Pareto Program (22)

Let μ^h_{cc-s} and $\mu^h_{\bar{u}}$ denote the Lagrange multipliers for the collateral constraint (20) for state s for an agent type h and for the participation constraint (23) for an agent type $h = 1, 2, \ldots, H$ with a normalization of $\mu^1_{\bar{u}} = 1$, respectively. Combining the first-order conditions with respect to c^h_{10}, k^h, and the complementarity slackness conditions for the collateral constraints gives:

$$\begin{aligned}
\frac{u^h_{20}}{u^h_{10}} &= \sum_s \pi_s \beta \frac{u^h_{2s}}{u^h_{10}} R_s + \sum_s \frac{\mu^h_{cc-s}}{\mu^h_{\bar{u}} u^h_{10}} p(z_s) R_s + \sum_s \frac{\alpha^h}{\mu^h_{\bar{u}} u^h_{10}} p'(z_s) \frac{\partial z_s}{\partial K} \sum_{\tilde{h}} \mu^{\tilde{h}}_{cc-s} \left[R_s k^h + \theta^{\tilde{h}}_s \right] \\
&= \sum_s \pi_s \beta \frac{u^h_{2s}}{u^h_{10}} R_s + \sum_s \frac{\mu^h_{cc-s}}{\mu^h_{\bar{u}} u^h_{10}} p(z_s) R_s - \sum_s \frac{\alpha^h}{\mu^h_{\bar{u}} u^h_{10}} \frac{p'(z_s)}{p(z_s)} \frac{\partial z_s}{\partial K} \sum_{\tilde{h}} \mu^{\tilde{h}}_{cc-s} \hat{\theta}^{\tilde{h}}_s, \qquad (72)
\end{aligned}$$

where the last equation follows from the complementarity slackness condition with respect to collateral constraints:

$$\mu^{\tilde{h}}_{cc-s} \left\{ p(z_s) \left[R_s k^h + \theta^{\tilde{h}}_s \right] + \hat{\theta}^h_s \right\} = 0 \Rightarrow \mu^{\tilde{h}}_{cc-s} \left[R_s k^h + \theta^{\tilde{h}}_s \right] = -\frac{\mu^{\tilde{h}}_{cc-s} \hat{\theta}^h_s}{p(z_s)}. \qquad (73)$$

Note that (72) is exactly the same as (24).

Optimal Conditions for a Collateral Equilibrium

Let γ_{cc-s} be the Lagrange multiplier for the collateral constraint for state s. Combining the first-order conditions with respect to c^h_{10} and k^h gives:

$$\frac{u^h_{20}}{u^h_{10}} = \sum_s \pi_s \frac{\beta u^h_{2s}}{u^h_{10}} R_s + \sum_s \frac{\gamma^h_{cc-s}}{u^h_{10}} p(z_s) R_s. \qquad (74)$$

We are ready to prove the lemma.

(i) ($\Longleftarrow$) Suppose that $\gamma^h_{cc-s} = \mu^h_{cc-s} = 0$ for all h and all s. We then can show that any competitive collateral equilibrium allocation will also solve the Pareto program (22) by matching all necessary and sufficient conditions. In particular, we can pick $\frac{\mu_{20}}{\mu_{10}} = P_{20}, \frac{\mu_{\hat{\theta}s}}{\mu_{10}} = \widehat{P}_{as}, \frac{\mu_{\theta s}}{\mu_{10}} = P_{as}$, and $\gamma^h_{cc-s} = \frac{\mu^h_{cc-s}}{\mu^h_{\bar{u}}} = 0$. In conclusion, any collateral equilibrium allocation is constrained optimal if $\gamma^h_{cc-s} = \mu^h_{cc-s} = 0$ for all h and all s.

(ii) ($\Longrightarrow$) Suppose that a competitive collateral equilibrium allocation is constrained optimal, i.e., solves the Pareto program (22). Hence, it must satisfy (72). Using the same matching conditions as above, this will be true only if the last terms in (72) is zero. We will prove this by a contradiction argument.

Suppose that there are some $\tilde{h}$ with $\mu_{cc-s}^{\tilde{h}} \neq 0$, and the last terms in (72) is zero:

$$\frac{\alpha^h}{\mu_{\bar{u}}^h u_{10}^h} \sum_s \frac{p'(z_s)}{p(z_s)} \frac{\partial z_s}{\partial K} \left(\sum_{\tilde{h}} \mu_{cc-s}^{\tilde{h}} \hat{\theta}_s^{\tilde{h}} \right) = 0. \tag{75}$$

This must be true for all h and $\tilde{h}$.

We will now argue that $\sum_{\tilde{h}} \mu_{cc-s}^{\tilde{h}} \hat{\theta}_s^{\tilde{h}}$ has the same *negative* sign for every state s. Using the first-order condition for the Pareto program with respect to $\hat{\theta}_s^h$, we can show that

$$\sum_{\tilde{h}} \mu_{cc-s}^{\tilde{h}} \hat{\theta}_s^{\tilde{h}} = \sum_{\tilde{h}} \mu_{1s} \alpha^{\tilde{h}} \hat{\theta}_s^{\tilde{h}} - \beta \pi_s \sum_{\tilde{h}} \mu_{\bar{u}}^{\tilde{h}} u_{1s}^{\tilde{h}} \hat{\theta}_s^{\tilde{h}}, \tag{76}$$

where μ_{1s} is the Lagrange multiplier for the resource constraint for $\hat{\theta}_s^h$. The resource constraint for $\hat{\theta}_s^h$, $\sum_{\tilde{h}} \alpha^{\tilde{h}} \hat{\theta}_s^{\tilde{h}} = 0$, then implies that $\sum_{\tilde{h}} \mu_{1s} \alpha^{\tilde{h}} \hat{\theta}_s^{\tilde{h}} = 0$ for all s. In addition, the first-order condition for the Pareto program with respect to c_{10}^h implies that $\mu_{\bar{u}}^{\tilde{h}} = \frac{\mu_{10} \alpha^{\tilde{h}}}{u_{10}^{\tilde{h}}}$. Thus, we now have

$$\sum_{\tilde{h}} \mu_{cc-s}^{\tilde{h}} \hat{\theta}_s^{\tilde{h}} = -\beta \pi_s \mu_{10} \sum_{\tilde{h}} \left(\frac{u_{1s}^{\tilde{h}}}{u_{10}^{\tilde{h}}} \right) \alpha^{\tilde{h}} \hat{\theta}_s^{\tilde{h}}. \tag{77}$$

The optimality requires that an agent with relative large IMRS, $\frac{u_{1s}^{\tilde{h}}}{u_{10}^{\tilde{h}}}$, will hold positive $\hat{\theta}_s^{\tilde{h}} \geq 0$ and vice versa. This implies that the positive term of $\alpha^{\tilde{h}} \hat{\theta}_s^{\tilde{h}} \geq 0$ will be weighted more than the negative one. Combining this result with the resource constraint for $\hat{\theta}_s^h$, $\sum_{\tilde{h}} \alpha^{\tilde{h}} \hat{\theta}_s^{\tilde{h}} = 0$, we can conclude that $\sum_{\tilde{h}} \left(\frac{u_{1s}^{\tilde{h}}}{u_{10}^{\tilde{h}}} \right) \alpha^{\tilde{h}} \hat{\theta}_s^{\tilde{h}} \geq 0, \forall s$, and therefore

$$\sum_{\tilde{h}} \mu_{cc-s}^{\tilde{h}} \hat{\theta}_s^{\tilde{h}} = -\beta \pi_s \mu_{10} \sum_{\tilde{h}} \left(\frac{u_{1s}^{\tilde{h}}}{u_{10}^{\tilde{h}}} \right) \alpha^{\tilde{h}} \hat{\theta}_s^{\tilde{h}} \leq 0, \forall s. \tag{78}$$

With strictly concave and identical homothetic utility function, we can show that $\frac{p'(z_s)}{p(z_s)} \frac{\partial z_s}{\partial K} < 0$, and therefore can conclude that

$$\frac{p'(z_s)}{p(z_s)} \frac{\partial z_s}{\partial K} \left(\sum_{\tilde{h}} \mu_{cc-s}^{\tilde{h}} \hat{\theta}_s^{\tilde{h}} \right) \geq 0, \ \forall s. \tag{79}$$

As a result, (75) will hold only if

$$\sum_{\tilde{h}} \mu^{\tilde{h}}_{cc-s}\hat{\theta}^{\tilde{h}}_s = -\beta\pi_s\mu_{10}\sum_{\tilde{h}}\left(\frac{u^{\tilde{h}}_{1s}}{u^{\tilde{h}}_{10}}\right)\alpha^{\tilde{h}}\hat{\theta}^{\tilde{h}}_s = 0, \forall s. \tag{80}$$

Given that $\sum_{\tilde{h}} \alpha^{\tilde{h}}\hat{\theta}^{\tilde{h}}_s = 0$, condition (80) implies that $\frac{u^{\tilde{h}}_{1s}}{u^{\tilde{h}}_{10}} = \frac{u^{h}_{1s}}{u^{h}_{10}}$, $\forall h, \tilde{h}; s$. Using the fact that $\frac{u^{h}_{2s}}{u^{h}_{1s}} = p(z_s)$ for all h, we can also show that $\frac{u^{\tilde{h}}_{1s}}{u^{\tilde{h}}_{1\hat{s}}} = \frac{u^{h}_{1s}}{u^{h}_{1\hat{s}}}$, $\forall h, \tilde{h}; s$. In words, the marginal rate of substitutions across times and states are equalized across agent types. Under the assumption 1, these equalities are necessary and sufficient conditions for first-best optimality, which in turn implies that all collateral constraints are not binding, i.e., $\gamma^{h}_{cc-s} = \mu^{h}_{cc-s} = 0$ for all h and all s. Hence, we can conclude that a collateral equilibrium is constrained optimal, solving the Pareto program (22), only if all collateral constraints are not binding.

