Risk-taking over the Life Cycle: Aggregate and Distributive Implications of Entrepreneurial Risk *

Dejanir H. Silva[†] Robert M. Townsend[‡]

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Abstract

We study the risk-taking behavior over the life cycle of entrepreneurs subject to partial insurance against idiosyncratic shocks. The model quantitatively accounts for the aggregate and idiosyncratic risk premia, the life-cycle profiles of consumption and risk-taking, and the patterns of wealth inequality observed in the data. A reform that relaxes the risk constraints reduces the idiosyncratic risk premium and induces an investment boom. Consistent with a Kuznets curve, inequality increases in the short run and declines in the long run. The initial generation of entrepreneurs benefits from better insurance, but future generations will be worse off after the reform.

KEYWORDS: Entrepreneurship, risk-taking, risk premium, insurance, inequality

JEL CLASSIFICATION: G11, G51, E44.

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[†]Purdue University, Krannert School of Management. Email: dejanir@purdue.edu.

[‡]Massachusetts Institute of Technology, Department of Economics. Email: rtownsen@mit.edu.

1 Introduction

Entrepreneurship is inherently a risky activity. Entrepreneurial risk is particularly important in developing economies, where the bulk of production occurs in privately owned businesses, and risk-sharing opportunities are limited. Given that entrepreneurs hold under-diversified portfolios, such risks have far-reaching implications.[1] At the individual level, entrepreneurial risk distorts investment and savings decisions. The importance of business income is highly heterogeneous and varies substantially over the life cycle, shaping wealth inequality patterns. At the aggregate level, imperfect insurance leads to an inefficient risk premium, which depresses the capital stock and hinders economic development. The pervasive effects of limited risk-sharing highlight the relevance of policy interventions that alleviate the consequences of entrepreneurs' lack of diversification.

To assess the aggregate impact of such interventions, it is crucial to determine the quantitative importance of the limits to risk sharing. Identifying such limits, however, is challenging. For instance, despite holding concentrated portfolios, entrepreneurs may engage in self-insurance or informal insurance arrangements, which attenuates the equilibrium impact of under-diversification. Such insurance arrangements are widespread in developing economies, as documented, e.g., by Kinnan and Townsend (2012). Moreover, it is essential to disentangle the effects of aggregate and idiosyncratic shocks. Lack of diversification is potentially less consequential if private businesses are mostly exposed to aggregate (non-diversifiable) risk. Therefore, the relevance of risk-sharing frictions cannot be directly inferred from the degree of observed under-diversification, which ignores unobserved transfers, or the volatility of business income, which is driven by both aggregate and idiosyncratic shocks.

This paper studies the aggregate and distributive implications of entrepreneurial risk in a quantitative life-cycle model with limited idiosyncratic insurance. We discipline the model using a rich dataset on small business owners in Thailand, which includes information on the returns of entrepreneurial activity and entrepreneurs' risk-taking and consumption.[2] The model captures several salient features of the data, including the risk premium on business returns, the life-cycle profile of entrepreneurs' risk-taking and consumption, and the patterns of inequality between- and within-age groups. By combining a theory of the idiosyncratic risk premium with detailed information on portfolios and business returns, we assess the extent of limits to risk sharing faced by entrepreneurs and, ultimately, determine the impact of reforms relaxing these constraints on investment, inequality, and economic development.

We start by studying empirically the determinants of expected entrepreneurial returns. Ex-

[1]For evidence on the under-diversification of entrepreneurs see, e.g., Moskowitz and Vissing-Jørgensen (2002), and Herranz et al. (2009) for a study focused on small businesses.

[2]A large literature studies the macroeconomic and asset-pricing implications of firm-level risk (see, e.g., Christiano et al. 2014, Gârleanu et al. 2015, Dou 2016, Herskovic et al. 2016, Di Tella 2017, Herskovic et al. 2018), but these studies usually rely on data of public companies to discipline their quantitative exercises.

pected returns are tightly connected to the marginal product of capital (MPK), so the required return on a project is potentially a key determinant of its scale.[3] Imperfect risk sharing implies that entrepreneurs require compensation for holding idiosyncratic risk, suggesting that average returns vary positively in the cross-section with the exposure to idiosyncratic shocks. As expected, we find that entrepreneurs with higher risk exposure receive higher expected returns. More importantly, a two-factor model explains most of the variation in entrepreneurial returns in our data. Differences in idiosyncratic volatility and exposure to aggregate risk explain almost 70% of the cross-sectional variation in average returns. Given idiosyncratic volatility and aggregate betas are measured with error, this captures only a lower bound on how much these two factors explain the variation in returns.[4] Therefore, a risk-based explanation can account for nearly all cross-section differences in expected entrepreneurial returns.

Moreover, while more than 90% of the variance is explained by idiosyncratic shocks, less than half of the expected return consists of compensation for holding idiosyncratic risk. Sharpe ratios then vary by source of risk in a way inconsistent with an autarky allocation, where only total variance would matter, and also inconsistent with perfect risk sharing, as the idiosyncratic risk premium would be zero. Therefore, the evidence suggests entrepreneurs are partially insuring idiosyncratic shocks.

Next, we consider the determinants of entrepreneurs' risk-taking decisions, as measured by the share of net worth invested in the business. Risk-taking depends not only on the risk and return properties of the project but also on entrepreneurs' attitudes towards risk. Households' characteristics potentially shape their risk preferences. In particular, we find substantial variation in risk-taking over the life cycle. Young entrepreneurs are nearly 40% more exposed to the business than old entrepreneurs. These differences in risk-taking cannot be explained by differences in expected returns, suggesting heterogeneity in risk tolerance across age groups.

To capture these motivating facts, we propose a general equilibrium model with two main ingredients: limited idiosyncratic insurance and finite lives with imperfect altruism. The economy is populated by entrepreneurs and wage earners. Entrepreneurs are the only ones with access to a production technology. The technology is exposed to aggregate and idiosyncratic shocks. Aggregate shocks are public information, so there are no frictions in sharing aggregate risk. In contrast, idiosyncratic shocks are private information. Entrepreneurs face a moral hazard problem, as they can divert a fraction of capital every period. Under the optimal contract, entrepreneurs are subject to a skin-in-the-game constraint, so they must bear a fraction of the idiosyncratic risk in equilibrium.

The moral hazard parameter controls the degree of partial insurance. We identify the

[3]See David et al. (2022) on how differences in (aggregate) risk premium lead to dispersion in MPK.

[4]When we sort entrepreneurs according to their exposure to aggregate and idiosyncratic risk, as is standard in empirical asset pricing, we obtain that nearly 90% of the variation can be explained by only these two factors.

moral hazard parameter by matching the decomposition of expected returns into aggregate and idiosyncratic components performed in the data. Intuitively, if entrepreneurs can diversify a significant fraction of the risk, expected returns would reflect primarily compensation for holding aggregate risk.

We show that the degree of partial insurance depends on three components: i) the idiosyncratic risk premium, ii) the share of the variance of returns due to idiosyncratic shocks, and iii) the share of the business on entrepreneurs' net worth. Entrepreneurs can diversify roughly 40% of the idiosyncratic volatility, so they have to absorb most of the idiosyncratic shocks. Therefore, entrepreneurs face relevant limitations in diversifying risks, significantly affecting their investment decisions.

Importantly, we assume that entrepreneurial households also receive some labor income from household members working outside the business, consistent with what we observe in our data. This fact implies that entrepreneurs' effective risk aversion depends inversely on the ratio of human wealth (present discounted value of future labor income) to financial wealth (investment in safe assets and the business). The human-financial wealth ratio declines over the life cycle in the data. This mechanism endogenously creates heterogeneity in risk aversion, which is essential to replicating the empirical life-cycle profiles.

The model quantitatively matches the decline in risk-taking over the life cycle, as young entrepreneurs are relatively risk-tolerant, given a higher human-financial ratio early in life.[5] Household savings have a significant life-cycle component. The consumption-wealth ratio initially declines with age as savings increase, and then it increases by the end of the life cycle. The model captures this pattern through the interplay of two different effects. First, the decline in the human-financial wealth ratio reduces the consumption-wealth ratio. Second, the marginal propensity to consume (MPC) increases with age due to imperfect altruism. This effect is stronger for older households, which explains the higher consumption-wealth ratio later in life.

To focus on the impact of limits to risk sharing, we initially assume the moral hazard problem is the only friction in the model. We then study the implications of introducing borrowing constraints into the model in two extensions. We first introduce limited pledgeability of physical capital by having collateral constraints. The life-cycle predictions are mostly unchanged, but collateral constraints have important implications for determining entrepreneurial returns. Collateral constraints sever the link between risk and return for constrained entrepreneurs. Differences in the business scale and expected returns are driven by differences in net worth for this group of entrepreneurs. Poorer entrepreneurs end up with higher MPK. We test these predictions in our data. Net worth explains only a negligible fraction of the cross-sectional

[5]This mechanism is reminiscent of work on portfolio choice with labor income (Bodie et al. 1992, Heaton and Lucas 1997, and Viceira 2001). See Huggett and Kaplan (2016) for a similar approach to valuing human wealth.

differences in average returns, and we find only weak support for the prediction that poorer entrepreneurs have higher expected returns after we control for risk exposure. Moreover, the relationship between risk and return is actually *stronger* for entrepreneurs with relatively low net worth. These results suggest that collateral constraints are not the primary driver of expected returns in our data.

In a second extension, we introduce limited pledgeability of human wealth by having uninsurable labor income risk. This version of the model allows us to study the interaction between uninsurable labor income risk, as in Aiyagari (1994), and uninsurable investment risk, as in Angeletos (2007). We obtain (approximate) analytical expressions despite having aggregate shocks and incomplete markets. The borrowing limit associated with uninsurable labor income risk has two effects on risk-taking. On the one hand, it tends to reduce risk-taking, as entrepreneurs have fewer resources, which limits the scale of the enterprise. On the other hand, it endogenously reduces risk aversion, which pushes entrepreneurs to be more exposed to risk. As entrepreneurs cannot borrow against a fraction of human wealth, future income acts as a buffer, making entrepreneurs more willing to take risks.

In contrast to the version with collateral constraints, variations in risk exposure always affect expected returns with limited borrowing against labor income. Moreover, the theory predicts that variations in idiosyncratic variance should significantly impact expected returns for poorer entrepreneurs, as they are effectively more risk-averse. We test this prediction and find support for it in our data. Overall, the hybrid model with both uninsurable labor and investment risk successfully captures different aspects of the data. As the main predictions of the baseline model do not change in the richer model, we focus on a setting with only investment risk.

Entrepreneurial risk has a significant effect on inequality. On average, wealth initially increases with age and then declines later in life. A similar inverted U-shape holds for the standard deviation of wealth conditional on age, a measure of within-group inequality. The initial increase in inequality is explained by a generalized "*r*-*g*" effect. A positive difference between the expected return on wealth, which includes the aggregate and idiosyncratic risk premium, and the economy's growth rate causes entrepreneurs to accumulate more wealth as they age. A countervailing force comes from the MPC increasing with age due to imperfect altruism. Even though this was not targeted during the calibration, the model roughly captures the between- and within-group inequality patterns observed in the data.

At the aggregate level, the idiosyncratic risk premium and the capital stock are simultaneously determined. The expected return on the business pins down the MPK. Increasing the idiosyncratic risk premium leads to a higher MPK and a smaller capital stock. Improving risk sharing leads to a decline in the idiosyncratic risk premium and an increase in the capital stock in the long run. The effect is quantitatively large, where a reform that reduces the risk premium by 140 basis points leads to an increase of the capital stock of 13%. This is consistent with the view that financial development, captured here by the extent of idiosyncratic risk sharing, is an important determinant of the level of economic development.[6]

We also consider the dynamic implications of relaxing risk constraints. Improving idiosyncratic insurance leads to an investment boom that lasts for a decade, accompanied by a sharp increase in the value of private businesses in the short run. Inequality falls in the long run, as entrepreneurs are less exposed to risk after the reform. In contrast, inequality increases in the short run due to a revaluation effect. The increase in the value of the business benefits entrepreneurs with larger initial investments, which are relatively richer before the reform. It takes a long time for inequality to converge to its new long-run level due to intergenerational links. This pattern is consistent with Kuznets's (1955) hypothesis over the relationship between inequality and economic development.[7]

Considering the transitional dynamics is also relevant to assessing the welfare effects of risk-sharing improvements. Entrepreneurs are worse off in the long run after the reform, despite the benefits of better diversification, as entrepreneurs accumulate less wealth over time. In contrast, the initial generation's welfare improves with the reform. They received their bequest before the intervention, and the value of their businesses increased substantially in the short run. Therefore, most of the gains of the reform are reaped by the initial generation of entrepreneurs and wage earners, who receive higher wages given the higher capital stock.

Related literature. This paper is related to several strands of literature in macroeconomics and finance. First, the work studying how firm-level uncertainty affects asset prices and the real economy (Gârleanu et al. 2015, Di Tella 2017, Dou 2016, Herskovic et al. 2016, Iachan et al. 2021).[8] While this literature focuses primarily on business-cycle fluctuations, we study how firm-level uncertainty affects the economy in the long run. Second, the literature on how the lack of diversification of entrepreneurs' portfolios affects several firm outcomes, including real investment (Panousi and Papanikolaou 2012), capital structure (Chen et al. 2010, Herranz et al. 2015), and risk-taking (Chen and Strebulaev 2019). This work is mainly in partial equilibrium and abstracts from the aggregate implications of entrepreneurial risk.

We also contribute to recent work on the importance of heterogeneous returns for inequality. This literature documents substantial heterogeneity in portfolio returns (Fagereng et al. 2019, Fagereng et al. 2020, Bach et al. 2020); it finds that private business wealth is one the main sources of wealth at the top (Smith et al. 2019, Smith et al. 2020), and that return hetero-

[6] See Levine (2005) for a literature review on the connections between financial and economic development.

[7] Greenwood and Jovanovic (1990) provides an early treatment of the Kuznets dynamics and its connection with financial development.

[8] A related literature studies the asset-pricing implications of labor income risk in infinite-horizon (Constantinides and Duffie 1996 and Heaton and Lucas 1996) and life-cycle models (Constantinides et al. 2002 and Storesletten et al. 2007).

geneity is important to quantitatively match the observed levels of inequality (Gomez 2017, Hubmer et al. 2021).[9] We study how the idiosyncratic risk premium in private business affects inequality between- and within-age groups and how changes in this premium affect the dynamics of inequality. Greenwald et al. (2021) show that financial wealth inequality increases after a decline in interest rates. We find similar effects for reducing the idiosyncratic risk premium in the short run, but inequality falls in the long run.

A related literature studies the extent to which households partially insure *labor income* shocks (Blundell et al. 2008 and Kaplan and Violante 2010). The work by, e.g., Krueger and Perri (2006), Attanasio and Pavoni (2011), and Heathcote et al. (2014) shows how the degree of partial insurance can be inferred with data on consumption and labor income. We focus on entrepreneurial risk and show that the degree of partial insurance can be identified using information on the idiosyncratic risk premium and the exposure of entrepreneurs' wealth to private businesses.

Our result that young entrepreneurs invest a larger fraction of their wealth in the business is closely related to the findings in the literature on portfolio choice over the life cycle (Jagannathan et al. 1996, Viceira 2001, Cocco et al. 2005), who found that young households should invest more in the stock market than old households.[10] In particular, we show that variations in the human-financial wealth ratio account quantitatively for the pattern of entrepreneurial risk-taking over the life cycle. Related work uses business income as a source of background risk to explain stock market investment (Heaton and Lucas 2000a, 2000b). In contrast, we endogenize the decision to invest in the business and show how non-business income is relevant in accounting for cross-sectional differences in risk-taking.

An extensive micro-development literature studies risk sharing (Townsend 1994, Morduch 1995) and the risk and return of production activities in developing economies (Udry and Anagol 2006, De Mel et al. 2008). Karlan et al. (2014) conducted a randomized control trial extending credit and insurance to farmers and found that the lack of insurance is the binding constraint to investment. Their results are consistent with our findings that relaxing risk constraints significantly impact investment. The macro-development literature studies the aggregate implications of credit constraints (Buera and Shin 2013, Midrigan and Xu 2014, Moll 2014). Our approach is complementary to theirs, as we focus instead on the role of risk constraints. Our work is closer to the original model of uninsurable investment risk by Angeletos (2007), which we extend to allow for partial idiosyncratic insurance, a rich demographic structure, and aggregate risk. These extensions are crucial to capture the patterns of consumption and risk-taking observed in the microdata and derive the dynamics of inequality in response to a relaxation of risk constraints.

[9] See De Nardi and Fella (2017) for a survey of quantitative Bewley models of wealth inequality.

[10] A related literature studies the life-cycle patterns of consumption and labor income (see, e.g., Deaton and Paxson 1994, Gourinchas and Parker 2002, and Storesletten et al. 2004).

2 Motivating evidence

In this section, we provide motivating evidence on entrepreneurial activity in the context of a developing country. First, we study the determinants of entrepreneurial returns. Second, we consider entrepreneurs' risk-taking and consumption decisions over the life cycle.

Data. We use data from the Townsend Thai Monthly Survey, an intensive monthly survey initiated in 1998 in four provinces of Thailand. Two provinces, Chachoengsao and Lopburi, are semi-urban in a more developed central region near the capital, Bangkok. The other two provinces are rural, Buriram and Srisaket, and are located in the less developed northeastern region by the border of Cambodia. In each of the four provinces, the survey is conducted in four villages, chosen at random within a given township. A detailed discussion of the Townsend Thai Monthly Survey can be found in Samphantharak and Townsend (2010).

Our sample covers 710 households and 14 years of monthly data, starting in January 1999. These economies were subject to various aggregate and idiosyncratic shocks during this time. Rice cultivation is affected by seasonal variations in rainfall and temperature. Restrictions on exports to the EU affected shrimp ponds. Milk cows' productivity varies substantially over time for a given animal and over the herd. The exposure to this rich set of shocks highlights the importance of entrepreneurial risk, enabling us to disentangle the role of aggregate and idiosyncratic shocks.

The data collected in the Townsend Thai Monthly Survey includes information on the net income generated by the business and household total assets and liabilities. The data is rich enough to construct a detailed balance sheet for these private businesses, which We can then compute the return on assets (ROA), measured as net profits income over business assets. We also measure the fraction of entrepreneurs' wealth invested in the business and the fraction invested in safe (real or financial) assets, providing us with a measure of risk-taking. Finally, the data includes household labor income and consumption information. This allows us to characterize the savings behavior and the importance of non-business income for entrepreneurial households. See Appendix B for a detailed description of these variables.[11]

2.1 The determinants of expected entrepreneurial returns

Whether and how much to invest in a business depends on its risk/return trade-off. We study this trade-off by considering the required risk compensation implicit in entrepreneurial returns. The case of perfect risk sharing provides a natural benchmark. Under this assumption,

[11]The household is the unit of measurement, even though households typically consist of multiple members doing separate or partially overlapping activities. We treat the household as a whole as unitary (see, e.g., Doepke and Tertilt (2016) for a discussion of unitary models of the household).

differences in expected returns are entirely driven by differences in the exposure to aggregate risk, as idiosyncratic shocks can be perfectly insured or diversified. On the other extreme, entrepreneurs can be in financial autarky, without access to any insurance. In this case, only the total amount of risk is relevant to entrepreneurs, causing expected returns to vary with both aggregate and idiosyncratic volatility. When idiosyncratic shocks can be partially insured or diversified, expected returns depend not only on the total variance of returns but also on the relative importance of each source of risk.

This discussion motivates using a two-factor model to explain the cross-section of expected returns of private businesses based on the exposure to aggregate and idiosyncratic shocks. This approach allows us to evaluate whether perfect risk sharing or autarky holds and, more importantly, whether a risk-based explanation can quantitatively account for the differences in entrepreneurial returns observed in the data. We estimate our factor model following the standard two-pass regression methodology developed by Fama and MacBeth (1973). In the first stage, we estimate entrepreneurs' exposure to aggregate risk by the slope of a time-series regression of returns for entrepreneur i, $R_{i,t}$, on the cross-sectional average return across all entrepreneurs in a given province, R_t^{agg}. Formally, we run the regression:

$$R_{i,t} = \alpha_i + \beta_i R_t^{agg} + \epsilon_{i,t},$$

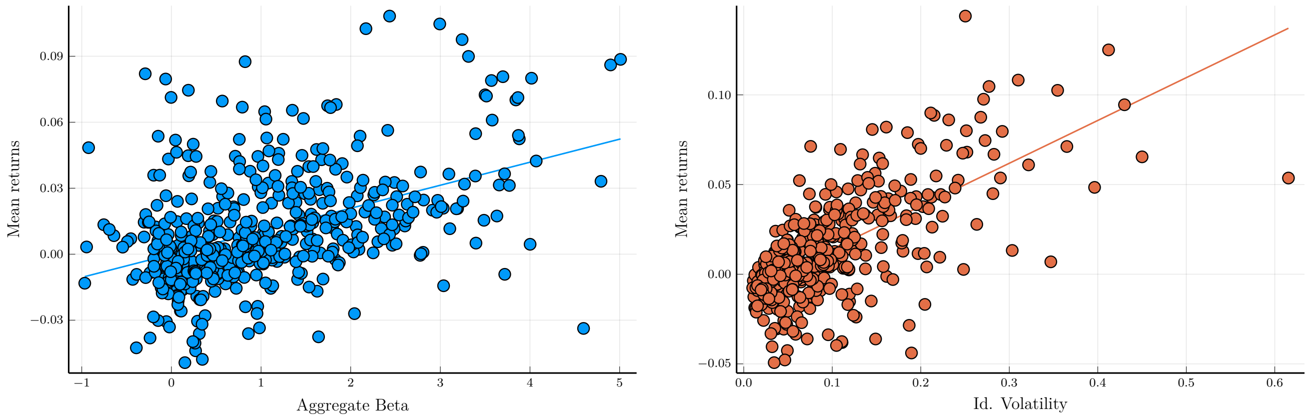
for each entrepreneur $i \in \{1, \ldots, N\}$. Notice that R_t^{agg} plays the role of the market portfolio in standard tests of the capital asset pricing model (CAPM). By averaging across entrepreneurs in a given region, idiosyncratic risk gets diversified, so β_i captures the exposure to aggregate risk for entrepreneur i.[12] We measure the exposure to idiosyncratic risk by the variance of the residuals in the above regression, $\sigma_i^2 = var[\epsilon_{i,t}]$.

Figure 1 shows how the exposure to each source of risk is related to entrepreneurs' average return. We find a strong association between average time-series returns and exposure to aggregate and idiosyncratic risk. The left panel on Figure 1 shows that entrepreneurs more exposed to aggregate risk have higher average returns. Similarly, the right panel shows a positive and significant relationship between average returns and idiosyncratic volatility, consistent with entrepreneurs' limited access to insurance.

One possible concern is that idiosyncratic volatility may be correlated with the aggregate beta, e.g., riskier projects could be more exposed to both sources of risk. The positive association between idiosyncratic risk and expected returns could still be consistent with perfect risk sharing. To address this point, we run a second-stage cross-sectional regression of average return for entrepreneur i, $\overline{R}_i = \frac{1}{T}\sum_{t=1}^{T} R_{i,t}$, on the estimated exposure to aggregate risk, $\hat{\beta}_i$,

[12] Following Samphantharak and Townsend (2018), we adopt the province as the relevant geographic unit. Moreover, we compute a "leave-one-out" mean of the returns, so we regress the return of entrepreneur i on the average return across all entrepreneurs other than i on a given province.

Figure 1: Average returns vs. aggregate and idiosyncratic risk



Note: The left (right) panel shows a scatter plot of average time-series returns for each entrepreneur against aggregate beta (idiosyncratic volatility). Aggregate beta is measured as the slope of the time-series regression of individual returns on the leave-one-out average return in the entrepreneur's province. Idiosyncratic volatility is calculated as the volatility of residuals from the same regression. To limit the influence of outliers, we trim 1% of the observations in the left and right tails.

and idiosyncratic risk $\hat{\sigma}_i^2$:

$$\overline{R}_i = \lambda_0 + \lambda_{ag}\hat{\beta}_i + \lambda_{id}\hat{\sigma}_i^2 + u_i.$$

The coefficients λ_{ag} and λ_{id} correspond to the *price of aggregate and idiosyncratic risk*, respectively. They capture the required compensation for one exposure unit to each source of risk.

Table 1 shows the results for the second-stage regression. We find a positive and significant price of risk for β_i and σ_i^2. These two factors account for a significant fraction of the cross-sectional variation in expected returns. Column 1 shows that the CAPM-inspired one-factor model explains a sizeable fraction of the variation in average portfolio returns, with an adjusted R^2 of 34%. Given the limited empirical success of the CAPM (see, e.g., Fama and French 1992), it is surprising that a single aggregate factor plays a significant role in explaining entrepreneurial returns. Column 2 shows that idiosyncratic risk explains an even larger fraction of the variation, with an adjusted R^2 of 58%. Together, these two factors explain most of the variation in expected returns, with an adjusted R^2 of 68%.

An important issue is that our risk exposure measures are noisy estimates of the actual beta and idiosyncratic volatility, so measurement error may bias our results. To deal with this issue, we embed our two-stage procedure into a generalized method of moments (GMM) framework and compute standard errors that account for the uncertainty in the estimation of $\hat{\beta}_i$ and $\hat{\sigma}_i^2$, as discussed in detail in Appendix B.2.[13] We also follow the standard procedure in empirical asset pricing (see, e.g., Black et al. 1972) and group entrepreneurs into bins, or portfolios, according to a 5×5 double sort based on aggregate beta and idiosyncratic risk.[14] The group-level betas and idiosyncratic variance are arguably better measured than

[13]See also Cochrane (2009) for a discussion on how to correct the standard errors in two-pass regressions.