The rest of the proof of is by contrapositive. Suppose a competitive collateral equilibrium is constrained optimal. The above result implies that a necessary and sufficient condition for a competitive collateral equilibrium to be constrained optimal is that all collateral constraints are not binding. No binding collateral constraints implies first-best optimality. In short, we have shown that first-best optimality is a necessary and sufficient condition for constrained optimality. Thus we can conclude that a competitive collateral equilibrium is constrained *suboptimal* if and only if it is not first-best optimal. □

Proposition 2: Under Assumption 1, if a competitive collateral equilibrium is not first-best optimal, then

(i) the equilibrium price of good 2 in period $t = 0$, P_{20}, is too high, i.e., $P_{20} > \left.\frac{u^{h}_{20}}{u^{h}_{10}}\right|_{op}$, and

(ii) the (endogenous) aggregate saving/collateral in a competitive collateral equilibrium, K^{ce}, is too large, i.e., $K^{ce} > K^{op}$.

Proof of Proposition 2. The proof is an immediate result of the proof of proposition 1 above. First, if a competitive collateral equilibrium is not first-best optimal, then (by proposition 1) we can show that the last term of (72) is strictly positive:

$$\sum_{s} \frac{\alpha^{h}}{\mu^{h}_{\bar{u}}u^{h}_{10}}\frac{p'(z_s)}{p(z_s)}\frac{\partial z_s}{\partial K}\sum_{\tilde{h}}\mu^{\tilde{h}}_{cc-s}\hat{\theta}^{\tilde{h}}_s > 0. \tag{81}$$

This implies that the marginal rate of substitution between good 1 and good 2 in period $t = 0$ at the competitive collateral equilibrium is larger than the optimal level of the marginal rate of substitution between good 1 and good 2 in period $t = 0$, i.e., $\left.\frac{u^h_{20}}{u^h_{10}}\right|_{ce} > \left.\frac{u^h_{20}}{u^h_{10}}\right|_{op}$. This implies that the equilibrium price of good 2 in period $t = 0$ is too high relative to its shadow price from the (constrained) optimal allocation $\left.\frac{u^h_{20}}{u^h_{10}}\right|_{op}$. In addition, given that the aggregate consumption of good 1 is fixed and preferences are identically homothetic, this result can be true only if the (endogenous) aggregate saving/collateral in a competitive collateral equilibrium, K^{ce}, is too large, i.e., $K^{ce} > K^{op}$. $\square$

B More Results

B.1 Default cannot Make Collateral Constraints Less Binding under Complete Contracts

This section shows that contracts that do actually default does not relax the collateral constraints (3); that is, contracts that do default are not necessary. They may exist and get traded, but we can support an equivalent allocation without them. In particular, we now derive a collateral constraint with contracts that do default, and then show that the same net-payoff and same collateral constraint can be reached using no-default contracts. This is, in fact, a result of Lemma 2 but it is nice to be explicit, as the result seems counterintuitive.

Let $\widehat{C}$, C, $\widehat{C}^\sigma$, and C^σ be the collateral levels of defaulting contracts promising to pay a unit of good 1 with good 2 as collateral, promising to pay a unit of good 2 with good 2 as collateral, promising to pay a unit of good 1 with financial assets as collateral, and promising to pay a unit of good 2 with financial assets as collateral, respectively. Note that, for expositional reasons, we assume that all contracts are contracts that do default. Accordingly, the payoffs of those contracts, which by construction with default, in state s are

$$\begin{aligned}
\widehat{D}_s &= \min\left(P_{2s}R_s\widehat{C},1\right) = P_{2s}R_s\widehat{C}, && (82)\\
D_s &= \min\left(R_sC,1\right) = R_sC, && (83)\\
\widehat{D}^\sigma_s &= \min\left(P_{2s}\widehat{C}^\sigma,1\right) = P_{2s}\widehat{C}^\sigma, && (84)\\
D^\sigma_s &= \min\left(\frac{C^\sigma}{P_{2s}},1\right) = \frac{C^\sigma}{P_{2s}}. && (85)
\end{aligned}$$

The collateral requirement condition for contracts using physical good 2 as collateral is given by

$$k^h \geq -\widehat{C}\min\left(0,\hat{\psi}^h_s\right) - C\min\left(0,\psi^h_s\right).$$

Multiplying by $P_{2s}R_s$ both sides gives

$$\begin{aligned}
P_{2s}R_sk^h &\geq -P_{2s}R_s\widehat{C}\min\left(0,\hat{\psi}^h_s\right) - P_{2s}R_sC\min\left(0,\psi^h_s\right)\\
&= -\widehat{D}_s\min\left(0,\hat{\psi}^h_s\right) - P_{2s}D_s\min\left(0,\psi^h_s\right). && (86)
\end{aligned}$$

where the last equality follows from (82)-(83).

The collateral requirement condition regarding contracts paying in good 1 using purchased assets as collateral can be written as

$$D_s \max\left(0, \psi_s^h\right) + D_s^\sigma \max\left(0, \sigma_s^h\right) \geq -\widehat{C}^\sigma \min\left(0, \hat{\sigma}_s^h\right), \tag{87}$$

which can be rearranged as

$$\begin{aligned} P_{2s} D_s \max\left(0, \psi_s^h\right) + P_{2s} D_s^\sigma \max\left(0, \sigma_s^h\right) &\geq -P_{2s}\widehat{C}^\sigma \min\left(0, \hat{\sigma}_s^h\right) \\ &= -\widehat{D}_s^\sigma \min\left(0, \hat{\sigma}_s^h\right). \end{aligned} \tag{88}$$

where the last equality follows from (84).

Similarly, the collateral requirement condition regarding contracts paying in good 2 using purchased assets as collateral can be written as

$$\begin{aligned} \widehat{D}_s \max\left(0, \hat{\psi}_s^h\right) + \widehat{D}_s^\sigma \max\left(0, \hat{\sigma}_s^h\right) &\geq -C^\sigma \min\left(0, \sigma_s^h\right) \\ &= -P_{2s} D_s^\sigma \min\left(0, \sigma_s^h\right), \end{aligned} \tag{89}$$

where the last equality follows from (85).

Summing conditions (86)-(89) gives the collateral constraint in state s, for an agent h,

$$\begin{aligned} P_{2s} R_s k^h &\geq -\widehat{D}_s\left[\max\left(0, \hat{\psi}_s^h\right) + \min\left(0, \hat{\psi}_s^h\right)\right] - \left[\max\left(0, \hat{\sigma}_s^h\right) + \widehat{D}_s^\sigma \min\left(0, \hat{\sigma}_s^h\right)\right] \\ &\quad -P_{2s} D_s\left[\max\left(0, \psi_s^h\right) + \min\left(0, \psi_s^h\right)\right] - P_{2s} D_s^\sigma\left[\max\left(0, \sigma_s^h\right) + \min\left(0, \sigma_s^h\right)\right] \\ &= -\left(\widehat{D}_s \hat{\psi}_s^h\right) - \left(\widehat{D}_s^\sigma \hat{\sigma}_s^h\right) - P_{2s}\left(\widehat{D}_s^\sigma \psi_s^h\right) - P_{2s}\left(D_s^\sigma \sigma_s^h\right). \end{aligned} \tag{90}$$

The above collateral constraint shows that what really matter for the collateral constraint is the total payoff of contracts. As a result, we can find equivalent contracts with no-default that satisfy the same collateral constraint, by re-normalizing the original contracts. In particular, consider contracts, with collateral $\widehat{C}' = \frac{1}{P_{2s}R_s}$, $C' = \frac{1}{R_s}$, $\widehat{C}'^\sigma = \frac{1}{P_{2s}}$, and $C'^\sigma = P_{2s}$. Hence, their payoffs are payoffs of those contracts in state s, respectively, are

$$\widehat{D}_s' = \min\left(P_{2s} R_s \widehat{C}', 1\right) = 1, \tag{91}$$

$$D_s' = \min\left(R_s C', 1\right) = 1, \tag{92}$$

$$\widehat{D}_s'^\sigma = \min\left(P_{2s}\widehat{C}'^\sigma, 1\right) = 1, \tag{93}$$

$$D_s'^\sigma = \min\left(\frac{C'^\sigma}{P_{2s}}, 1\right) = 1. \tag{94}$$

Note that these are no-default contracts.

In order to reach the same total payoff as originally, let the agent hold securities $\hat{\psi}_s'^h = \widehat{D}_s \hat{\psi}_s^h$, $\hat{\sigma}_s'^h = \widehat{D}_s^\sigma \hat{\sigma}_s^h$, $\psi_s'^h = D_s \psi_s^h$, and $\sigma_s'^h = D_s^\sigma \sigma_s^h$. As a result, the collateral constraint (90) becomes

$$P_{2s} R_s k^h + \left(\hat{\psi}_s'^h + \hat{\sigma}_s'^h \right) + P_{2s} \left(\psi_s'^h + \sigma_s'^h \right) \geq 0, \tag{95}$$

which is identical to the collateral constraint (3), derived from no-default contracts only.

B.2 Details of the Building Blocks of the Collateral Constraints

This section precisely defines directly collateralized and asset-back securities (pyramiding), and derives the unified collateral constraints (3) by considering the collateral constraints of each type of securities one at a time and adding them up (and disaggregating back down).

Collateral Constraints on Directly Collateralized Securities

To generalize a bit, let $\hat{\psi}^h \equiv \left(\hat{\psi}_s^h \right)_{s=1}^S \in \mathbb{R}^S$ and $\psi^h \equiv \left(\psi_s^h \right)_{s=1}^S \in \mathbb{R}^S$ denote agent h's portfolios of securities demanded, held at the end of period 0 paying in good 1 and in good 2, both with good 2 as collateral directly, respectively. Again, we adopt the convention that positive means demand and negative means sale. So, holding a positive amount of a security paying good 2 in state s, $max\left(0, \psi_s^h\right) = \psi_s^h$, a positive number, is equivalent to buying that security (or lending) while holding a negative amount of a security, $min\left(0, \psi_s^h\right) = \psi_s^h$, a negative number, is equivalent to selling that security (or borrowing). In short, the max and min operators pick off demand and supply, respectively. A wedge is created by the need to back the supply by collateral but not the demand.