[14]As these are private businesses, the portfolios are not tradeable. Forming portfolios allows us to reduce the

Dependent Variable:	mean_roa			
Model:	(1)	(2)	(3)	(4)
Variables				
(Intercept)	-0.0071	0.0093***	-0.0026	-0.0036
	(0.0045)	(0.0015)	(0.0037)	(0.0023)
beta	0.0195***		0.0117***	0.0090**
	(0.0044)		(0.0039)	(0.0032)
sigma2		0.1246***	0.1031***	0.2127***
		(0.0280)	(0.0292)	(0.0718)
Fit statistics				
Observations	541	541	541	24
R^2	0.344	0.578	0.685	0.892
Adjusted R^2	0.343	0.577	0.684	0.882

*Signif. Codes: ***: 0.01, **: 0.05, *: 0.1*

Table 1: Estimation of two-factor model

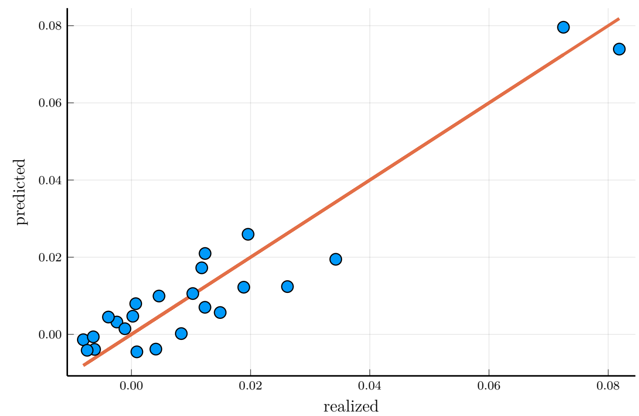


Figure 2: Realized vs. predicted returns

Note: The left panel shows a cross-sectional regression of average returns on the aggregate beta or idiosyncratic variance (or both). Standard errors account for the uncertainty in estimating the regressors, and they are robust to contemporaneous correlations between error terms across entrepreneurs. The right panel shows the 45° degree line and a scatter plot of the predicted returns of the two-factor model and actual average returns for the portfolio-level analysis.

their individual-level counterparts, which limits the impact of measurement error. Column 4 shows the results. We obtain a similar price of aggregate risk and a larger price of idiosyncratic risk, consistent with some attenuation bias in this case. The two risk factors account for 88% of the variation in group-level returns. Figure 2 shows the fit of the two-factor model by comparing predicted and realized average returns. The model successfully accounts for the significant variation in expected returns observed in our data. Both factors are necessary to achieve this success. We show in Appendix B.3 that a single-factor model, based on either an aggregate or idiosyncratic factor, performs substantially worse than the two-factor model. In particular, we can reject the hypothesis of a single-factor model based on either source of risk. Therefore, this evidence suggests that a two-factor model is necessary to explain the cross-section of entrepreneurial returns.

To better grasp the role of each risk factor, we provide a decomposition of entrepreneurial returns into an aggregate and idiosyncratic component. Table 2 decomposes the risk premium and variance of returns using the results from Table 1. We find that most risk is idiosyncratic, accounting for 94% of the variance, while idiosyncratic risk accounts for slightly less than half of the risk premium. This implies that the Sharpe ratio of the aggregate component is nearly five times larger than the Sharpe ratio of the idiosyncratic component.[15]

The results from Table 2 are inconsistent with the perfect risk sharing and autarky benchmarks. The fact that the idiosyncratic risk premium explains almost half of total expected

noise in the estimation and better capture the risk/return relationship. To avoid within-portfolio diversification, we assign to the portfolio the average idiosyncratic volatility instead of the volatility of the average.

[15] Samphantharak and Townsend (2018) computes a similar decomposition of expected returns and variance based on entrepreneurs' total portfolio returns, including risky and safe assets. The decomposition of the portfolio's risky component, which is our focus, is the relevant one to map into the structural model in Section 3.

Table 2: Aggregate and idiosyncratic components of risk and return

	Risk premium	% of returns	Volatility	% of variance	Sharpe ratio
Total returns	4.4%	100%	21.0%	100%	0.21
Aggregate component	2.4%	54.7%	5.1%	6.0%	0.47
Idiosyncratic component	2.0%	45.3%	20.3%	94.0%	0.10

Note: The aggregate component is measured as the coefficient on the aggregate beta on the cross-sectional regression times the average beta across entrepreneurs. The idiosyncratic component is measured as the cross-sectional regression coefficient on idiosyncratic risk times the idiosyncratic variance averaged across entrepreneurs. The variance decomposition is computed at the individual level based on the results from the first-stage regression and then averaged across all entrepreneurs.

returns indicates that limits to diversification are substantial, which allows us to reject the perfect risk-sharing benchmark. Financial autarky is also inconsistent with the observed pattern in the Sharpe ratio. As shown in Appendix E.4, the share of variance of the idiosyncratic component should coincide with its share of expected returns under autarky. Equivalently, the Sharpe ratio should be proportional to the volatility.[16] This implies that the Sharpe ratio for the idiosyncratic component should be four times *larger* than the one for the aggregate component, but we observe the opposite pattern in the data.[17] These results suggest that partial insurance is relevant to explain the empirical patterns involving risk and return. We revisit this decomposition through the lenses of our structural model in Section 5.2.

2.2 Risk-taking and savings over the life cycle

Next, we consider entrepreneurs' risk-taking and savings decisions. The decision of how much to invest depends on the risk and return of the entrepreneurial activity and households' attitude towards risk. Behavior towards risk is potentially shaped by households' characteristics, such as age and household size. We study then how risk-taking and consumption decisions depend on expected returns, age, and a range of demographic controls.

To ensure stationarity, we consider the behavior of the ratio of the value invested in the business to the entrepreneur's financial wealth, our measure of risk-taking, and the ratio of consumption to financial wealth. By looking at ratios instead of levels, we limit the influence of aggregate shocks and focus on potentially more stable relationships. We consider five age groups, ranging from 25 to 80 years old, where group 1 is the youngest and group 5 is the oldest. The cutoffs for each group are chosen such that we have roughly the same number of households in each group. We cluster standard errors by household and year to account

[16]In a two-factor model, the risk premium is given by $p^{ag}\sigma_{ag} + p^{id}\sigma_{id}$, where (p^{ag}, p^{id}) is the return per unit of risk (the prices of risk) and $(\sigma_{ag}, \sigma_{id})$ the volatilities. Perfect risk sharing corresponds to $p^{id} = 0$. In autarky, the prices of risk (or Sharpe ratio) are $p^{ag} = \gamma\sigma_{ag}$ and $p^{id} = \gamma\sigma_{id}$, so the risk premium is $\gamma(\sigma_{ag}^2 + \sigma_{id}^2)$.

[17]Financial autarky implies that the idiosyncratic Sharpe ratio relative to the aggregate Sharpe ratio should be $20.3/5.1 \approx 4.0$, while we observed a ratio of $0.10/0.47 \approx 0.21$, almost a twentyfold difference.

Table 3: Risk-taking and consumption behavior over the life cycle

	Risk-taking				Consumption-wealth ratio			
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
Age group: 1	0.30***	0.31***	0.30***	0.27***	0.14***	0.14***	0.12***	0.10***
	(0.01)	(0.01)	(0.02)	(0.02)	(0.009)	(0.008)	(0.008)	(0.007)
Age group: 2	0.27***	0.28***	0.27***	0.25***	0.13***	0.13***	0.12***	0.11***
	(0.01)	(0.01)	(0.01)	(0.01)	(0.009)	(0.007)	(0.007)	(0.007)
Age group: 3	0.27***	0.27***	0.26***	0.25***	0.12***	0.12***	0.11***	0.11***
	(0.01)	(0.01)	(0.01)	(0.01)	(0.008)	(0.007)	(0.008)	(0.007)
Age group: 4	0.23***	0.23***	0.23***	0.22***	0.10***	0.10***	0.09***	0.08***
	(0.02)	(0.01)	(0.01)	(0.01)	(0.007)	(0.006)	(0.006)	(0.006)
Age group: 5	0.22***	0.22***	0.23***	0.21***	0.11***	0.11***	0.10***	0.09***
	(0.01)	(0.01)	(0.01)	(0.01)	(0.008)	(0.008)	(0.008)	(0.007)
Year FE		Yes	Yes	Yes		Yes	Yes	Yes
Demographic controls			Yes	Yes			Yes	Yes
Risk and return controls				Yes				Yes
N	8,680	8,680	7,713	6,499	8,548	8,548	7,693	6,495
Adjusted R^2	0.015	0.025	0.107	0.139	0.010	0.020	0.062	0.070

Clustered (year & household) standard-errors in parentheses
*Signif. Codes: ***: 0.01, **: 0.05, *: 0.1*

Note: Age effects correspond to the conditional expectation of the outcome variable evaluated at the mean of the continuous controls and averaged over the year, province, and sector fixed-effects (when included). Demographic controls include dummies for the province, household size, the number of kids, and a set of sector affiliation dummies. Risk and return controls include the entrepreneur's average return on the business and the exposure to aggregate and idiosyncratic risk.

for shocks potentially correlated over time or across households. We aggregate the data to the annual level to better isolate the age effects. To limit the influence of outliers, we trim the data at the 1% level at both tails. The details of the empirical methodology are described in Appendix B.4.

We compute age effects controlling for year fixed-effects and additional controls following the methodology of Kaplan (2012). In particular, for an outcome variable $z_{i,a,r,t}$ for entrepreneur i of age a at region r and year t, we compute the age effects as the conditional expectation for a particular age group marginalized with respect to all controls:

$$age_a^{year\ FE+dem} = \frac{1}{T}\frac{1}{R}\sum_{t=1}^{T}\sum_{r=1}^{R}\mathbb{E}[z_{i,a,r,t}|a,r,t,x_{i,a,r,t} = \overline{x}],$$

where $x_{i,a,r,t}$ represents the vector of continuous control variables and $\overline{x}$ denotes its mean.

Table 3 presents the results. We find that risk-taking has a strong life-cycle pattern. Column 1, the case without any additional controls, shows that the amount invested in the business declines sharply with age, with the coefficient for the oldest group being roughly 30% smaller than for the youngest group. A similar pattern holds as we add more controls. Time fixed-effects do not change the results, as shown in column 2, suggesting that aggregate shocks affect the amount invested in the business and financial wealth but leave their ratio roughly unchanged. Column 3 includes demographic and geographic controls, including the household size, the number of children, the province the household is located in, and her sector affiliation, with similar patterns. Column 4 adds the average return on the business and the exposure to aggregate and idiosyncratic risk as controls. We find once again substantial variation in risk-taking over the life cycle, indicating these patterns are driven by differences in attitudes toward risk instead of differences in risk or expected returns.

The consumption-wealth ratio also has a strong life-cycle pattern. However, instead of declining monotonically with age as our risk-taking measure, we find a U-shaped pattern: consumption-wealth initially decreases with age but increases for the oldest group.

We derive three main conclusions from these results. First, the cross-sectional variation in risk-taking and consumption has a substantial life-cycle component. Second, the life-cycle patterns in risk-taking are mainly driven by differences in risk appetite instead of differences in risk or average return. Third, year fixed-effects do not significantly affect the life-cycle patterns. This suggests a relatively stationary environment despite the presence of aggregate shocks. These facts, in conjunction with the importance of partial insurance discussed in Section 2.1, motivate the ingredients of our theoretical model.

3 A life-cycle model of entrepreneurial risk taking

In this section, we consider a model of entrepreneurial activity with two main ingredients: i) imperfect idiosyncratic insurance and ii) finite lives with imperfect altruism. These two ingredients play a crucial role in capturing the empirical patterns described in Section 2.

3.1 Environment

Time is continuous, and the economy is populated by two types of households: entrepreneurs and wage earners. Population grows at rate g, and the share of entrepreneurs in the population is constant and given by χ_e. The set of entrepreneurs and wage earners alive at period t are denoted by $\mathcal{E}_t$ and $\mathcal{W}_t$, respectively. Entrepreneurs live for T periods and leave bequests to their offspring. For simplicity, we assume that wage earners have an infinite horizon. While all households receive labor income, only entrepreneurs can access a production technology. Production is exposed to both aggregate and idiosyncratic shocks. Households buy and sell aggregate insurance in a frictionless market, but they have access only to imperfect idiosyncratic insurance. Households borrow and lend at a riskless rate r_t. We now describe in detail the technology, preferences, and financial frictions both types of households face.