More generally, a security paying a unit of good 1 in state s backed by good 2 pays the minimum of 1 unit of good 1 or the value of its collateral in state s. By an argument similar to the one given earlier, the minimum no-default collateral is $\frac{1}{p(z_s)R_s}$ per unit. Similarly, with no-default and no-over-collateralization, a security paying in good 2 in state s requires $\frac{1}{R_s}$ units of good 2 as collateral. The results so far are summarized in the first two rows of the Table 9 with collateral requirement in the last column.

Table 9: Collateral requirements for each type of securities.

	payment unit	collateral unit	issued liabilities	purchased assets available as collateral	total collateral requirement for no default securities
$\hat{\psi}_s^h$	good 1	good 2	$-\min\left(0, \hat{\psi}_s^h\right)$	$\max\left(0, \hat{\psi}_s^h\right)$	$-\left(\frac{1}{R_s p(z_s)}\right)\min\left(0, \hat{\psi}_s^h\right)$
ψ_s^h	good 2	good 2	$-\min\left(0, \psi_s^h\right)$	$\max\left(0, \psi_s^h\right)$	$-\left(\frac{1}{R_s}\right)\min\left(0, \psi_s^h\right)$
$\hat{\sigma}_s^h$	good 1	securities paying in good 2	$-\min\left(0, \hat{\sigma}_s^h\right)$	$\max\left(0, \hat{\sigma}_s^h\right)$	$-\left(\frac{1}{p(z_s)}\right)\min\left(0, \hat{\sigma}_s^h\right)$
σ_s^h	good 2	securities paying in good 1	$-\min\left(0, \sigma_s^h\right)$	$\max\left(0, \sigma_s^h\right)$	$-p(z_s)\min\left(0, \sigma_s^h\right)$
$\hat{\nu}_s^h$	good 1	securities paying in good 1	$-\min\left(0, \hat{\nu}_s^h\right)$	$\max\left(0, \hat{\nu}_s^h\right)$	$-\min\left(0, \hat{\nu}_s^h\right)$
ν_s^h	good 2	securities paying in good 2	$-\min\left(0, \nu_s^h\right)$	$\max\left(0, \nu_s^h\right)$	$-\min\left(0, \nu_s^h\right)$

For securities $\left(\hat{\psi}_s^h, \psi_s^h\right)$ with good 2 as collateral, paying in good 1 and good 2, respectively, agent h must hold good 2 at the end of period 0 no less than the collateral requirement in any state (shown in Table 9):

$$k^h \geq -\min\left(0, \hat{\psi}_s^h\right)\left(\frac{1}{R_s p(z_s)}\right) - \min\left(0, \psi_s^h\right)\left(\frac{1}{R_s}\right),\ \forall s, \tag{96}$$

which can be rewritten as

$$p(z_s)R_s k^h + \min\left(0, \hat{\psi}_s^h\right) + p(z_s)\min\left(0, \psi_s^h\right) \geq 0,\ \forall s. \tag{97}$$

These are *state-contingent collateral requirement constraints* with directly collateralized securities. We incorporate asset-backed securities in the next section.

Note that when an agent h's collateral requirement constraints (96) are not binding for every state s (i.e., the LHS of (96) exceeds its RHS or (96) holds with strict inequality for every state s), then the agent h holds collateral k^h more than needed to back issued securities. The extra part of collateral is normal saving.

Pyramiding: Asset-Backed Securities

In real world economies, agents are allowed to use the *promises to receive* goods of others as collateral to back their own promises. This is termed *pyramiding.* In other words, there are two types of collateral, good 2 itself (described in the preceding section) and "assets" backed by such collateral. The prototypical example of an asset-backed promise in this paper is an ex-ante agreement for an agent to give up good 1 in the spot market in state s backed by someone else's promise, a receipt of good 2, or vice versa. The promise of receipt is the asset, and this backs the promise to pay. Indeed, if the planned spot-market trade is at equilibrium price of $p(z_s)$, then one is moving along a budget line and so the value of collateral, the good to be recovered, exactly equals the promise and there is no need for additional underlying collateral.

With two physical commodities, there are four possible types of asset-backed securities, summarized in the last four rows of Table 9. For example, a unit of an asset-backed security $\hat{\sigma}_s$ paying in good 1 in state s needs $\frac{1}{p(z_s)}$ units of assets paying in good 2 as collateral. The value of the payoff of $\frac{1}{p(z_s)}$ units of securities paying in good 2 in state s equals $p(z_s)\times\frac{1}{p(z_s)} = 1$ unit of good 1, which is exactly the face-value promise to pay. These collateral requirements are minimum no-default levels.

As shown in the third row of Table 9 (see the column titled total collateral requirement), an asset-backed security paying a unit of good 1 in state s, $\hat{\sigma}_s^h$, requires that the total amount of purchased assets paying in good 2 in state s is no less than $-\left(\frac{1}{p(z_s)}\right)\min\left(0,\hat{\sigma}_s^h\right)$. Similarly, an asset-backed security ν_s^h requires that the total amount of purchased assets paying in good 2 in state s is no less than $-\min\left(0,\nu_s^h\right)$ (see the last row of Table 9). On the other hand, the total amount of purchased assets paying in good 2 is $\max\left(0,\psi_s^h\right)+\max\left(0,\sigma_s^h\right)+\max\left(0,\nu_s^h\right)$, as shown in the second, fourth and last rows of Table 9 (see the next-to-last column titled purchased assets). Hence, the collateral requirement condition regarding issued securities $\hat{\sigma}_s^h$ and ν_s^h that require financial assets paying in good 2 as collateral can be written as, for any state s,

$$\max\left(0,\psi_s^h\right)+\max\left(0,\sigma_s^h\right)+\max\left(0,\nu_s^h\right) \quad\geq\quad -\left(\frac{1}{p(z_s)}\right)\min\left(0,\hat{\sigma}_s^h\right)-\min\left(0,\nu_s^h\right).$$

This states that the agent purchases enough assets or promises paying in good 2, $\theta_s^h,\sigma_s^h,\nu_s^h$, to back up her own asset-backed securities or issued promises $\hat{\sigma}_s^h, \nu_s^h$. The above condition can be rearranged as

$$p(z_s) \max\left(0, \psi_s^h\right) + p(z_s) \max\left(0, \sigma_s^h\right) + p(z_s)\nu_s^h \quad \geq \quad -\min\left(0, \hat{\sigma}_s^h\right), \tag{98}$$

where we applies the fact that $\max\left(0, \nu_s^h\right) + \min\left(0, \nu_s^h\right) = \nu_s^h$.

Similarly, the collateral requirement condition for issued securities that require financial assets paying in good 1 as collateral is given by

$$\max\left(0, \hat{\psi}_s^h\right) + \max\left(0, \hat{\sigma}_s^h\right) + \hat{\nu}_s^h \quad \geq \quad -p(z_s) \min\left(0, \sigma_s^h\right), \ \forall s, \tag{99}$$

where the right-hand-side comes from the fourth and fifth rows of Table 9.

We now show that the collateral constraints

$$p(z_s)R_s k^h + \hat{\theta}_s^h + p(z_s)\theta_s^h \quad \geq \quad 0, \ \forall s \tag{100}$$

are equivalent to collateral requirement conditions (with three types of collateral), (97), (98), and (99). In other words, there is no loss of generality to use the collateral constraints (100); an allocation is attainable under (100) if and only if it is so under (97), (98), and (99).

To be more precise, let $\hat{\theta}_s^h = \hat{\psi}_s^h + \hat{\sigma}_s^h + \hat{\nu}_s^h$ and $\theta_s^h = \psi_s^h + \sigma_s^h + \nu_s^h$ be contingent securities paying in good 1 and in good 2 in state s, respectively, which can be backed either by good 2 or purchased assets (other people's promises). Note that $\hat{\theta}_s^h$ and θ_s^h include both directly collateralized and asset-backed securities. An attainable allocation under (97), (98), and (99) can be defined similarly to the one under (3) by replacing (12)-(13) the following resource constraints:

$$\sum_h \alpha^h \hat{\psi}_s^h = \sum_h \alpha^h \psi_s^h = \sum_h \alpha^h \hat{\sigma}_s^h = \sum_h \alpha^h \sigma_s^h = \sum_h \alpha^h \hat{\nu}_s^h = \sum_h \alpha^h \nu_s^h = 0, \ \forall s. \tag{101}$$

The collateral constraint (100) results from summing (97), (98), and (99) altogether, and then applying $\max(0, x) + \min(0, x) = x$ to get rid of max and min operators. In addition, the proof of this lemma also shows how to recover contract allocation $\left(\hat{\psi}_s^h, \psi_s^h, \hat{\sigma}_s^h, \sigma_s^h, \hat{\nu}_s^h, \nu_s^h\right)_h$ from $\left(\hat{\theta}_s^h, \theta_s^h\right)$.

Lemma 4. *The following statements are true:*

(i) if $\left(\mathbf{c}_0^h, k^h, \hat{\psi}_s^h, \psi_s^h, \hat{\sigma}_s^h, \sigma_s^h, \hat{\nu}_s^h, \nu_s^h\right)_h$ *is attainable, then the collateral constraint (100) and the market-clearing conditions (12)-(13) hold, and*

(ii) if $\left(k^h, \hat{\theta}_s^h, \theta_s^h\right)_h$ *is attainable, then there exists a collateral and security allocation* $\left(k^h, \hat{\psi}_s^h, \psi_s^h, \hat{\sigma}_s^h, \sigma_s^h, \hat{\nu}_s^h, \nu_s^h\right)_h$ *that satisfies collateral requirement conditions (97), (98), (99) and the market-clearing conditions (101).*

Proof. The first statement can be proved as follows. First, it is clear that conditions (101) imply (12)-(13). We now only need to show that (97), (98), and (99) imply (100). Summing up all collateral requirement conditions, (97), (98), and (99), and using the fact that $\max(0, x) + \min(0, x) = x$ give, for an agent h in state s,

$$p(z_s)R_s k^h + \left[\hat{\psi}_s^h + \hat{\sigma}_s^h + \hat{\nu}_s^h\right] + p(z_s)\left[\psi_s^h + \sigma_s^h + \nu_s^h\right] \;\; \geq \;\; 0,$$

which is the collateral constraint for an agent h in state s where $\hat{\theta}_s^h = \hat{\psi}_s^h + \hat{\sigma}_s^h + \hat{\nu}_s^h$ and $\theta_s^h = \psi_s^h + \sigma_s^h + \nu_s^h$.