Technology. Entrepreneur i combines capital $k_{i,t}$ and hired labor $l_{i,t}$ to produce a final homogeneous good $\tilde{y}_{i,t}$, the numeraire in this economy, using the technology:

$$\tilde{y}_{i,t} = A_t k_{i,t}^{\alpha} l_{i,t}^{1-\alpha}, \tag{1}$$

and we denote scaled output by $y_t = \tilde{y}_t / A_t$.[18]

Productivity A_t is subject to aggregate shocks and follows a geometric Brownian motion:

$$\frac{dA_t}{A_t} = \mu_A dt + \sigma_A dZ_t, \tag{2}$$

where Z_t is a standard Brownian motion.

Entrepreneurs can adjust their capital stock by investing in new capital or buying capital from another entrepreneur. The investment technology is risky and subject to adjustment costs. In particular, given a total investment of $\iota_{i,t} A_t k_{i,t}$, capital evolves according to

$$\frac{dk_{i,t}}{k_{i,t}} = (\Phi(\iota_{i,t}) - \delta)\, dt + \sigma_{id} dZ_{i,t}. \tag{3}$$

$Z_{i,t}$ is an idiosyncratic Brownian motion for entrepreneur i, which is independent across entrepreneurs. The investment function $\Phi(\cdot)$ satisfies $\Phi(0) = 0$, $\Phi'(\cdot) > 0$, and $\Phi''(\cdot) < 0$.

The concavity of $\Phi(\cdot)$ captures the presence of adjustment costs. Notably, investment is risky and subject to idiosyncratic shocks. Entrepreneurs can also adjust their capital stock by buying capital from other entrepreneurs at the price $\tilde{q}_t = q_t A_t$. The market value of the business is given by $\tilde{q}_t k_{i,t}$. In equilibrium, q_t will be non-stochastic, so the relative price of capital $\tilde{q}_t$ moves proportionally with aggregate productivity shocks.

The evolution of the aggregate capital stock, $k_t = \int_{\mathcal{E}_t} k_{i,t} di$, is not affected by idiosyncratic shocks, as these shocks get diversified in the aggregate:

$$dk_t = \left[\int_{\mathcal{E}_t} (\Phi(\iota_{i,t}) - \delta)\, k_{i,t} di \right] dt. \tag{4}$$

The return on investing in the project can be written as the sum of the dividend yield, i.e., profits net of investment expenses relative to the value of the business, and capital gains:

$$\begin{aligned} dR_{i,t} &= \frac{\tilde{y}_{i,t} - \tilde{w}_t l_{i,t} - \iota_{i,t} A_t k_{i,t}}{\tilde{q}_t k_{i,t}} dt + \frac{d(\tilde{q}_t k_{i,t})}{\tilde{q}_t k_{i,t}} \\ &= \mu_{i,t}^R dt + \sigma_A dZ_t + \sigma_{id} dZ_{i,t}, \end{aligned}$$

where $\tilde{w}_t = w_t A_t$ denotes the wage rate and $\mu_{i,t}^R$ is the expected return.

[18] We adopt this convention throughout the paper: variables that grow with aggregate productivity A_t are denoted with a tilde and the corresponding scaled variable are denoted without a tilde.

Using Ito's lemma to compute expected capital gains, the expected return on the project is

$$\mu_{i,t}^R \equiv \underbrace{\frac{y_{i,t} - w_t l_{i,t} - \iota_{i,t} k_{i,t}}{q_t k_{i,t}}}_{\text{dividend yield}} + \underbrace{\frac{\dot{q}_t}{q_t} + \mu_A + \Phi(\iota_{i,t}) - \delta}_{\text{expected capital gain}}. \tag{5}$$

Preferences and labor supply. Entrepreneurs live for T periods. They have isoelastic preferences over consumption $\tilde{c}_{i,t}$ with curvature parameter γ and derive utility of leaving bequests:

$$\mathbb{E}_{s_i}\left[\int_{s_i}^{s_i+T} e^{-\rho(t-s_i)}\frac{\tilde{c}_{i,t}^{1-\gamma}}{1-\gamma}dt + e^{-\rho(T-s_i)}(1-\psi)^{\gamma} V^* \frac{\tilde{n}_{i,s_i+T}^{1-\gamma}}{1-\gamma}\right], \tag{6}$$

$\tilde{n}_{i,t}$ denotes financial wealth (or net worth), and s_i denotes the entrepreneur's birthdate.[19]

The parameter ψ measures the strength of the bequest motive. If $\psi = 1$, entrepreneurs give no weight to their offspring. If $\psi = 0$, the behavior of entrepreneurs coincides with the one of an agent with an infinite horizon.[20] The case $0 < \psi < 1$ captures imperfect altruism.

In addition to business income, entrepreneurial households are allowed to receive labor income. This is consistent with the observation that households have multiple sources of income in our data. Labor is supplied inelastically, it is denoted by $\bar{l}_{i,t}$, and it can vary deterministically over the life cycle.[21] As discussed in Section 3.3, there is significant variation in the importance of labor income over the life cycle for entrepreneurs.

Financial friction. Entrepreneurs face a moral hazard problem, similar to the one in He and Krishnamurthy (2012) and Di Tella (2017). We follow Di Tella (2017) and restrict attention to short-term contracts. Aggregate shocks are perfectly observable by all households, while idiosyncratic shocks are only observed by the entrepreneur. An entrepreneur can divert capital, but a fraction $1-\phi$ of the diverted capital is lost in the process. The parameter $\phi \in (0,1)$ controls the severity of the moral hazard problem.

Following the literature on dynamic moral hazard, we show in Appendix C.1 that the solution to the contracting problem can be implemented by a market structure where entrepreneurs have access to a riskless asset with return r_t and both aggregate and idiosyncratic insurance. There is no limit on aggregate insurance, as aggregate shocks are perfectly observable. The quantity of idiosyncratic insurance is limited for incentive purposes. Formally, entrepreneur i pays $p_t^{ag}\tilde{\theta}_{i,t}^{ag}$ to reduce aggregate volatility by $\tilde{\theta}_{i,t}^{ag}$, where p_t^{ag} denotes the price of aggregate insurance. In contrast, entrepreneur i can buy idiosyncratic insurance $\tilde{\theta}_{i,t}^{id}$ at zero

[19]Financial wealth equals the entrepreneur's assets, including the value of the business, net of liabilities.

[20]The constant V^*, given in Appendix A.1, equals the value function coefficient for an infinite-horizon agent.

[21]Notice that $\bar{l}_{i,t}$ denotes the amount of labor supplied by the household, while $l_{i,t}$ is the amount of labor demanded by the entrepreneur to run her project.

cost in equilibrium, as providers of insurance can perfectly diversify across entrepreneurs. However, the amount of idiosyncratic insurance is limited by the *skin-in-the-game* constraint:

$$\tilde{\theta}^{id}_{i,t} \leq (1-\phi)\tilde{q}_t k_{i,t}\sigma_{id}. \tag{7}$$

This particular market structure represents one possible implementation of the optimal contract allocation. For instance, instead of formal insurance contracts, this implementation may capture the presence of informal insurance arrangements. As documented by, e.g., Kinnan and Townsend (2012), kinship networks play an important role in allowing households to partially share idiosyncratic risk. The insurance constraint can also be interpreted as an equity constraint, where entrepreneurs are unable to freely sell claims on their business to a diversified set of investors, as in e.g. Chen et al. (2010) or Panousi and Papanikolaou (2012).

Constraint (7) is binding in equilibrium, causing entrepreneurs to be *insurance-constrained*. To focus on entrepreneurial risk, we initially abstract from borrowing constraints. We discuss the role of borrowing constraints in Section 3.4. Households face a natural borrowing limit:

$$\tilde{n}_{i,t} \geq -\tilde{h}_{i,t}, \tag{8}$$

where $\tilde{h}_{i,t} \equiv \mathbb{E}_t\left[\int_t^{s+T} \frac{\pi_z}{\pi_t}\tilde{w}_z\bar{l}_{i,z}dz\right]$ denotes human wealth, and π_t denotes a stochastic discount factor (SDF) for this economy.[22] The SDF evolves according to $d\pi_t = -r_t\pi_t dt - p^{ag}_t\pi_t dZ_t$. As $\tilde{n}_{i,t} = \tilde{q}_t k_{i,t} + \tilde{b}_{i,t}$, where $\tilde{b}_{i,t}$ denotes the value of riskless bonds, condition (8) can be written as $-\tilde{b}_{i,t} \leq \tilde{q}_t k_{i,t} + \tilde{h}_{i,t}$. Entrepreneurs can borrow freely against the business and human wealth.

Entrepreneurs' problem. Entrepreneur i with age a_i chooses a vector of stochastic processes $(\tilde{c}_i, \tilde{\theta}^{ag}_i, \tilde{\theta}^{id}_i, k_i, l_i, \iota_i)$, taking prices $(\tilde{q}, \tilde{w}, r, p^{ag})$ as given, to solve the following problem:

$$\tilde{V}_t(\tilde{n}_i, a_i) = \max_{\tilde{c}_i, \tilde{\theta}^{ag}_i, \tilde{\theta}^{id}_i, k_i, l_i, \iota_i} \mathbb{E}_t\left[\int_0^{T-a_i} e^{-\rho z}\frac{\tilde{c}^{1-\gamma}_{i,t+z}}{1-\gamma}dz + e^{-\rho(T-a_i)}(1-\psi)^{\gamma}V^*\frac{\tilde{n}^{1-\gamma}_{i,t+T-a_i}}{1-\gamma}\right], \tag{9}$$

subject to (7), (8), non-negativity constraints $c_{i,t}, k_{i,t} \geq 0$, and the law of motion of $\tilde{n}_{i,t}$

$$d\tilde{n}_{i,t} = \left[(\tilde{n}_{i,t} - \tilde{q}_t k_{i,t})r_t + \tilde{q}_t k_{i,t}\mu^R_{i,t} - p^{ag}_t\tilde{\theta}^{ag}_{i,t} + \tilde{w}_t\bar{l}_{i,t} - \tilde{c}_{i,t}\right]dt + \left(\tilde{q}_t k_{i,t}\sigma_A - \tilde{\theta}^{ag}_{i,t}\right)dZ_t + \left(\tilde{q}_t k_{i,t}\sigma_{id} - \tilde{\theta}^{id}_{i,t}\right)dZ_{i,t},$$

given initial financial wealth $\tilde{n}_{i,t} = \tilde{n}_i > -\tilde{h}_{i,t}$.

The term in brackets in the expression above is the expected growth rate of financial wealth. The entrepreneur invests $\tilde{n}_{i,t} - \tilde{q}_t k_{i,t}$ in the riskless asset, with rate of return r_t, and she invests the amount $\tilde{q}_t k_{i,t}$ in the risky business technology, with expected rate of return $\mu^R_{i,t}$.

[22]Even though markets are (endogenously) incomplete, households agree on the valuation of variables not exposed to idiosyncratic risk. Therefore, there is no ambiguity in defining the relevant SDF for aggregate payoffs.

The cost of aggregate insurance is $p_t^{ag}\tilde{\theta}_{i,t}^{ag}$. The entrepreneur receives labor income $\tilde{w}_t\bar{l}_{i,t}$ and consumes $\tilde{c}_{i,t}$. The last two terms represent the exposure to aggregate and idiosyncratic risk, which equals the risk exposure from the business net of insurance.

Wage earners' problem. In contrast to entrepreneurs, wage earners do not have access to a production technology. To simplify exposition, we assume they have an infinite horizon. Allowing for finite lives and a bequest motive does not change our main results, as shown in Appendix E.3. Wage earners and entrepreneurs share a per-period isoelastic utility function with curvature parameter γ.[23] As often assumed in models with heterogeneous returns, as e.g. Kiyotaki and Moore (1997), wage earners and entrepreneurs have different discount rates.

The problem of wage earner $j \in \mathcal{W}_t$ is given by

$$\tilde{V}_t^w(\tilde{n}_j) = \max_{\tilde{c}_j, \tilde{\theta}_j^{ag}} \mathbb{E}_t \left[\int_t^\infty e^{-\rho_w(z-t)} \frac{\tilde{c}_{j,z}^{1-\gamma}}{1-\gamma} dz \right], \tag{10}$$

subject to non-negativity constraint $\tilde{c}_{j,t} \geq 0$, $\tilde{n}_{j,t} \geq \tilde{h}_{j,t}$, where $\tilde{h}_{j,t}$ denotes wage earner j's human wealth, and the law of motion of financial wealth $\tilde{n}_{j,t}$

$$d\tilde{n}_{j,t} = \left[\tilde{n}_{j,t}r_t - p_t^{ag}\tilde{\theta}_{j,t}^{ag} + \tilde{w}_t\bar{l}_{j,t} - \tilde{c}_{j,t}\right] dt - \tilde{\theta}_{j,t}^{ag} dZ_t,$$

given initial financial wealth $\tilde{n}_{j,t} = \tilde{n}_j > -\tilde{h}_{j,t}$. $\tilde{\theta}_{j,t}^{ag}$ can take positive or negative values, so wage earners can choose to either buy or provide aggregate insurance to entrepreneurs.[24]

Equilibrium. We provide below a definition of the competitive equilibrium.