The second statement is proved as follows. Consider an allocation $\left(k^h, \hat{\theta}_s^h, \theta_s^h\right)_h$ that satisfies (100) and (12)-(13). We will now choose a corresponding allocation $\left(k^h, \hat{\psi}_s^h, \psi_s^h, \hat{\sigma}_s^h, \sigma_s^h, \hat{\nu}_s^h, \nu_s^h\right)_h$ that satisfies $\hat{\theta}_s^h = \hat{\psi}_s^h + \hat{\sigma}_s^h + \hat{\nu}_s^h$, $\theta_s^h = \psi_s^h + \sigma_s^h + \nu_s^h$, the collateral requirement conditions (97), (98), (99), and the market-clearing conditions (101). Consider the following candidate allocation:

$$\hat{\psi}_s^h = \hat{\theta}_s^h + p(z_s)\theta_s^h, \tag{102}$$

$$\psi_s^h = \hat{\nu}_s^h = \nu_s^h = 0, \tag{103}$$

$$\hat{\sigma}_s^h = \hat{\theta}_s^h - \hat{\psi}_s^h = -p(z_s)\theta_s^h, \tag{104}$$

$$\sigma_s^h = \theta_s^h. \tag{105}$$

(103) implies that agents hold no $\psi_s^h, \hat{\nu}_s^h, \nu_s^h$; they will borrow or lend through directly collateralized contract paying in good 1 $\hat{\psi}_s^h$ only.

It is straightforward to show that resource constraints (101) hold. Since the resource constraints are satisfied and the collateral allocations k^h are the same, the market fundamentals are the same. We now would like to show that collateral requirement conditions (97), (98), (99) also hold. First, we will show that (98) and (99) hold. There are two cases to consider;

(i) $\theta_s^h > 0$, (ii) $\theta_s^h < 0$. Case I: Suppose that $\theta_s^h > 0$. Using (105), this implies that $\sigma_s^h > 0$, which in turn leads to $\min\left(0, \sigma_s^h\right) = 0$. On the other hand, it is true that

$$\max\left(0, \hat{\psi}_s^h\right) + \max\left(0, \hat{\sigma}_s^h\right) = \max\left(0, \hat{\psi}_s^h\right) + \max\left(0, \hat{\sigma}_s^h\right) + \hat{\nu}_s^h \geq 0,$$

where the first equality follows from (103). Since $\min\left(0, \sigma_s^h\right) = 0$, we have

$$\max\left(0, \hat{\psi}_s^h\right) + \max\left(0, \hat{\sigma}_s^h\right) = \max\left(0, \hat{\psi}_s^h\right) + \max\left(0, \hat{\sigma}_s^h\right) + \hat{\nu}_s^h \geq -p(z_s)\min\left(0, \sigma_s^h\right),$$

which is (99). On the other hand, (104) implies that $\hat{\sigma}_s^h < 0$ when $\theta_s^h > 0$. As a result, $\min\left(0, \hat{\sigma}_s^h\right) = \hat{\sigma}_s^h$. Using (103), (104), (105), we then can show that

$$\begin{aligned} & p(z_s)\max\left(0, \psi_s^h\right) + p(z_s)\max\left(0, \sigma_s^h\right) + p(z_s)\nu_s^h + \min\left(0, \hat{\sigma}_s^h\right) \\ & \quad = 0 + p(z_s)\sigma_s^h + 0 + \hat{\sigma}_s^h = p(z_s)\theta_s^h - p(z_s)\theta_s^h = 0, \end{aligned}$$

where the second equality follows from (104) and (105). This shows that (98) holds.

Case II: Suppose that $\theta_s^h < 0$. (104) and (105) imply that $\max\left(0, \hat{\sigma}_s^h\right) = \hat{\sigma}_s^h = -p(z_s)\theta_s^h$ and $\min\left(0, \sigma_s^h\right) = \sigma_s^h = \theta_s^h$, respectively. We then can write

$$\max\left(0, \hat{\psi}_s^h\right) + \max\left(0, \hat{\sigma}_s^h\right) + \hat{\nu}_s^h = \max\left(0, \hat{\psi}_s^h\right) - p(z_s)\theta_s^h \geq -p(z_s)\theta_s^h = -p(z_s)\min\left(0, \sigma_s^h\right),$$

which is exactly (99). Note that the first equality follows from (103), the second inequality follows from the fact that $\max\left(0, \hat{\psi}_s^h\right) \geq 0$. Similarly, using , we can show that $\max\left(0, \sigma_s^h\right) = \min\left(0, \hat{\sigma}_s^h\right) = 0$. This implies that

$$p(z_s)\max\left(0, \psi_s^h\right) + p(z_s)\max\left(0, \sigma_s^h\right) + p(z_s)\nu_s^h + \min\left(0, \hat{\sigma}_s^h\right) = 0 + 0 + 0 + 0 = 0,$$

which is exactly (98).

Similarly, we can now show that (97) also holds. There are two cases to be considered as well.

Case I: suppose that $\hat{\theta}_s^h + p(z_s)\theta_s^h < 0$. (102) implies that $\hat{\psi}_s^h < 0$, which in turn implies that $\min\left(0, \hat{\psi}_s^h\right) = \hat{\psi}_s^h = \hat{\theta}_s^h + p(z_s)\theta_s^h$. Using (103), we now can show that

$$p(z_s)R_s k^h + \min\left(0, \hat{\psi}_s^h\right) + p(z_s)\min\left(0, \psi_s^h\right) = p(z_s)R_s k^h + \hat{\theta}_s^h + p(z_s)\theta_s^h + 0 \geq 0,$$

where the last inequality follows (100). This implies that (97) holds.

Case II: we can use a similar argument to show that (97) holds when $\hat{\theta}_s^h + p(z_s)\theta_s^h = \hat{\psi}_s^h > 0$. In summary, we have show that all collateral requirement conditions hold. $\square$

B.3 Pooling Collateral versus Tranching

This section shows that the markets economize on collateral; that is, there is no gain from pooling collateral across agents type h. Let the collateral constraints with pooling be:

$$p(z_s) R_s K \geq -\sum_h \alpha^h p(z_s) \min\left\{0, \psi_s^h\right\} - \sum_h \alpha^h \min\left\{0, \hat{\psi}_s^h\right\}, \tag{106}$$

where the average collateral $K = \sum_h \alpha^h k^h$. We then show that the group collateral constraint is equivalent to individuals collateral constraints (3).

Lemma 5. *For any allocation $\left(k^h, \psi_s^h, \tau_s^h, \hat{\tau}_s^h\right)$ satisfying the collateral constraints (106), then there exists there exists an equivalent allocation $\left(k'^h, \psi_s'^h, \tau_s^h, \hat{\tau}_s^h\right)$ with*

$$\begin{aligned} k'^1 &= \frac{\sum_h \alpha^h k_s^h}{\alpha^1}, \text{ and } k'^h = 0 \text{ for } h \neq 1, &(107)\\ \psi_s'^h &= \left(R_s k_s^h + \psi_s^h\right) - R_s k'^h, &(108)\end{aligned}$$

where $k_s^h = \frac{-p(z_s)\min\left(0,\psi_s^h\right) - \min\left(0,\hat{\psi}_s^h\right)}{p(z_s) R_s}, \forall s.$

Proof. This result can be proved in two steps: (i) show that the collateral constraints (106) hold if and only if there exists k_s^h such that (3) hold, (ii) then show that any allocation with state-contingent collateral, k_s^h, can be replicated by an allocation with fixed collateral allocation k^h.

Step I: $\Longrightarrow$ Suppose that collateral constraints (106) hold. Now consider an alternative allocation with

$$k_s^h = \frac{-p(z_s)\min\left(0, \psi_s^h\right) - \min\left(0, \hat{\psi}_s^h\right)}{p(z_s) R_s}, \forall s. \tag{109}$$

This clearly implies no default. We then only need to show that the average collateral needed $\sum_h \alpha^h k_s^h$ is no larger than K. Summing the above equation over h with weight α^h gives, for each s,

$$\begin{aligned} \sum_h \alpha^h k_s^h &= \sum_h \alpha^h \frac{-p(z_s)\min\left(0, \psi_s^h\right) - \min\left(0, \hat{\psi}_s^h\right)}{p(z_s) R_s} \\ &\leq K, &(110)\end{aligned}$$

where the last inequality follows from the group collateral constraints (106).

$\Longleftarrow$ This can be done by summing over the individuals collateral constraints with weight α^h.