Definition 1. *The competitive equilibrium is a set of aggregate stochastic processes: the aggregate capital stock k, the interest rate r, the wage rate $\tilde{w}$, the relative price of capital $\tilde{q}$, and the price of aggregate insurance p^{ag}; a set of stochastic processes for each entrepreneur $i \in \mathcal{E}_t$ and wage earner $j \in \mathcal{W}_t$: consumption $\tilde{c}_i$, financial wealth $\tilde{n}_i$, capital k_i, labor l_i, aggregate insurance $\tilde{\theta}_i^{ag}$, and idiosyncratic insurance $\tilde{\theta}_i^{id}$ for $i \in \mathcal{E}_t$; consumption $\tilde{c}_j$ and aggregate insurance $\tilde{\theta}_j^{ag}$ for $j \in \mathcal{W}_t$ such that:*

(a) *Aggregate capital stock satisfies the law of motion (4), given the initial capital stock k_0.*

(b) *$(\tilde{c}_i, \tilde{\theta}_i^{ag}, \tilde{\theta}_i^{id}, k_i, l_i, \iota_i)$ solve entrepreneurs' problem (9), given $(\tilde{q}, \tilde{w}, r, p^{ag})$.*

(c) *$(\tilde{c}_j, \tilde{\theta}_j^{ag})$ solve wage earners' problem (10), given $(\tilde{w}, r, p^{ag})$.*

[23] As shown by Swanson (2012), and consistent with Lemma 1 below, the coefficient of relative risk aversion in the presence of labor is affected by, but is not equal to, γ.

[24] Wage earners can also provide idiosyncratic insurance. In our notation, we have already imposed that they can diversify the exposure to idiosyncratic risk, so their financial wealth is only exposed to aggregate risk.

(d) *Markets clear for all $t \geq 0$:*

i. Market for goods

$$\int_{\mathcal{E}_t} \tilde{c}_{i,t} di + \int_{\mathcal{W}_t} \tilde{c}_{j,t} dj + \int_{\mathcal{E}_t} \iota_{i,t} A_t k_{i,t} di = \int_{\mathcal{E}_t} \tilde{y}_{i,t} di.$$

ii. Markets for capital and labor

$$\int_{\mathcal{E}_t} k_{i,t} di = k_t, \qquad \int_{\mathcal{E}_t} l_{i,t} di = \int_{\mathcal{E}_t} \bar{l}_{i,t} di + \int_{\mathcal{W}_t} \bar{l}_{j,t} dj.$$

iii. Markets for aggregate insurance and riskless bonds

$$\int_{\mathcal{E}_t} \tilde{\theta}^{ag}_{i,t} di + \int_{\mathcal{W}_t} \tilde{\theta}^{ag}_{j,t} dj = 0, \qquad \int_{\mathcal{E}_t} [\tilde{n}_{i,t} - \tilde{q}_t k_{i,t}] di + \int_{\mathcal{W}_t} \tilde{n}_{j,t} dj = 0.$$

3.2 Solution to entrepreneurs' problem

We describe next the solution to the entrepreneurs' problem. We focus on a stationary equilibrium, where scaled aggregate variables are constant, that is, $w_t = w$ and $q_t = q$. In Section 6, we consider a non-stationary environment where aggregate variables are time-varying.

Maximizing expected returns. As $(l_{i,t}, \iota_{i,t})$ enters the maximization problem in Equation (9) only through the expected return on the business, given in Equation (5), entrepreneurs choose these variables to maximize expected returns. Labor demand assumes the usual form:

$$w = (1-\alpha)\left(\frac{k_{i,t}}{l_{i,t}}\right)^{\alpha}. \tag{11}$$

The capital-labor ratio is equalized across entrepreneurs and coincides with the aggregate capital-labor ratio $K_t \equiv k_t/\bar{l}_t$, where k_t denotes the aggregate capital stock and $\bar{l}_t$ denotes the aggregate labor supply. In a stationary equilibrium, capital grows at the same rate as labor supply, which grows with the population at rate g.

The investment rate $\iota_{i,t}$ is given by

$$\Phi'(\iota_{i,t}) = \frac{1}{q} \Rightarrow \iota_{i,t} = (\Phi')^{-1}\left(\frac{1}{q}\right) \equiv \iota(q), \tag{12}$$

where $\iota(q)$ is increasing in q, given the concavity of $\Phi(\cdot)$. From Equation (5), an increase in the investment rate $\iota_{i,t}$ raises the expected capital gain by $\Phi'(\iota_{i,t})$, but it reduces the dividend yield by $1/q$. Expected returns are maximized when $\Phi'(\iota_{i,t}) = 1/q$, i.e., Equation (12) is satisfied.

Substituting Equations (11) and (12) into Equation (5), we obtain

$$\mu^R = \frac{\alpha K^{\alpha-1} - \iota(q)}{q} + \mu_A + \Phi(\iota(q)) - \delta, \tag{13}$$

where $\mu^R_{i,t} = \mu^R$ in a stationary equilibrium, so expected returns are equalized across entrepreneurs. Realized returns are, of course, still heterogeneous.

Human and total wealth. The lemma below shows that the relevant notion of wealth is *total wealth*, $\omega_{i,t} \equiv n_{i,t} + h_{i,t}$, the sum of financial wealth and human wealth. In particular, the entrepreneur's value function depends on total wealth $\omega_{i,t}$ and on age $a_{i,t} = t - s_i$.

Lemma 1. *Suppose the economy is in a stationary equilibrium. Then,*

(a) *Human wealth evolves according to*

$$\frac{\partial h(a)}{\partial a} = \left(r + p^{ag}\sigma_A - \mu_A\right) h(a) - w\bar{l}(a), \tag{14}$$

given $h(T) = 0$. Human wealth is then given by

$$h(a) = \int_0^{T-a} e^{-(r+p^{ag}\sigma_A-\mu_A)z} w\bar{l}(a+z)dz. \tag{15}$$

(b) *The (scaled) value function is given by*[25]

$$V(n,a) = \zeta(a)^{-\frac{1}{\gamma}} \frac{\left(n + h(a)\right)^{1-\gamma}}{1-\gamma}, \tag{16}$$

where $\zeta(a)$ equals the consumption ratio to total wealth. The entrepreneur's effective risk aversion is given by

$$-\frac{V_{nn}n}{V_n} = \frac{\gamma}{1 + \frac{h(a)}{n}}. \tag{17}$$

(c) *Capital stock and the aggregate and idiosyncratic insurance solve the mean-variance problem:*

$$\max_{k_i,\theta_i^{ag},\theta_i^{id}} \frac{qk_{i,t}}{n_{i,t}}(\mu^R - r) - \frac{p^{ag}\theta^{ag}_{i,t}}{n_{i,t}} - \frac{1}{2}\frac{\gamma}{1+\frac{h_{i,t}}{n_{i,t}}}\left[\left(\frac{qk_{i,t}+h_{i,t}}{n_{i,t}}\sigma_A - \frac{\theta^{ag}_{i,t}}{n_{i,t}}\right)^2 + \left(\frac{qk_{i,t}}{n_{i,t}}\sigma_{id} - \frac{\theta^{id}_{i,t}}{n_{i,t}}\right)^2\right], \tag{18}$$

subject to (7).

Proof. See Appendix A.1. □

[25]Scaled value function is related to the original value function by the condition $\tilde{V}_t(\tilde{n}, a) = A_t^{1-\gamma} V\left(\frac{\tilde{n}}{A_t}, a\right)$.

The first part of Lemma 1 shows that human wealth is the present discounted value of future labor income, where the discount rate incorporates the aggregate risk premium $p^{ag}\sigma_A$.[26] This extra discount is required as wages move with aggregate productivity, making labor income risky. Human wealth being a risky asset is consistent with Benzoni et al. (2007), which shows that human wealth becomes highly correlated with stocks when aggregate output and labor income are cointegrated, as in our model. As human wealth depends only on the entrepreneur's age, we drop the dependence on the household, i.e. $h_{i,t} = h(a_{i,t})$.

The second part of Lemma 1 gives the value function, an age-dependent CRRA function of total wealth. Importantly, the entrepreneur's effective risk aversion depends on $h_{i,t}/n_{i,t}$, the *human-financial wealth* ratio, which varies substantially over the life cycle in the data.

The final part of the lemma shows that entrepreneurs' portfolio choice reduces to a mean-variance problem, given the effective risk aversion $\gamma/(1+\frac{h_{i,t}}{n_{i,t}})$. This follows from the continuous-time formulation, as Equation (18) comes from rearranging the Hamilton-Jacobi-Bellman (HJB) equation of the entrepreneur's problem. Problem (18) is subject to the skin-in-the-game constraint (7) and the Lagrange multiplier to this constraint, which we call the *shadow price of idiosyncratic insurance*, plays an important role in the entrepreneurs' risk-taking decision.

Policy functions. We next consider entrepreneurs' policy functions.

Proposition 1. *Suppose the economy is in a stationary equilibrium. Then,*

i. Demand for capital is given by

$$\frac{qk_{i,t}}{n_{i,t}} = \frac{1+\frac{h_{i,t}}{n_{i,t}}}{\gamma}\frac{p^{id}}{\phi\sigma_{id}}, \tag{19}$$

where p^{id} is the shadow price of idiosyncratic insurance, given by

$$p^{id} = \frac{\mu^R - r - p^{ag}\sigma_A}{\phi\sigma_{id}}. \tag{20}$$

ii. The demand for aggregate insurance is given by

$$\frac{\theta^{ag}_{i,t}}{n_{i,t}} = \left(\frac{qk_{i,t}}{n_{i,t}} + \frac{h_{i,t}}{n_{i,t}}\right)\sigma_A - \frac{1+\frac{h_{i,t}}{n_{i,t}}}{\gamma}p^{ag}. \tag{21}$$

iii. The consumption-wealth ratio is given by

$$\frac{c_{i,t}}{n_{i,t}} = \frac{\bar{r}}{1-\psi e^{-\bar{r}(T-a_i)}}\left(1+\frac{h_{i,t}}{n_{i,t}}\right), \tag{22}$$

[26] We can write Equation (14) alternatively as $w\bar{l}(a) + \mu_A + \frac{1}{h(a)}\frac{\partial h(a)}{\partial a} - r = p^{ag}\sigma_A$, so the expected excess return on human wealth equals the aggregate risk premium $p^{ag}\sigma_A$.

where $\bar{r} \equiv \frac{1}{\gamma}\rho + \left(1 - \frac{1}{\gamma}\right) r^{MV}$ *and* $r^{MV} \equiv r + \frac{(p^{ag})^2 + (p^{id})^2}{2\gamma}$.

Proof. See Appendix A.1. □

The demand for capital has three components: entrepreneur's effective risk aversion, $\gamma/(1 + \frac{h_{i,t}}{n_{i,t}})$, the price of idiosyncratic insurance, p^{id}, and the quantity of non-diversified risk, $\phi\sigma_{id}$. The cross-sectional dispersion in risk-taking is captured by differences in effective risk aversion, which are driven by the human-financial wealth ratio. Entrepreneurial risk-taking then inherits the life-cycle patterns of the human-financial wealth ratio.

Equation (20) shows that the shadow price of idiosyncratic insurance, the Lagrange multiplier on the skin-in-the-game constraint, is equalized across entrepreneurs. Moreover, it equals the return per unit of risk (the Sharpe ratio) of an investor who fully insures the project against aggregate risk. In equilibrium, this Sharpe ratio is positive, so the skin-in-the-game constraint is always binding, that is, $\theta^{id}_{i,t} = (1 - \phi) q_t k_{i,t} \sigma_{id}$. Intuitively, entrepreneurs purchase as much idiosyncratic insurance as possible given it has zero cost.

Rearranging Equation (20), we can express expected excess returns as follows:

$$\mu^R - r = \underbrace{p^{ag}\sigma_A}_{\text{agg. risk premium}} + \underbrace{p^{id}\phi\sigma_{id}}_{\text{id. risk premium}}.$$

Hence, we obtain a two-factor model for expected returns, corresponding to the compensation for aggregate and idiosyncratic risk, consistent with the evidence in Section 2. In the absence of aggregate risk, growth, and adjustment costs, this expression simplifies to $\alpha K^{\alpha - 1} - r = p^{id}\phi\sigma_{id}$, so the MPK is not equal to the interest rate when $\phi > 0$, even though entrepreneurs face no borrowing constraints. Entrepreneurs do not expand the business, despite $\mu^R > r$, to limit their exposure to risk. In equilibrium, the wedge $\mu^R - r$ depends on entrepreneurs' risk appetite and their ability to diversify these risks, as captured by ϕ.