Step II: Let $\left(k_s^h, \psi_s^h, \tau_s^h, \hat{\tau}_s^h\right)$ be an attainable allocation with contingent collateral; that is, it satisfies the collateral constraint for each h and s:

$$R_s k_s^h \geq -\min\left(0, \psi_s^h\right), \tag{111}$$

and the average collateral is the same in every state; $K = \sum_h \alpha^h k_s^h$ for all s. In addition, the consumption allocation of agent h in state s is given by

$$\begin{aligned} c_{1s}^h &= e_{1s}^h + \hat{\tau}_s^h, && (112) \\ c_{2s}^h &= e_{2s}^h + \left(R_s k_s^h + \psi_s^h\right) + \tau_s^h, && (113) \end{aligned}$$

where the spot trade satisfies:

$$\hat{\tau}_s^h + p(z_s)\tau_s^h = 0. \tag{114}$$

Now consider a candidate allocation $\left(k'^h, \psi_s'^h, \tau_s'^h, \hat{\tau}_s'^h\right)$ with

$$\begin{aligned} k'^1 &= \frac{K}{\alpha^1}, \text{ and } k'^h = 0 \text{ for } h \neq 1, && (115) \\ \psi_s'^h &= \left(R_s k_s^h + \psi_s^h\right) - R_s k'^h, && (116) \\ \hat{\tau}_s'^h &= \hat{\tau}_s^h, \text{ and } \tau_s'^h = \tau_s^h. && (117) \end{aligned}$$

Using (115), we can write the securities as

$$\begin{aligned} \psi_s'^1 &= \left(R_s k_s^1 + \psi_s^1\right) - k'^1, && (118) \\ \psi'^h &= R_s k_s^h + \psi_s^h \text{ for } h \neq 1. && (119) \end{aligned}$$

Using the collateral constraint (111) we can show that for each $h \neq 1$:

$$\psi_s'^h = R_s k_s^h + \psi_s^h \geq R_s k_s^h + \min\left\{0, \psi_s^h\right\} \geq 0, \tag{120}$$

where the last inequality follows from the collateral constraint (111). This, $\psi_s'^h \geq 0$, implies that the collateral constraint for any $h \neq 1$ holds (since he does not issue securities at all).

We hence only need to show that the collateral constraint also holds for $h = 1$. We can rewrite (118) as

$$k'^1 = \left(R_s k_s^1 + \psi_s^1\right) - \psi_s'^1 \geq -\psi_s'^1, \tag{121}$$

where the last inequality follows from the collateral constraint (111) for $h = 1$. This shows that the collateral constraint also holds for $h = 1$.

Given that $\sum_h \alpha^h k_s'^h = K = \sum_h \alpha^h k_s^h$, the market fundamentals are the same for every state. With the same market fundamental, z_s, the spot trade is satisfied, using (117).

Now we will show that the consumption allocations are also the same.

$$c_{1s}'^h = e_{1s}^h + \hat{\tau}_s'^h = e_{1s}^h + \hat{\tau}_s^h = c_{1s}^h, \tag{122}$$

where the second equality follows from (117), and the last one follows from (112). Similarly,

$$\begin{aligned} c_{2s}'^h &= e_{2s}^h + \left(R_s k_s'^h + \psi_s'^h\right) + \tau_s'^h = e_{2s}^h + \left(R_s k_s'^h + \left(R_s k_s^h + \psi_s^h - R_s k'^h\right)\right) + \tau_s^h \\ &= e_{2s}^h + \left(R_s k_s^h + \psi_s^h\right) + \tau_s^h = c_{2s}^h, \end{aligned} \tag{123}$$

where the second equality follows from (116) and (117), and the last one follows from (113).

□

B.4 Ex-ante Contracting versus Ex-post Spot Trading

Thus far we implicitly shut down trade in the spot markets in each state. This section shows that the spot markets are redundant when all types of contracts are available (see Lemma 6 below). In other words, agents do not need to trade in spot markets, though they may well do so. Importantly, the spot markets are open and deliver the spot price $p(z_s)$. In addition, we also show that the asset-backed securities are not necessary when the spot markets are open and active (see Lemma 7 below). Put differently, agents simply are indifferent between trading in spot markets or ex-ante asset-backed securities.

When the spot markets are open, each agent h can trade $\hat{\tau}_s^h$ units of good 1 for τ_s^h units of good 2 at a spot price $p(z_s)$ according to the spot-trade constraint:

$$\hat{\tau}_s^h + p(z_s)\tau_s^h = 0. \tag{124}$$

Recall that the spot price function, $p(z_s)$, is the price such that the spot markets for both goods clear:

$$\sum_h \alpha^h \hat{\tau}_s^h = 0, \tag{125}$$

$$\sum_h \alpha^h \tau_s^h = 0. \tag{126}$$

Hence, an attainable allocation with the spot markets is defined by adding the spot-trade constraint (124) and market-clearing constraints (125)-(126) to Definition 2.

To be more precise, an allocation is said to be *equivalent* to an attainable allocation if it is attainable and generates the same consumption allocation and market fundamental in each state s as the original attainable allocation.

Lemma 6. *For any attainable allocation* $\left(\mathbf{c}_0^h, k^h, \hat{\theta}^h, \theta^h, \hat{\tau}^h, \tau^h\right)_h$*, there exists an* ***equivalent*** *allocation* $\left(\mathbf{c}_0^h, k^h, \hat{\theta}'^h, \theta'^h, \hat{\tau}'^h, \tau'^h\right)_h$ *such that*

$$\hat{\tau}_s'^h = \tau_s'^h = 0, \forall s, h. \tag{127}$$

Proof. Let $\left(\mathbf{c}_0^h, k^h, \hat{\theta}^h, \theta^h, \hat{\tau}^h, \tau^h\right)_h$ be an attainable allocation. We will show that we can find an equivalent allocation with no spot trade, i.e., $\hat{\tau}_s'^h = \tau_s'^h = 0$. Consider the following candidate allocation (with $'$)

$$\mathbf{c}_0'^h = \mathbf{c}_0^h, \forall h, \tag{128}$$

$$\hat{\theta}_s'^h = \hat{\theta}_s^h + \hat{\tau}_s^h, \forall s, h, \tag{129}$$

$$\theta_s'^h = \theta_s^h + \tau_s^h, \forall s, h. \tag{130}$$

Note that agents here acquire or issue securities on good 1 and good 2 in state s rather than waiting for trade in spot markets. The rest of the proof is similar to the proof of Lemma 4, and hence is omitted (it is available in our Working Paper version). □

Condition (127) in Lemma 6 implies that the spot markets in period 1 are redundant when all securities are allowed; that is, anything that can be done through the spot markets and one set of securities is feasible under another set of securities without spot markets. Henceforth (and previously), the ex-post spot trade transfers will be (were) set to zero, ($\hat{\tau}^h = 0, \tau^h = 0$ as in (127)) and the spot-trade constraints (124) will be (were) neglected, unless stated otherwise.

Lemma 7. *For any attainable allocation* $\left(\mathbf{c}_0^h, k^h, \hat{\psi}_s^h, \psi_s^h, \hat{\sigma}_s^h, \sigma_s^h, \hat{\nu}_s^h, \nu_s^h, \hat{\tau}^h, \tau^h\right)_h$*, there exists an* ***equivalent*** *allocation* $\left(\mathbf{c}_0^h, k^h, \hat{\psi}_s'^h, \psi_s'^h, \hat{\sigma}_s'^h, \sigma_s'^h, \hat{\nu}_s'^h, \nu_s'^h, \hat{\tau}'^h, \tau'^h\right)_h$ *such that*

$$\hat{\sigma}_s'^h = \sigma_s'^h = \hat{\nu}_s'^h = \nu_s'^h = 0, \forall s, h. \tag{131}$$

Proof. Suppose $\left(\mathbf{c}_0^h, k^h, \hat{\psi}_s^h, \psi_s^h, \hat{\sigma}_s^h, \sigma_s^h, \hat{\nu}_s^h, \nu_s^h, \hat{\tau}^h, \tau^h\right)_h$ is attainable. Consider the following alternative allocation (with $'$) $\left(\mathbf{c}_0^h, k^h, \hat{\psi}_s'^h, \psi_s'^h, \hat{\sigma}_s'^h, \sigma_s'^h, \hat{\nu}_s'^h, \nu_s'^h, \hat{\tau}'^h, \tau'^h\right)_h$ such that

$$\hat{\sigma}_s'^h = \sigma_s'^h = \hat{\nu}_s'^h = \nu_s'^h = 0, \psi_s^h = 0, \ \forall h, s, \tag{132}$$

$$\hat{\psi}_s'^h = \left(\hat{\psi}_s^h + \hat{\sigma}_s^h + \hat{\nu}_s^h\right) + p(z_s)\left(\psi_s^h + \sigma_s^h + \nu_s^h\right), \tag{133}$$

$$\hat{\tau}_s'^h = -p(z_s)\left(\psi_s^h + \sigma_s^h + \nu_s^h\right) + \hat{\tau}_s^h, \tag{134}$$

$$\tau_s'^h = \left(\psi_s^h + \sigma_s^h + \nu_s^h\right) + \tau_s^h. \tag{135}$$

Note that at the alternative allocation, agents will do in spot markets what they might have done in asset-backed security markets. In addition, with active spot markets, there is no need to trade in collateral-backed securities paying in good 2 (trade in the ones paying in numeraire good only). The rest of the proof is similar to the proof of Lemma 6, and hence is omitted (it is available in our Working Paper version). □

It is worthy of emphasis that Lemma 6 and Lemma 7 imply that the asset-backed securities that we need in this model are the ones that replicate spot markets. In other words, the asset-backed securities in this model (with tranching) are simply substitutes for spot markets. Henceforth, we let asset-backed securities play this role and shut down active trade in spot markets. The result is summarized in the following corollary.

Corollary 1. *Asset-backed securities and the spot markets are perfect substitute in this model.*

B.5 Spot Markets and Security Prices: No-Arbitrage Condition

The pyramiding mechanism puts a restriction on the prices of contracts traded within each security exchange. The ratio of the equilibrium prices of the securities in security exchange z_s in state s, $\frac{P_a(z_s,s)}{\widehat{P}_a(z_s,s)}$, must be equal to the marginal rate of substitution or the spot price in the security exchange, $p(z_s)$. Otherwise, there will be an arbitrage possibility (by keeping the collateral constraints satisfied with pyramiding). The result is summarized in the following lemma.

Lemma 8. *In a competitive equilibrium, for each s and z_s,*

$$P_a(z_s, s) = p(z_s)\widehat{P}_a(z_s, s). \tag{136}$$

Using the no-arbitrage condition (136), the collateral constraints (100) can be rewritten as

$$P_a(z_s, s) R_s k^h + \widehat{P}_a(z_s, s)\hat{\theta}_s^h + P_a(z_s, s)\theta_s^h \geq 0, \forall s. \tag{137}$$

These constraints state that the value in units of good 1 at $t = 0$ of all ex ante securities held (RHS) cannot exceed the value of collateral held (LHS). These constraints are applicable when the spot markets are not available but the ex-ante asset-backed securities can be traded.

B.6 Prices of the Right to Trade

Trading in security exchanges also imposes a restriction on collateral, contract and prices of rights to trade in security exchanges, $P_{20}, P_a(z_s, s), P_\Delta(z_s, s)$. Even though collateral and securities are indeterminate (see Lemma 10 below), holding collateral additionally impacts the spot price $p(z_s)$. Therefore, the equilibrium price of collateral must reflect the role of collateral on the spot price in each security exchange.

Again a no-arbitrage condition requires that the prices of two different bundles that result in the same consumption allocation for an agent h must have the same prices. Using the profit maximization condition of a broker-dealer (49) and some algebra, we can prove the following equation must hold.

$$P_{20} + \sum_{s=1}^{S} P_\Delta(z_s, s) z_s R_s = \sum_{s=1}^{S} P_a(z_s, s) R_s. \tag{138}$$

The RHS is the price of contracts paying R_s units of good 2 in every state s. On the other hand, the LHS is the total cost of the same return, received by buying and holding a unit of collateral. The first term on the LHS is the price of the collateral good. The second term on the LHS comes from the fact that holding more a unit of good 2 increases Δ in every state s by the amount $z_s R_s$. In particular, an agent holding an additional unit of collateral must pay for the marginal impact $z_s R_s$ at price $P_\Delta(z_s, s)$. This term prices the impact of collateral on the market fundamental. In equilibrium, these two values must be the same.

Lemma 9. *In a competitive equilibrium, for each set of security exchanges* $\mathbf{z} = (z_s)_s$*, (138) holds.*

B.7 An Indeterminacy of Collateral Allocations

Using a similar argument to proof of Lemma 6, we show that the collateral and securities paying in good 2 allocations are indeterminate; that is, neither k nor θ_s can be pinned down (but the net-claim of good 2, $R_s k + \theta_s$, will be uniquely determined). Roughly speaking, agents are indifferent between buying contracts ($\theta_s > 0$), and holding collateral ($k > 0$) and selling contracts against it ($\theta_s < 0$) as long as they lead to the same consumption allocation in period 1 over state s. Note that storage technology is linear and there is no direct utility per se from holding collateral. The formal result is summarized in the following lemma.

Lemma 10. *For a feasible bundle of an agent type* h $\left(c_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right)$*, any bundle* $\left(c_0', k', \hat{\theta}', \theta', \mathbf{z}', \Delta'^h\right)$ *such that (i)* $c_0' = c_0$*, (ii)* $\mathbf{z}' = \mathbf{z}$*, (iii)* $\hat{\theta}' = \hat{\theta}$*, and (iv)* $R_s k' + \theta_s' = R_s k + \theta_s$*,* $\forall s$*, is also be feasible for the agent* h*, and leads to the same consumption allocation as the original bundle.*

Proof. The proof is similar to the one of Lemma 6. □

Condition (iv) implies that there is some indeterminacy between k and θ_s. In particular, if we set $k' = 0$, then we can reach the same consumption allocation by setting the security position to be $\theta_s' = R_s k + \theta_s$. This implies that there is no loss of generality to assume that all collateral is held by an unconstrained type, and the others including constrained agents hold no collateral, $k = 0$, and therefore we will do so, unless stated otherwise.

Thus a constrained agent may hold no collateral and therefore issue no directly-collateralized securities, yet her collateral constraint is binding still. The fact that a constrained agent is effectively not borrowing at all in equilibrium seems counterintuitive at first. In a partial equilibrium setting when the price of collateral good is fixed exogenously, one would imagine that the agent will try to buy more of the collateral good and then borrow against to increase current consumption. In this *general equilibrium* setting where collateral price is determined endogenously, however, the price of the collateral good rises so in effect those transactions will offset each other and lead to a zero net transfer.

B.8 Trading in Security Exchanges Generates Intertemporal Transfers

Trading in security exchanges can generate additional intertemporal transfers. For example, a constrained agent would like to smooth consumption by issuing securities or borrowing to transfer future resources back to period 0 but cannot do so much because of the limited commitment. Trading in security exchanges facilitates more consumption smoothing by generating period 0 transfer for a constrained agent.

For the sake of discussion, we will consider a case with two types of agents one of which is constrained, and without uncertainty (i.e., $S = 1$). In addition, as shown in the previous section, we assume that a constrained agent holds no collateral, $k = 0$. Using (25), (49), and (136), an agent type h's budget constraint (44) can be rewritten as

$$\sum_b x^h(b)\left[c_{10} + P_{20}c_{20}\right] \le e^h_{10} + P_{20}e^h_{20} + \sum_b x^h(b)\,\widehat{P}_a(z_1, 1)\left[-\hat{\theta}_1 - p(z_1)\theta_1\right]$$
$$+\sum_b x^h(b)\,P_\Delta(z_1, 1)\left[\frac{e^h_{11}}{e^h_{21}} - z_1\right]e^h_{21}.$$

The third term on the RHS is the revenue from borrowing via $\left(\hat{\theta}_1, \theta_1\right)$. Using the collateral constraint (40), $p(z_1)R_1k + \hat{\theta}_1 + p(z_1)\theta_1 \geq 0$. Since the constrained agent holds no collateral, $k = 0$, her collateral constraint becomes $\hat{\theta}_1 + p(z_1)\theta_1 = 0$. Of course, this constrained agent would like to go short on the contracts (i.e., having $\hat{\theta}_1 + p(z_1)\theta_1 < 0$) but cannot do so because she holds no collateral. In other words, with zero collateral, the agent cannot borrow from trading in contracts.

Of special interest, the last term on the RHS shows that the constrained agent could potentially receive positive period 0 wealth by trading in security exchanges. In particular, a constrained agent could smooth consumption intertemporally by trading in security exchanges in such a way that this term is positive, giving her more resources to purchase date zero consumption. For example, if $P_\Delta(z_s, s) > 0$, then the constrained agent will buy a security exchange z_s whose market fundamental is lower than her own endowment, i.e., $z_s < \frac{e^h_{1s}}{e^h_{2s}}$, and vice versa.

On the other hand, an unconstrained agent will potentially hold strictly positive amount of collateral, $k > 0$. She will in fact transfer out period 0 wealth from trading in security exchanges. For example, if a constrained agent has $z_s < \frac{e^h_{1s}}{e^h_{2s}}$, then the consistency constraint (58) implies that an unconstrained agent in security exchange z_s must have $z_s > \frac{e^h_{1s}}{R_s k + e^h_{2s}}$. Hence, if $P_\Delta(z_s, s) > 0$, the last term on the RHS will be negative for an unconstrained agent.

C Detailed Derivations

C.1 Derivation of a Competitive Equilibrium with the Externality in Environment 1

The endowment profile and the first-best allocation suggest that agent 2 would like to move resources forward from $t = 1$ to $t = 0$, and therefore will be constrained. Hence, we will assume that agents type 2 hold no collateral, i.e. $k^1 = k$ and $k^2 = 0$. We now solve for an equilibrium k. From the market clearing conditions of contracts, we can set $\hat{\theta}_1^1 = \hat{\theta} = -\hat{\theta}_1^2$ and $\theta_1^1 = \theta = -\theta_1^2$. Note that this does not mean agent 1 is demanding both securities. In addition, using the specified collateral allocation, the market fundamental in period $t = 1$ is now $z = \frac{4}{4+k}$ (the ratio of endowment of good 1 to the sum of endowment of good 2 and saving), and consequently the spot price of good 2 in period 1 is $p(z) = \left(\frac{4}{4+k}\right)^2$.

With homothetic preferences, the first-order conditions of the problem (7) for both types imply that in spot markets at date $t = 0$

$$P_{20} = \left(\frac{c_{10}^1}{c_{20}^1}\right)^2 = \left(\frac{c_{10}^2}{c_{20}^2}\right)^2 = \left(\frac{4}{4-k}\right)^2. \tag{139}$$

Since agent 1's collateral constraint is not binding, the first-order conditions of her utility-maximization problem (7) with respect to θ_1^1 and c_{10}^1 lead to

$$P_1 = \frac{u_{21}^1}{u_{10}^1} = \left(\frac{c_{10}^1}{c_{21}^1}\right)^2, \tag{140}$$

where $u_{it}^h = \frac{\partial u^h}{\partial c_{it}}$ is the marginal utility with respect to c_{it}, and P_1 is the price of a security paying in good 2 in period $t = 1$, θ_1^h. Note that we put superscript h on the utility function for clarity. Further, the first-order conditions of the consumer's problem (7) with respect to θ_1^1 and k^1 (interior solutions) lead to

$$P_{20} = P_1. \tag{141}$$

Intuitively, this is the case because their payoffs are identical and both are collateralizable. Using (139) and (140), condition (141) implies that

$$\frac{c_{10}^1}{c_{20}^1} = \frac{c_{10}^1}{c_{21}^1} \Longrightarrow c_{20}^1 = c_{21}^1. \tag{142}$$

That is, an unconstrained agent consumes the same amount of good 2 in both periods.

Substituting (139) and (140) into (141) gives

$$\begin{aligned}\left(\frac{4}{4-k}\right)^2 &= \left(\frac{c_{10}^1}{c_{21}^1}\right)^2; \\ \frac{4}{4-k} &= \frac{c_{10}^1}{1+k+\theta} \Longrightarrow (4-k)\, c_{10}^1 = 4+4k+4\theta, \qquad (143)\end{aligned}$$

where we use $c_{21}^1 = 1+k+\theta$.

On the other hand, an agent type 2's collateral constraint is binding; with $k^2 = 0$,

$$\hat{\theta}^2 + p(z)\theta^2 = 0 \Longrightarrow -\hat{\theta} - p(z)\theta = 0 \Longrightarrow \hat{\theta} = -\left(\frac{4}{4+k}\right)^2 \theta, \qquad (144)$$

where the second and the last equations use $\hat{\theta}^2 = -\hat{\theta}$ and $\theta^2 = -\theta$, and $p(z) = \left(\frac{4}{4+k}\right)^2$, respectively.