Interestingly, the demand for capital depends on the price and quantity of *idiosyncratic* risk. Access to aggregate insurance leads to a separation between the choice of the scale of the business and how much aggregate risk entrepreneurs are willing to hold. The choice of how much to invest in the business is essentially a decision of how much idiosyncratic risk to bear.

The demand for aggregate insurance, given in Equation (21), is decreasing in p^{ag} with a slope given by the inverse of the effective risk aversion. In equilibrium, we obtain $p^{ag} = \gamma\sigma_A$, so the demand for aggregate insurance simplifies to $\theta^{ag}_{i,t} = (qk_{i,t} - n_{i,t})\sigma_A$.[27] Poor entrepreneurs buy aggregate insurance, $\theta^{ag}_{i,t} > 0$, while rich entrepreneurs provide insurance, $\theta^{ag}_{i,t} < 0$. This arrangement can be implemented by having richer entrepreneurs send transfers to poor entrepreneurs as an indemnity after a negative aggregate shock, with transfers

[27] The result $p^{ag} = \gamma\sigma_A$ is standard in asset pricing and can be obtained by combining Equation (21) with the market clearing condition for aggregate insurance, as shown in Appendix C.3.

in the opposite direction after a positive aggregate shock. Note that if entrepreneurs on average have positive savings on the risk-free asset, such that $\int_{\mathcal{E}_t}(qk_{i,t} - n_{i,t})di < 0$, then they are net sellers of insurance. In this case, wage earners are borrowers in equilibrium. As they effectively hold a leveraged position on their risky human wealth, wage earners are disproportionately exposed to risk and demand aggregate insurance in equilibrium.

The consumption-wealth ratio is given in Equation (22). The first term $\bar{r}/(1-\psi e^{-\bar{r}(T-a)})$ represents the marginal propensity to consume (MPC).[28] It is increasing in age, as it is typical in finite-horizon problems, and the bequest motive parameter ψ controls the strength of this effect. If $\psi = 0$, the MPC is constant, recovering a standard result in infinite-horizon problems. If $\psi = 1$, then the MPC gets arbitrarily large as the entrepreneur approaches the end of life, so the stock of wealth is fully consumed at the final age T, as in Merton (1969). When $\gamma > 1$, an increase in risk-adjusted returns raises the average MPC. Finally, the consumption-wealth ratio depends on the human-financial wealth ratio, which varies over the life cycle.

3.3 Quantitative implications

We consider next the quantitative implications of the model for the life-cycle behavior of entrepreneurs. We start describing the calibration of the model and consider next the model's performance in accounting for the empirical life cycle patterns.

Technology, preferences, and demographics. We adopt the following calibration, which is summarized in Table 4. The capital share is set to $\alpha = 0.33$. The average growth rate of productivity is set to $\mu = 0.003$, following the evidence provided by Jeong and Townsend (2007) for Thailand. The investment function assumes the functional form $\Phi(\iota) = \sqrt{\overline{\Phi}^2 + 2\iota} - \overline{\Phi}$. This corresponds to the case of quadratic adjustment costs, as the investment rate required for capital to grow at rate g is $\iota = \overline{\Phi}(g+\delta) + 0.5(g+\delta)^2$. The coefficient of the investment function is chosen to match a long-run relative price of capital q of one. The depreciation rate is set to $\delta = 0.10$. The discount rate of wage earners is chosen to match a risk-free rate of $r = 3.5\%$, consistent with the average real rate for Thailand over the last two and half decades. The discount rate of entrepreneurs and the bequest motive parameters are chosen to match the consumption-wealth ratio at the beginning and end of life. The life horizon is set to $T = 55$, so it covers the life span from 25 to 80 years old, and the population growth is set to $g = 0.3\%$, the most recent value for population growth in Thailand. The parameter χ_e is chosen to match the average ratio of the value of the business relative to the entrepreneurs' financial wealth.

[28]The MPC is defined as the change in consumption in response to an increase in financial wealth, that is, the MPC is given by $\frac{\partial c_{i,t}}{\partial n_{i,t}} = \frac{\bar{r}}{1-\psi e^{-\bar{r}(T-a_i)}}$.

Table 4. Calibrated parameters

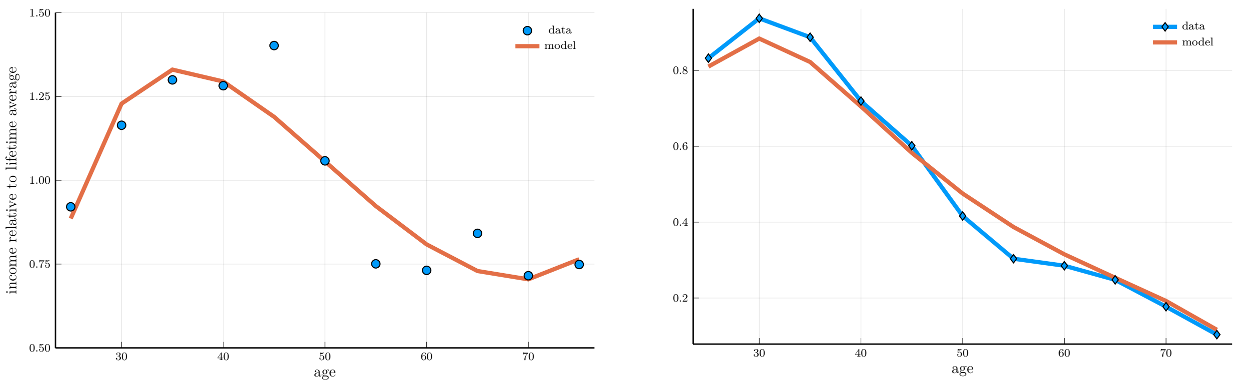
	Parameter	Choice
	Preferences	
ρ	Entrepreneur's rate of time preference	0.141
ρ_w	Wage earner's rate of time preference	0.127
ψ	Bequest motive	0.37
γ	Risk aversion	9.23
	Technology & financial friction	
μ_A	Average productivity growth rate	0.003
σ_A	Aggregate volatility	0.051
σ_{id}	Idiosyncratic volatility	0.203
ϕ	Moral hazard parameter	0.571
$\overline{\Phi}$	Adjustment cost parameter	0.90
α	Capital share in production function	0.33
δ	Depreciation rate	0.10
	Demographics	
g	Population growth	0.003
χ_e	Share of entrepreneurs in the population	0.46
T	Life span (adult life)	55

Risk, return, and the moral hazard parameter. We choose the risk aversion coefficient, the aggregate and idiosyncratic volatility, and the moral hazard parameter to match the decomposition of risk and return provided in Table 2. The volatility parameters, σ_A and σ_{id}, are chosen to match the aggregate and idiosyncratic components of total volatility. The aggregate risk premium is given by $p^{ag}\sigma_A = \gamma\sigma_A^2$, so we choose γ to match the aggregate risk premium. The idiosyncratic risk premium is informative about ϕ. If $\phi = 0$, we obtain the complete-market solution, so the idiosyncratic risk premium equals zero. As we raise ϕ, the importance of the idiosyncratic risk premium increases. Using expression (19) to solve for p^{id}, we can write the idiosyncratic risk premium in terms of the moral hazard parameter ϕ and observable quantities, namely the level of idiosyncratic volatility and the exposure of entrepreneurs' total wealth to the private business.[29] Given these quantities, we can identify ϕ.

Measuring human wealth. It remains to specify the labor supply parameters. We follow Gârleanu and Panageas (2015) and assume that $\bar{l}_{i,t}$ is a function of the entrepreneur's age: $\bar{l}_{i,t} = \sum_{l=1}^{L}\Gamma_l e^{\varphi_l a_i}$, where we normalize the average labor supply of entrepreneurs to one, that is, $\frac{\int_{\mathcal{E}_t}\bar{l}_{i,t}di}{\int_{\mathcal{E}_t}di} = 1$. This functional form is flexible enough to capture the empirical labor income dynamics while being analytically tractable. We estimate the parameters $(\Gamma_l, \varphi_l)_{l=1}^{L}$ by non-

[29]See Section 5 for a discussion of the determination of p^{id} in equilibrium.

Figure 3: Life Cycle Profiles: Labor Income and Human-Financial Ratio



(a) Labor Income

(b) Human-financial wealth ratio

linear least squares such that the distribution of labor income across age groups matches the one observed in the data. We set the number of exponential terms to $L = 3$. The wage earner's labor supply is constant and given by $\bar{l}_{j,t} = 1$. The left panel of Figure 3 shows that the functional form does a good job of approximating the empirical labor income profile.

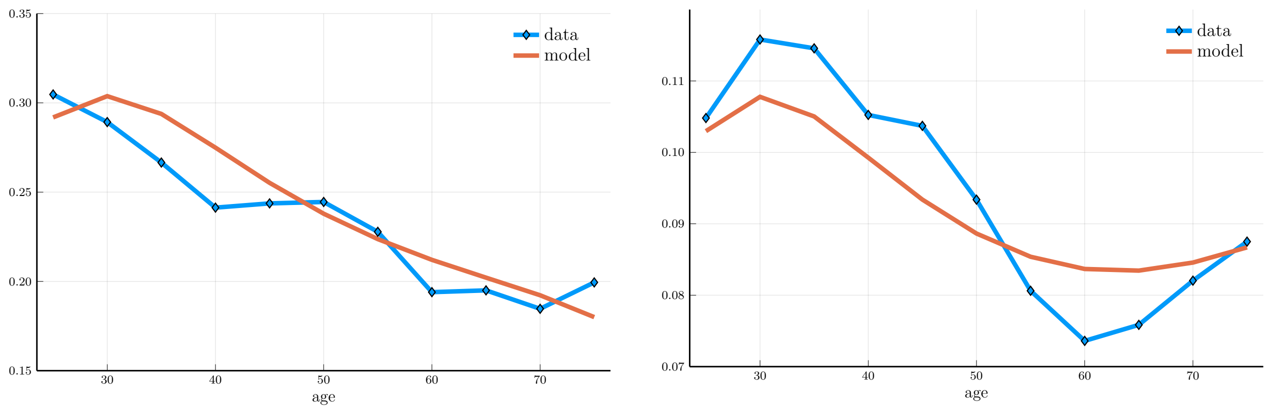
From Equation (15), human wealth can be computed as follows:

$$h_{i,t} = \int_0^{T-a_i} e^{-(r+p^{ag}\sigma_A-\mu_A)z} w \sum_{l=1}^{L} \Gamma_l e^{\varphi_l(z+a_i)} dz.$$

Given the discount rate and the labor income profile, we can compute the human-financial wealth $h_{i,t}/n_{i,t}$, both in the data and in the model. Appendix B discusses the construction of the empirical life-cycle profiles in detail. The right panel of Figure 3 shows that the human-financial wealth ratio declines over the life cycle. This is the result of labor income being relatively high at the beginning of the life cycle and that households have fewer years of future income as time goes by. Quantitatively, human wealth is nearly as important as financial wealth at the beginning of life. By the age of 50, human wealth is less than half the financial wealth. Even though this was not a calibration target, the model closely matches the evolution of the human-financial wealth ratio over the life cycle.

Implications for risk-taking and savings. We consider next the evolution of risk-taking and consumption over the life cycle. The left panel of Figure 4 shows that the share of wealth invested in the business declines with age, consistent with the evidence in Table 3. In a stationary environment, this pattern cannot be explained by differences in expected returns, which are assumed to be constant. The model generates this pattern by having the effective risk aversion decreasing in the human-financial wealth ratio, consistent with the evidence the life-cycle patterns were not driven by differences in returns or risk exposure. Given the pres-

Figure 4: Life Cycle Profiles: Risk-taking and Consumption-Wealth Ratio



(a) Share of wealth invested in the business

(b) Consumption-wealth ratio

ence of human wealth, young entrepreneurs are endogenously less risk-averse than older entrepreneurs, so they invest a larger fraction of their wealth in the business. Notice that the ratio of the risk-taking measure at the beginning and at the end of life is entirely determined by the human-financial wealth ratio, which was calibrated independently of any information on the cross-section of entrepreneurs' risk-taking.

The consumption-wealth ratio is roughly U-shaped as a function of age. This non-monotonic behavior is the result of two forces. First, the human-financial wealth ratio declines with age, which induces households to reduce consumption. Second, the MPC increases with age, which induces households to consume more. The first effect dominates at the beginning of the life cycle, and the second effect dominates as the entrepreneur gets older.

3.4 The role of borrowing constraints and occupational choice

We consider next three extensions of our baseline model capturing the role of different forms of borrowing constraints and endogenous occupational choice.

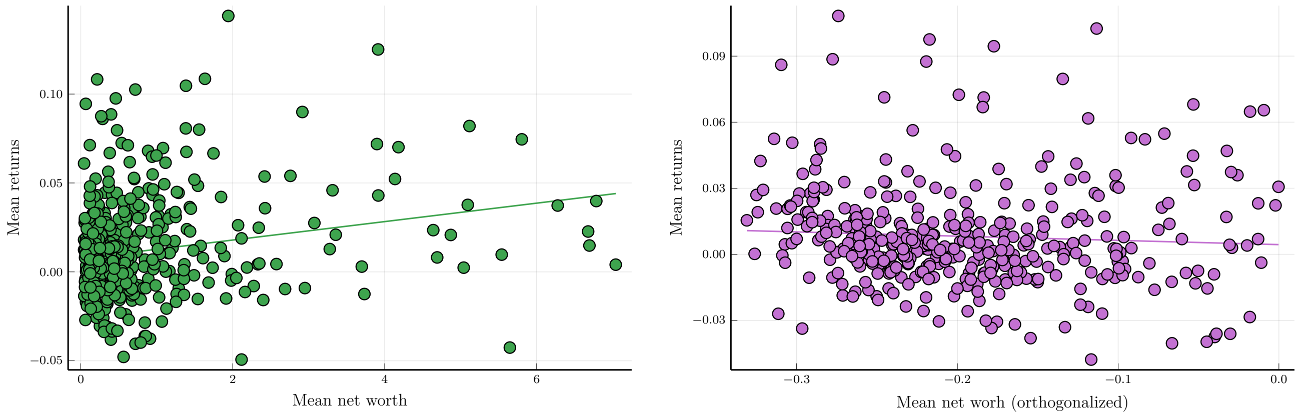
Collateral constraints. In our baseline model, risk is the only factor preventing entrepreneurs from scaling up their operations. In Appendix E.1, we consider the effects of limited pledgeability of physical capital by introducing collateral constraints. We also introduce decreasing returns to scale in production and heterogeneous idiosyncratic volatility. This allows us to study how differences in both risk exposure and net worth affect expected returns in the presence of collateral constraints. The Lagrange multiplier on the insurance constraint varies across entrepreneurs, and it depends on the degree of pledgeability of physical capital. Leverage constraints do not change our predictions for risk-taking over the life cycle. The demand for capital (19) holds exactly when we aggregate across entrepreneurs of a given age. Allowing for collateral constraints and heterogeneity changes the interpretation of p^{id}, which corresponds now to the average price of idiosyncratic risk, but does not affect the life-cycle patterns.

The main difference relative to the baseline model is that expected returns depend on entrepreneurs' financial wealth. Among constrained entrepreneurs, the business scale is determined by their net worth, so expected returns are independent of the risk exposure. Differences in expected returns are then entirely driven by differences in net worth. Therefore, if leverage constraints are a major factor limiting the activity of entrepreneurs, we should observe a strong negative association between financial wealth and expected returns.

We test this prediction in our data. The left panel of Figure 5 shows the (cross-sectional) scatter plot of average returns and average net worth. We find a non-significant positive relationship between returns and net worth. The theory predicts the association between wealth and expected returns should hold only for entrepreneurs with relatively low net worth, after controlling for differences in risk exposure, so a sharper test would focus only on poorer entrepreneurs. The right panel shows the relationship between average returns and the component of net worth that is orthogonal to aggregate beta and idiosyncratic volatility for entrepreneurs with below-average (orthogonalized) net worth. We observe a weak and non-significant association between net worth and expected returns. In both cases, net worth explains only a negligible fraction of the variation in the cross-section of entrepreneurial returns, with an adjusted R^2 of less than half a percentage point. Moreover, the model with leverage constraints predicts that there should be no association between expected returns and risk exposure for entrepreneurs with sufficiently low net worth. We test this prediction by running the same cross-sectional regression of average returns on the exposure to aggregate and idiosyncratic risk but restricted to a sample of entrepreneurs with below-average net worth. We find that the association between risk exposure and returns is actually *stronger* for this subsample, where the R^2 of the cross-sectional regression is 0.82, compared to $R^2 = 0.68$ for the whole sample. These results suggest that collateral constraints are not the main driver of expected entrepreneurial returns in our sample.

Uninsurable labor income risk and borrowing constraints. We consider next the implications of limited pledgeability of human wealth. In Appendix E.2, we introduce uninsurable labor income risk, where labor (or non-business) income is subject to a Poisson disability shock that permanently reduces labor income by a fraction $\xi_d \in [0, 1]$. The magnitude of the shock limits how much entrepreneurs can borrow against future income, where entrepreneurs cannot borrow if $\xi_d = 1$, and we recover the baseline case of full pledgeability when $\xi_d = 0$. Even though a closed-form solution is not available with both insurance and borrowing con-

Figure 5: Average returns vs. net worth



Note: The left panel shows a scatter plot of the average net worth against the average return for each entrepreneur. The level of net worth is normalized by its cross-sectional mean. The right panel shows a scatter plot of the residuals of a cross-sectional regression of average net worth on aggregate beta and idiosyncratic variance against average time-series returns for a sample of entrepreneurs with below-average orthogonalized net worth. To limit the influence of outliers, we trim 1% of the observations in the left and right tails.

straints, we provide an analytical characterization using perturbation techniques.[30] As it is well known in this class of models, the consumption function is concave on wealth: $c_{i,t} \propto \omega_{i,t}^{\psi_c}$, where $\psi_c \in (0,1)$ when $\xi_d > 0$. The parameter ψ_c controls the concavity of the consumption function. Our main finding is that the inability to borrow against future income has two opposing effects on the demand for capital, as shown in the following expression:

$$\frac{qk_{i,t}}{n_{i,t}} = \frac{1 + (1-\xi_d)\frac{h_{i,t}}{n_{i,t}}}{\gamma\psi_c} \frac{p^{id}}{\phi\sigma_{id}}. \tag{23}$$

When $\xi_d = 0$, the consumption function is linear, $\psi_c = 1$, and we recover Equation (19). When $\xi_d > 0$, entrepreneurs have fewer resources available to invest due to limited credit, which tends to reduce the scale of the business. This effect is intuitive and particularly important for low net-worth entrepreneurs. However, limited pledgeability of human wealth implies that entrepreneurs are effectively less risk averse, which tends to *increase* the scale of the business. As entrepreneurs cannot borrow against a fraction of human wealth, future income acts as a buffer, making entrepreneurs more willing to take risks. This second effect is particularly relevant for richer entrepreneurs. In combination, these two effects imply that borrowing constraints increase the dispersion of capital holdings, but it has a more muted effect on average capital holdings. As in the baseline model, entrepreneurs reduce their exposure to the business as they get older due to a declining human-financial wealth ratio.

In contrast to the version with collateral constraints, the model predicts a positive asso-

[30] Our method extends the one used by Viceira (2001), and it is similar to risky steady-state approximations (see e.g. Coeurdacier et al. 2011). Despite having aggregate risk and labor income risk, we do not approximate the wealth distribution with finite moments as in Krusell and Smith (1998). See Appendix E.2 for details.

ciation between risk exposure and expected returns, as in the baseline setting. Moreover, this relationship should be stronger for entrepreneurs with low net worth, as they are more risk averse. We test this prediction in the data. As shown in the appendix, average returns are more sensitive to changes in idiosyncratic risk for entrepreneurs with below-average net worth. Overall, the model with borrowing constraints generates predictions similar to the baseline model regarding the life cycle and the risk-return trade-off. It generates a new layer of predictions regarding the different behaviors of entrepreneurs with similar demographics and risk exposure but different net worth. Therefore, uninsurable labor income does not substantially change the model's life-cycle implications.

Endogenous occupational choice. In our baseline model, a household does not choose to be an entrepreneur or a wage earner, as the fraction of entrepreneurial households is set exogenously. In Appendix E.3, we introduce an endogenous occupational choice. We also assume that wage earners have finite lives and imperfect altruism, like entrepreneurs. To become an entrepreneur, the household must pay a fixed cost at the beginning of life. Households draw a cost parameter from a given distribution. The threshold to become an entrepreneur depends on financial wealth, such that a household that receives a larger bequest is more likely to become an entrepreneur, and the shadow price of idiosyncratic risk. We show in the appendix that our results carry through essentially unchanged to this setting, where the fraction of entrepreneurs in the economy is endogenous.

4 Distributive implications of entrepreneurial risk

In this section, we consider how entrepreneurial risk affects wealth inequality between and within age groups. The main object of interest in this section is the joint distribution of (scaled) financial wealth and age, which we denote by $f_t(n,a)$. Let $f(a)$ be the age distribution in the population. As the population grows at rate g and entrepreneurs live for T periods, $f(a)$ follows an exponential distribution truncated at age T. Given the joint distribution, we obtain the average wealth conditional on age, $n_t(a)$, and the average wealth of all entrepreneurs, $n_{e,t}$:

$$n_t(a) = \int_{-h_t(a)}^{\infty} n f_t(n|a) dn, \qquad n_{e,t} = \int_0^T n_t(a) f(a) da,$$

where $f_t(n|a) = f_t(n,a)/f(a)$. We focus on the stationary distribution, such that $f_t(n,a) = f(n,a)$ for all t, which allow us to drop time subscripts, $n_t(a) = n(a)$ and $n_{e,t} = n_e$.

4.1 Between-group inequality

The next proposition provides a characterization of between-group inequality.

Proposition 2. *Suppose the economy is in a stationary equilibrium. Then,*
i) Between-group inequality: *The share of wealth held by entrepreneurs of age a, $\frac{f(a)n(a)}{n_e}$, satisfies*

$$\log\frac{f(a)n(a)}{n_e} = \log\frac{f(0)n(0)}{n_e} + \underbrace{\log\left(\frac{1+\frac{h(0)}{n(0)}}{1+\frac{h(a)}{n(a)}}\right)}_{\substack{\text{human-to-financial}\\ \text{wealth effect}}} + \underbrace{\left[r + \frac{(p^{ag})^2}{\gamma} + \frac{(p^{id})^2}{\gamma} - (g+\mu_A)\right]a}_{\text{generalized "r-g" effect}} - \underbrace{\int_0^a \frac{\bar{r}}{1-\psi e^{-\bar{r}(T-a')}}da'}_{\text{average MPC effect}}. \tag{24}$$

where

$$n(0) = \frac{e^{\left(r+\frac{(p^{ag})^2}{\gamma}+\frac{(p^{id})^2}{\gamma}-(g+\mu_A)-mpc_e\right)T}}{1-e^{\left(r+\frac{(p^{ag})^2}{\gamma}+\frac{(p^{id})^2}{\gamma}-(g+\mu_A)-mpc_e\right)T}}h(0), \tag{25}$$

and $mpc_e = \frac{1}{T}\int_0^T \frac{\bar{r}}{1-\psi e^{-\bar{r}(T-a)}}da$ is the average MPC across the life cycle.
ii) Average financial wealth: *The average financial wealth of entrepreneurs is given by*

$$n_e = f(0)(n(0)+h(0))\int_0^T e^{\left(r+\frac{(p^{ag})^2}{\gamma}+\frac{(p^{id})^2}{\gamma}-(g+\mu_A)\right)a}\frac{e^{-\bar{r}a}-\psi e^{-\bar{r}T}}{1-\psi e^{-\bar{r}T}}da - h_e, \tag{26}$$

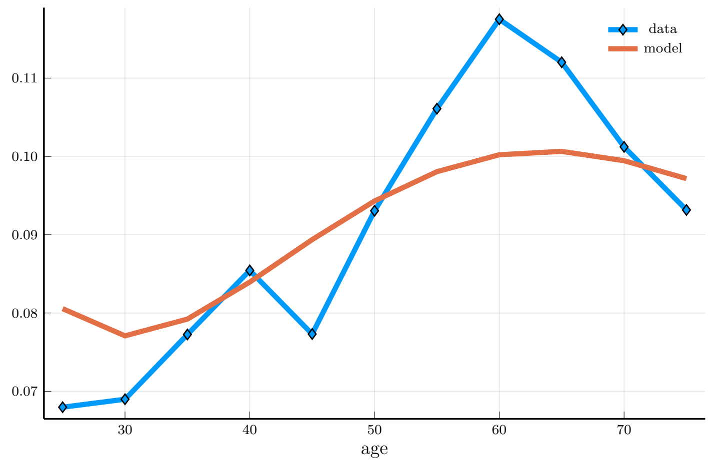
where $h_e = \int_0^T f(a)h(a)da$.

Proof. See Appendix A.2. □

The first part of Proposition 2 decomposes the distribution of wealth across age groups into three effects. First, a human-to-financial wealth effect. For older entrepreneurs, human wealth has been mostly converted into financial wealth, that is, labor income accelerates the accumulation of financial wealth. The second term is a generalized "$r-g$" effect.[31] The first component is the return on total wealth: $r + (p^{ag})^2/\gamma + (p^{id})^2/\gamma$. The correct notion of return in this context takes into account the aggregate and idiosyncratic risk premium. The second component is the growth rate of the economy, $g+\mu_A$, the sum of population and productivity growth. The generalized "$r-g$" effect implies that the wealth share tends to increase with age if the return on total wealth exceeds the growth rate of the economy. The third term is the average MPC effect. It captures the fact that wealth accumulated at age a depends on the entrepreneur's past consumption decisions. Entrepreneurs accumulate wealth when young, but eventually consumption increases until wealth achieves a desired bequest level.