The budget constraint of an agent 1 (9) can be written as

$$c_{10}^1 - 3 + P_{20}\left[c_{20}^1 + k - 3\right] + \hat{P}_1\hat{\theta} + P_1\theta = 0. \qquad (145)$$

A standard no-arbitrage argument (similar to the one used in Lemma 8) implies that

$$P_1 = p(z)\widehat{P}_1. \qquad (146)$$

It thus true from (146) that

$$\hat{P}_1\hat{\theta} + P_1\theta = \widehat{P}_1\hat{\theta} + \widehat{P}_1 p(z)\theta = \widehat{P}_1\left[\hat{\theta} + p(z)\theta\right] p(z) = 0, \qquad (147)$$

where the last equation follows the fact that the term in the bracket is zero, from (144). Now the LHS of the budget constraint (145) can be rewritten as

$$c_{10}^1 + P_{20}\left[c_{20}^1 + k - 3\right] = 3. \qquad (148)$$

Using (139), we can replace c_{20}^1 by $\left(\frac{4-k}{4}\right) c_{10}^1$. Then using $P_{20} = \left(\frac{4}{4-k}\right)^2$ gives

$$\begin{aligned}c_{10}^1 + \left(\frac{4}{4-k}\right)^2\left[\left(\frac{4-k}{4}\right) c_{10}^1 + k - 3\right] &= 3 \\ \Longrightarrow (4-k)\, c_{10}^1 &= \frac{3k^2 - 40k + 96}{8-k}. \qquad (149)\end{aligned}$$

Substituting (143)into (149) gives

$$\frac{3k^2 - 40k + 96}{8 - k} \quad = \quad 4 + 4\theta + 4k \Longrightarrow 4\theta + 4k = \frac{3k^2 - 36k + 64}{8 - k}. \tag{150}$$

With the identical homothetic preferences, the period $t = 1$ consumption allocations must satisfy

$$z = \frac{4}{4 + k} = \frac{c_{11}^1}{c_{21}^1} \Longrightarrow \frac{4}{4 + k} = \frac{1 + \hat{\theta}}{1 + k + \theta}. \tag{151}$$

Substitute (144) into (151) gives

$$4\theta + 4k = -3k\left(\frac{4 + k}{8 + k}\right) + 4k. \tag{152}$$

Using (150) and (152), we have

$$\frac{3k^2 - 36k + 64}{8 - k} = -3k\left(\frac{4 + k}{8 + k}\right) + 4k \Longrightarrow 4k^3 - 384k + 512 = 0. \tag{153}$$

There are three roots for equation (153). Using the condition that $0 \leq k \leq 4$, there is only one feasible solution, i.e. $k \approx 1.3595$. To sum up, the equilibrium collateral allocation is $k^1 = k = 1.3595$ and $k^2 = 0$.

C.2 Derivation of a Competitive Equilibrium with the Externality in Environment 2

First of all, the price of good 2 in period $t = 0$ is given by

$$P_{20} = \left(\frac{2}{2 - k}\right)^2 \tag{154}$$

Similarly, the market fundamental in each state s is $z_s = \frac{2}{2+k}$. Hence, the spot price of good 2 in each state s is given by

$$p(z_s) = \left(\frac{2}{2 + k}\right)^2, \forall s. \tag{155}$$

Further, the price of a (collateralized) security paying in good 2 in state s is given by

$$P_s = \max_h \left(\frac{\pi_s u_{2s}^h}{u_{10}^h}\right), \ \forall s. \tag{156}$$

The endowment structure implies that agents type 2 will have higher MRS $\frac{\pi_s u^h_{2s}}{u^h_{10}}$ in state 1, and vice versa. Hence, (156) can be rewritten as

$$P_1 = \frac{\pi_s u^2_{21}}{u^2_{10}} = \frac{1}{2}\left(\frac{2}{1+k+\theta}\right)^2 = \frac{\pi_s u^1_{22}}{u^1_{10}} = P_2. \tag{157}$$

Note that the symmetry also implies that $P_1 = P_2$. Using the optimal conditions with respect to k^h and θ^h_s, we can show that

$$P_{20} = P_1 + P_2 \Longrightarrow \left(\frac{2}{2-k}\right)^2 = \left(\frac{2}{1+k+\theta}\right)^2. \tag{158}$$

Next, with the homotheticity of preferences, the ratio of consumption in each state of each agent must be equal to the market fundamental; that is,

$$\frac{1+\hat{\theta}}{1+k+\theta} = z_s = \frac{2}{2+k}. \tag{159}$$

Furthermore, the collateral constraint in state $s = 1$ of an agent type $h = 1$ is binding, i.e.

$$p(z_1)k - \hat{\theta} - p(z_1)\theta = 0 \Longrightarrow \hat{\theta} = \left(\frac{2}{2+k}\right)^2 (k-\theta). \tag{160}$$

Note that the same equation can be derived from the binding collateral constraint in state $s = 2$ for an agent type $h = 2$.

We can compute a collateral equilibrium using (158), (159), and (160) to solve for $\left(k, \theta, \hat{\theta}\right)$. We can rewrite (158) as

$$2 - k = 1 + k + \theta \Longrightarrow \theta = 1 - 2k. \tag{161}$$

In addition, Substituting (160) into (159) gives

$$1 + \left(\frac{2}{2+k}\right)^2 (k-\theta) = \left(\frac{2}{2+k}\right)(1+k+\theta). \tag{162}$$

Then, substituting (161) into (162) will give

$$\begin{aligned} 1 + \left(\frac{2}{2+k}\right)^2 (k-1+2k) &= \left(\frac{2}{2+k}\right)(1+k+1-2k) \\ &\Longrightarrow 3k^2 + 16k - 8 = 0. \end{aligned} \tag{163}$$

The unique feasible (positive) solution to the above quadratic equation is $k \approx 0.4603$.

C.3 Derivation of a Competitive Equilibrium with the Externality in Environment 3

We restrict our attention to a symmetric allocation of each type. Using Lemma , we assume that all constrained agents hold no collateral, i.e., $k^h = 0$ for $h = 2, 3$. Let $k^1 = k$.

First, the first-order conditions of the consumer's problem (7) result in

$$\frac{c_{10}^1}{c_{20}^1} = \frac{c_{10}^2}{c_{20}^2} = \frac{c_{10}^3}{c_{20}^3} = \frac{12.5}{12.5 - k}. \tag{164}$$

From the endowment profile, it is clear that an agent 1 will not be constrained. The first-order conditions of the consumer's problem (7) with respect to θ^1 and c_{10}^1 lead to

$$\frac{u_{21}^1}{u_{10}^1} = P. \tag{165}$$

Further, the first-order conditions of the consumer's problem (7) with respect to θ^1 and k^1 (interior solutions) lead to

$$P_{20} = P. \tag{166}$$

Combining (165), (166) and the utility function (37), gives

$$P_{20} = \left(\frac{12.5}{12.5 - k}\right)^2 \quad = \quad P = \frac{u_{21}^1}{u_{10}^1} = \left(\frac{c_{10}^1}{c_{21}^1}\right)^2. \tag{167}$$

This implies that

$$\frac{12.5}{12.5 - k} \quad = \quad \frac{c_{10}^1}{c_{21}^1} = \frac{c_{10}^1}{0.5 + k + \theta^1} \Longrightarrow (12.5 - k)\, c_{10}^1 = 12.5 \left(0.5 + k + \theta^1\right), \tag{168}$$

where we use $c_{21}^1 = 0.5 + k + \theta^1$.

In addition, the market fundamental in period $t = 1$ is $z = \frac{12.5}{12.5+k}$, and consequently the spot price of good 2 in period $t = 1$ is $\left(\frac{12.5}{12.5+k}\right)^2$. The bindingness of the collateral constraints of agent 2 and agent 3, combining with the market-clearing conditions of securities, imply that

$$\hat{\theta}^1 = -\left(\frac{12.5}{12.5 + k}\right)^2 \theta^1. \tag{169}$$

A standard no-arbitrage argument (similar to the one used in Lemma 8) implies that

$$P_1 = p(z)\widehat{P}_1, \tag{170}$$

which can be used to show that

$$\hat{P}_1\hat{\theta}^1 + P_1\theta^1 = \widehat{P}_1\hat{\theta}^1 + \widehat{P}_1 p(z)\theta^1 = \widehat{P}_1\left[\hat{\theta}^1 + p(z)\theta^1\right]p(z) = 0, \tag{171}$$

where the last equation follows the bindingness of the collateral constraints of agent 2 and agent 3, combining with the market-clearing conditions of securities. The budget constraint of an agent 1 (9) can be written as

$$c_{10}^1 - e_{10}^1 + P_{20}\left[c_{20}^1 + k - e_{20}^1\right] \;\; = \;\; 0. \tag{172}$$

Substituting (164) and (167) into (172), we have

$$(12.5-k)\,c_{10}^1 \;\; = \;\; \frac{12.5^2\left(e_{20}^1 - k\right) + e_{10}^1\left(12.5-k\right)^2}{25-k}. \tag{173}$$

Substituting (168) into (173), we have

$$12.5\left(0.5 + k + \theta^1\right) \;\; = \;\; \frac{12.5^2\left(e_{20}^1 - k\right) + e_{10}^1\left(12.5-k\right)^2}{25-k}. \tag{174}$$

With the identical homothetic preferences, the period $t = 1$ consumption allocations must satisfy

$$z = \frac{12.5}{12.5+k} = \frac{c_{11}^1}{c_{21}^1} \Longrightarrow \frac{12.5}{12.5+k} = \frac{0.5+\hat{\theta}^1}{0.5+k+\theta^1}, \tag{175}$$

where the equality follows (169). This can be rewritten as

$$12.5\left(0.5+k+\theta^1\right) = (12.5+k)\left(0.5 - \left(\frac{12.5}{12.5+k}\right)^2\theta^1\right). \tag{176}$$

Solving (174) and (176) for k and θ^1, with $e_{10}^1 = 4.2631$ and $e_{20}^1 = 11.5$, gives one feasible solution $(0 \le k \le 12.5)$ $k = 7.2836, \theta^1 = -4.2849$. To sum up, the competitive collateral equilibrium allocation is $k^1 = k = 7.2836$, and $k^2 = k^3 = 0$.

C.4 Computing Competitive Equilibrium with Segregated Security Exchanges

All numerical solutions for competitive equilibrium with segregated security exchanges in section 4 are computed using the following programming problem:

Program 3.

$$\max_{\left(\mathbf{x}^h \in \widetilde{X}^h\right)_h} \sum_h \lambda^h \alpha^h \sum_{\mathbf{c}_0,k,\hat{\theta},\theta,\mathbf{z},\Delta} x^h\left(\mathbf{c}_0,k,\hat{\theta},\theta,\mathbf{z},\Delta\right)\left\{u\left(c_{10}^h,c_{20}^h\right)+\beta\sum_s \pi_s u\left(e_{1s}^h+\hat{\theta}_s, e_{2s}^h+R_s k+\theta_s\right)\right\}$$

subject to

$$\begin{aligned}
\sum_h \sum_{\mathbf{c}_0,k,\hat{\theta},\theta,\mathbf{z},\Delta} \alpha^h x^h\left(\mathbf{c}_0,k,\hat{\theta},\theta,\mathbf{z},\Delta\right) &= 1, \\
\sum_h \sum_{\mathbf{c}_0,k,\hat{\theta},\theta,\mathbf{z},\Delta} \alpha^h x^h\left(\mathbf{c}_0,k,\hat{\theta},\theta,\mathbf{z},\Delta\right) c_{10} &\leq \sum_h \alpha^h e_{10}^h, \\
\sum_h \sum_{\mathbf{c}_0,k,\hat{\theta},\theta,\mathbf{z},\Delta} \alpha^h x^h\left(\mathbf{c}_0,k,\hat{\theta},\theta,\mathbf{z},\Delta\right)\left[c_{20}+k\right] &\leq \sum_h \alpha^h e_{20}^h, \\
\sum_h \sum_{\mathbf{c}_0,k,\hat{\theta},\theta,\mathbf{z}_{-s},\Delta} \alpha^h x^h\left(\mathbf{c}_0,k,\hat{\theta},\theta,\mathbf{z}_{-s},z_s,\Delta\right)\hat{\theta}_s &= 0 \ ,\forall s; z_s, \\
\sum_h \sum_{\mathbf{c}_0,k,\hat{\theta},\theta,\mathbf{z}_{-s},\Delta} \alpha^h x^h\left(\mathbf{c}_0,k,\hat{\theta},\theta,\mathbf{z}_{-s},z_s,\Delta\right)\theta_s &= 0 \ ,\forall s; z_s,
\end{aligned}$$

where $\widetilde{X}^h$ is the restricted consumption possibility set for an agent type h in which we allow a positive mass only on a bundle $\left(\mathbf{c}_0,k,\hat{\theta},\theta,\mathbf{z},\Delta\right)$ that satisfies a market fundamental ratio condition: $z_s=\frac{e_{1s}^h+\hat{\theta}_s}{e_{2s}^h+R_s k+\theta_s}$. This in turn puts restrictions on grids of securities $\hat{\theta}_s$ and θ_s for each security exchange.

Pareto program 3 and the Pareto program 2 are equivalent but the former is more computationally tractable. The restrictions on grids of securities do not affect the results because each agent type must optimally choose securities (from a complete set of securities) in such a way that his consumption ratio equals to the market fundamental $z_s=\frac{e_{1s}^h+\hat{\theta}_s}{e_{2s}^h+R_s k+\theta_s}$. These restrictions imply that the consistency constraints (64) are redundant.

We can then recover all prices including $\widehat{P}_a(z_s,s), P_a(z_s,s), P_\Delta(z_s,s)$ from the Lagrange multipliers of the securities $\widetilde{\widehat{P}}_a(z_s,s)$ and $\widetilde{P}_a(z_s,s)$ (normalized by the Lagrange multiplier of c_{10}) from Pareto program 3 as follows. First, the no-arbitrage condition (136) applies here:

$$P_a(z_s,s) \;=\; p(z_s)\,\widehat{P}_a(z_s,s). \tag{177}$$

Second, the equivalence between Pareto program 3 and the Pareto program 2, and the second welfare theorem implies that equilibrium prices $\widehat{P}_a(z_s,s), P_a(z_s,s), P_\Delta(z_s,s)$ and the

Lagrange multipliers of the securities paying in good 1 and good 2, respectively, $\widetilde{\widehat{P}}_a(z_s,s)$ and $\widetilde{P}_a(z_s,s)$ from Pareto program 3 must satisfy the following relationships:

$$\widetilde{\widehat{P}}_a(z_s,s) = \widehat{P}_a(z_s,s) + P_\Delta(z_s,s), \tag{178}$$

$$\widetilde{P}_a(z_s,s) = P_a(z_s,s) - z_s P_\Delta(z_s,s). \tag{179}$$

Solving the last three equations simultaneously give equilibrium prices as follows:

$$\widehat{P}_a(z_s,s) = \frac{z_s\widetilde{\widehat{P}}_a(z_s,s) + \widetilde{P}_a(z_s,s)}{z_s + p(z_s)}, \tag{180}$$

$$P_a(z_s,s) = p(z_s)\frac{z_s\widetilde{\widehat{P}}_a(z_s,s) + \widetilde{P}_a(z_s,s)}{z_s + p(z_s)}, \tag{181}$$

$$P_\Delta(z_s,s) = \frac{p(z_s)\widetilde{\widehat{P}}_a(z_s,s) - \widetilde{P}_a(z_s,s)}{z_s + p(z_s)}. \tag{182}$$

C.5 Competitive Equilibrium with Segregated Security Exchanges with Three Agent Types but Two Constrained Types are Almost Identical

This environment is similar to Environment 3. The key difference is in the endowments in period $t = 1$ for agent types $h = 2, 3$. The endowment profile is given in Table 10 below. Note that type 2 and type 3 are almost identical. As in Environment 3, we assume that the Pareto weights are $\lambda^1 = 0.8, \lambda^2 = \lambda^3 = 0.1$.

Table 10: Endowment profiles of the agents.

Type of Agents	e^h_{10}	e^h_{20}	e^h_{11}	e^h_{21}
$h = 1$	11.56	11.5	0.5	0.5
$h = 2$	0.37	0.5	7.0	5.0
$h = 3$	0.58	0.5	6.9	5.0

They key point is that the competitive equilibrium with segregated security exchanges still has two active exchanges as in Environment 3. Of course, the active exchanges are closer to each other than the ones in Environment 3. It is worthy of emphasis that the number of active exchanges in equilibrium equals to the number of constrained types again. See also Appendix C.6 below for a similar result with four agent types, three of which are constrained.

Table 11: Equilibrium allocation of (non-zero-mass) lotteries. There are two active security exchanges; $z = \mathbf{0.8883}$ and $z = \mathbf{0.8913}$.

	$h=1$		$h=2$	$h=3$
k	5.6933	5.6735	0	0
$\hat{\tau}$	2.3528	2.3575	-1.1987	-1.1565
τ	-2.9817	-2.9676	1.5089	1.4657
c_{10}	7.3228	7.3228	2.5886	2.5886
c_{20}	3.9939	3.9939	1.4114	1.4114
c_{11}	2.8528	2.8528	5.8013	5.7435
c_{21}	3.2116	3.2060	6.5089	6.4657
z	**0.8883**	**0.8913**	**0.8913**	**0.8883**
Δ	5.0015	5.0025	-2.5435	-2.4685
x^h	0.4916	0.5084	1.0000	1.0000

C.6 Competitive Equilibrium with Segregated Security Exchanges with Four Agent Types and Three Constrained Types

The endowment profile is given in Table 12 below. There are four agent types of equal mass. This example is constructed in such a way that agent types $h = 2, 3, 4$ are constrained in equilibrium. We assume that the Pareto weights are $\lambda^1 = 0.5, \lambda^2 = 0.2, \lambda^3 = 0.1, \lambda^4 = 0.2$.

Table 12: Endowment profiles of the agents.

Type of Agents	e^h_{10}	e^h_{20}	e^h_{11}	e^h_{21}
$h=1$	2.06	11.5	0.5	0.5
$h=2$	1.94	0.5	6.0	4.0
$h=3$	3.52	0.5	4.0	5.0
$h=4$	4.98	1.0	2.0	3.0

They key point is that the number of active exchanges in equilibrium equals to the number of constrained types again. Specifically, with four agent types, three of which are constrained, there are three active exchanges in equilibrium.

Table 13: Equilibrium allocation of (non-zero-mass) lotteries. There are three active security exchanges; $z =$ **0.6034**, $z =$ **0.6719**, and $z =$ **0.8710**.

	$h = 1$			$h = 2$	$h = 3$	$h = 4$
k	5.8556	5.1771	3.9264	0	0	0
$\hat{\tau}$	1.2550	1.3320	1.5620	-1.1713	-0.2574	-0.0714
τ	-3.4470	-2.9505	-2.0590	1.5439	0.5702	0.1962
c_{10}	4.6096	4.6096	4.6096	2.9144	2.0616	2.9144
c_{20}	3.0344	3.0344	3.0344	1.9174	1.3529	1.9174
c_{11}	1.7550	1.8320	2.0620	4.8287	3.7426	1.9286
c_{21}	2.9086	2.7266	2.3674	5.5439	5.5702	3.1962
z	**0.6034**	**0.6719**	**0.8710**	**0.8710**	**0.6719**	**0.6034**
Δ	3.3350	3.3145	3.3554	-2.5160	-0.6405	-0.1898
x^h	0.0569	0.1932	0.7498	1.0000	1.0000	1.0000