[31]The importance of $r-g$ in determining wealth inequality has been emphasized by Piketty (2014). See Benhabib et al. (2011) for the implications to the tail of the distribution and Jones (2015) for a literature review.

Figure 6: Financial wealth distribution across age groups



The wealth of newborn agents, that is the bequest they receive, can be written as

$$n(0) = e^{\left(r+\frac{(p^{ag})^2}{\gamma}+\frac{(p^{id})^2}{\gamma}-(g+\mu_A)-mpc_e\right)T}(n(0)+h(0)).$$

The exponential term captures the net accumulation rate of entrepreneurs' wealth over their lifetime. The amount of bequests is increasing in returns and decreasing in the growth rate of the economy and the average MPC.

The second part of Proposition 2 characterizes the average financial wealth of entrepreneurs. For a given level of the capital stock, this captures the wealth distribution between the two types of households, as $\frac{\chi_e n_e}{qK}$ equals the share of financial wealth held by entrepreneurs and $1-\frac{\chi_e n_e}{qK}$ the share of financial wealth held by wage earners. As n_e is the average of the financial wealth conditional on age $n(a)$, the same effects that shape the distribution of wealth across age groups pin down the overall level of wealth held by entrepreneurs.

Figure 6 shows how financial wealth varies across age groups. The model captures the inverted U-pattern of financial wealth. At the beginning of the life cycle, both the human-financial wealth effect and the "$r-g$" effect dominate the average MPC effect, so the wealth share initially increases with age. The average MPC effect dominates later in the life cycle, bringing down the wealth share.

4.2 Within-group inequality

We turn next to the characterization of the wealth distribution conditional on age. Let $\mu_{n,t}(n,a)$ denote the expected change and $\sigma_{n,t}(n,a)$ the instantaneous volatility of financial wealth for an entrepreneur with financial wealth n and age a. The evolution of the distribution of wealth conditional on age is given by the Kolmogorov Forward Equation, as shown in Lemma 2.

Lemma 2 (Kolmogorov Forward Equation)**.** *The conditional distribution of financial wealth $f_t(n|a)$ satisfies the partial differential equation*

$$\frac{\partial f_t(n|a)}{\partial t} + \frac{\partial f_t(n|a)}{\partial a} = -\frac{\partial [f_t(n|a)\mu_{n,t}(n,a)]}{\partial n} + \frac{1}{2}\frac{\partial^2 \left[f_t(n|a)\sigma_{n,t}^2(n,a)\right]}{\partial n^2}, \tag{27}$$

and the boundary condition $f_t(e^{-gT}n|0) = f_t(n|T)$, given an initial condition $f_0(n|a)$.

Proof. See Appendix A.3. □

Despite the complexity created by the dependency on age, it is possible to solve for the conditional distribution of financial wealth in closed form for the special case where entrepreneurs leave no bequests, that is, $\psi = 1$.

Proposition 3 (Within-group inequality: no bequests)**.** *Suppose $\psi = 1$, $r + \frac{(p^{ag})^2}{\gamma} + \frac{(p^{id})^2}{\gamma} > \mu_A$.*
i) Shifted log-normal distribution. *Conditional on age, financial wealth follows a shifted log-normal distribution with support $(-h(a), \infty)$, i.e., $n + h(a)$ follows a log-normal distribution.*
ii) Mean and variance by age. *The expected value and variance of n conditional on age are given by*

$$\mathbb{E}[n|a] = h(0)e^{\left(r+\frac{(p^{ag})^2}{\gamma}+\frac{(p^{id})^2}{\gamma}-\mu_A\right)a}\frac{e^{-\bar{r}a} - e^{-\bar{r}T}}{1-e^{-\bar{r}T}} - h(a) \tag{28}$$

$$\mathbb{V}[n|a] = \left[e^{\left(\frac{p^{id}}{\gamma}\right)^2 a} - 1\right]\left[h(0)e^{\left(r+\frac{(p^{ag})^2}{\gamma}+\frac{(p^{id})^2}{\gamma}-\mu_A\right)a}\frac{e^{-\bar{r}a} - e^{-\bar{r}T}}{1-e^{-\bar{r}T}}\right]^2. \tag{29}$$

iii) Inverted-U shape of inequality over the life cycle. *There exists $0 < \hat{a} < T$ such that $\mathbb{V}[n|a]$ is increasing in a for $a < \hat{a}$ and decreasing for $a > \hat{a}$.*

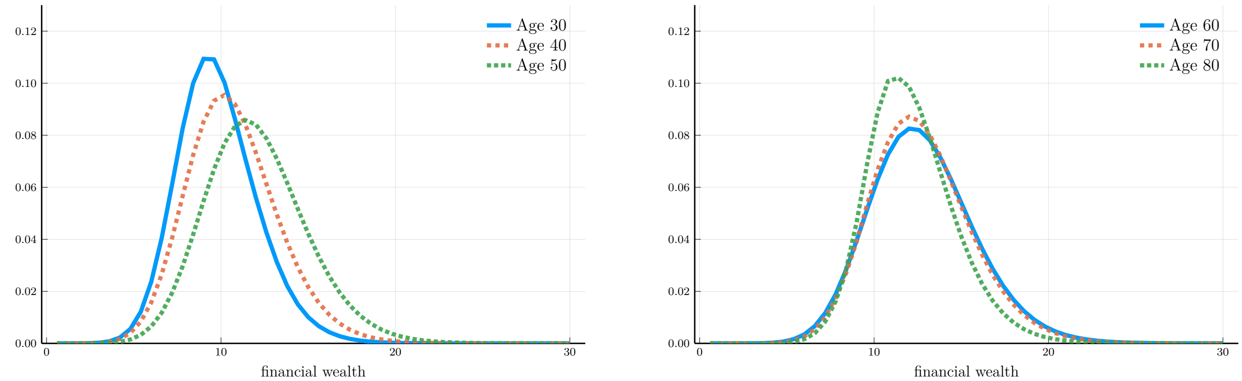
Proof. See Appendix A.4. □

Proposition 3 gives a complete characterization of the distribution of wealth conditional on age. Wealth has a *shifted log-normal* distribution, with an age-dependent shifter $-h(a)$. Since entrepreneurs can borrow, financial wealth clearly cannot be log-normally distributed, as n can take on negative values. However, financial wealth cannot go below the natural borrowing limit $-h(a)$, so total wealth assumes only positive values. Total wealth follows a log-normal distribution with mean and variance dependent on age.

The expression for the conditional mean $\mathbb{E}[n|a] = n(a)$ is essentially the same as the one in Equation (24), after rearranging and specializing to the case $\psi = 1$. As we have seen, average wealth increases with age at the beginning of life, and it goes down by the end of the life cycle.

The expression (29) shows how the variance of wealth evolves over the life cycle. Without bequests, the variance is zero at ages $a = 0$ and $a = T$. The wealth dispersion increases at the

Figure 7: Stationary distribution of financial wealth conditional on age



(a) Ages 30, 40, and 50

(b) Ages 60, 70, and 80

beginning of life, as some entrepreneurs receive a series of positive shocks while others suffer negative shocks. This force increases with the exposure to idiosyncratic risk, as measured by p^{id}/γ. The increasing MPC provides a countervailing force, as the level of wealth is brought down at the end of the life cycle.

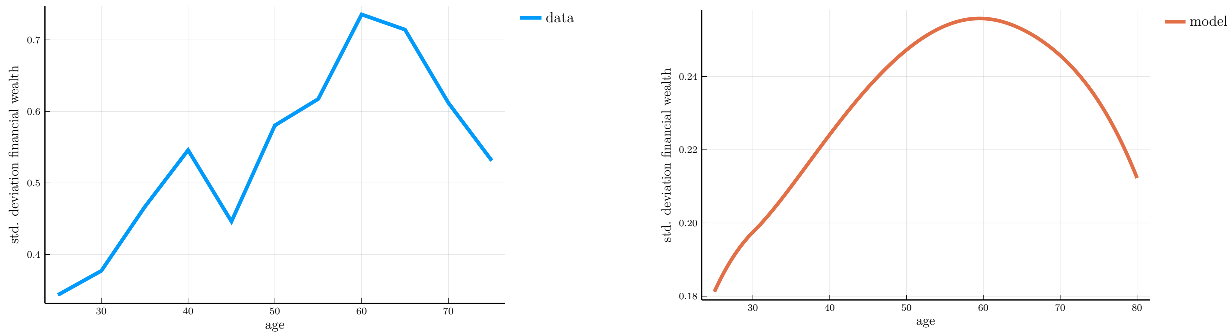
The results in Proposition 3 assume no bequests. In the general case, we must use numerical methods, as discussed in Appendix C.2. Figure 7 shows the stationary distribution of financial wealth for selected ages. The mean and the dispersion of the distribution initially increase with age, then eventually both start to decline as entrepreneurs get older.

The same inverted U-pattern on wealth inequality can be found in the data. Figure 8 shows the evolution of within-group inequality over the life cycle in our data. To make the units easier to interpret, we divide financial wealth by the average wealth for all entrepreneurs. The figure shows that the standard deviation of n/n_e increases sharply until roughly age 60 and then declines until the end of the life cycle. Quantitatively, the model generates a substantial increase in inequality over the life cycle, over 40% from early in life until the peak, even though idiosyncratic shocks to the business are the only source of heterogeneity across entrepreneurs. The increase in inequality in the data is even more pronounced. Introducing differences in preferences or labor income across households could potentially bring the level of within-group inequality closer to the one in the data.

Wealth inequality and risk sharing. As shown in Appendix A.4, the within-group variance of log total wealth, and ultimately log consumption, increases with age:

$$\mathbb{V}[\log \omega_{i,t}|a] = \mathbb{V}[\log c_{i,t}|a] = \left(\frac{p^{id}}{\gamma}\right)^2 a.$$

Figure 8: Standard deviation of financial wealth within age groups



(a) Within-group inequality: data

(b) Within-group inequality: model

Note: Panel (a) shows the standard deviation of financial wealth by age in the data. Panel (b) shows the standard deviation of financial wealth by age in the model economy. In both cases, we normalize the standard deviation by entrepreneurs' average wealth.

The steepness of the variance life cycle profile depends on p^{id}, so it is a function of the degree of risk sharing in the economy. This is in line with the literature using the dispersion in consumption over the life cycle to infer how imperfect is risk sharing (see e.g. Storesletten et al. 2004). In our setting with entrepreneurial risk, the shadow price of idiosyncratic insurance controls the level of consumption inequality in the economy.

5 Aggregate implications of entrepreneurial risk

In this section, we study the impact of entrepreneurial risk on the long-run level of the aggregate capital stock. We find that financial development, captured by the magnitude of the moral hazard parameter ϕ, is tightly linked to the level of economic development.

5.1 Equilibrium characterization

We present next the determination of aggregate variables in a stationary equilibrium. The detailed derivations are provided in Appendix C.3.

Aggregate risk premium, interest rate, and the price of capital. Equating supply and demand for aggregate insurance, we obtain the price of aggregate insurance,

$$p^{ag} = \gamma \sigma_A, \tag{30}$$

The interest rate is given by the standard condition

$$r = \rho_w + \gamma \mu_A - \frac{\gamma(\gamma+1)}{2} \sigma_A^2, \tag{31}$$

From Equation (12), we obtain $q = \overline{\Phi} + g + \delta$, where we choose $\overline{\Phi}$ such that $q = 1$. Note that p^{ag}, r, and q are independent of the moral hazard parameter ϕ, so they coincide with their value at the complete-market economy.

Aggregate capital stock and idiosyncratic risk premium. From Equation (20), we obtain an expression for the expected return on the business

$$r + p^{ag}\sigma_A + p^{id}\phi\sigma_{id} = \frac{\alpha K^{\alpha-1} - \iota(q)}{q} + g + \mu_A. \tag{32}$$

The left-hand side captures the required rate of return to invest in the business. The right-hand side gives the actual expected return of investing in the business, a function of the marginal product of capital (MPK) net of adjustment costs. Equation (32) generalizes the standard textbook relation between MPK and interest rates to an environment with growth, risk, and adjustment costs, as $r = \alpha K^{\alpha-1} - \delta$ in the absence of these three elements.

The expression (32) gives an inverse relation between the idiosyncratic risk premium $p^{id}\phi\sigma_{id}$ and the capital-labor ratio K. This downward-sloping relationship is represented by the solid blue line in the left panel of Figure 9, which we refer to as the MPK schedule.

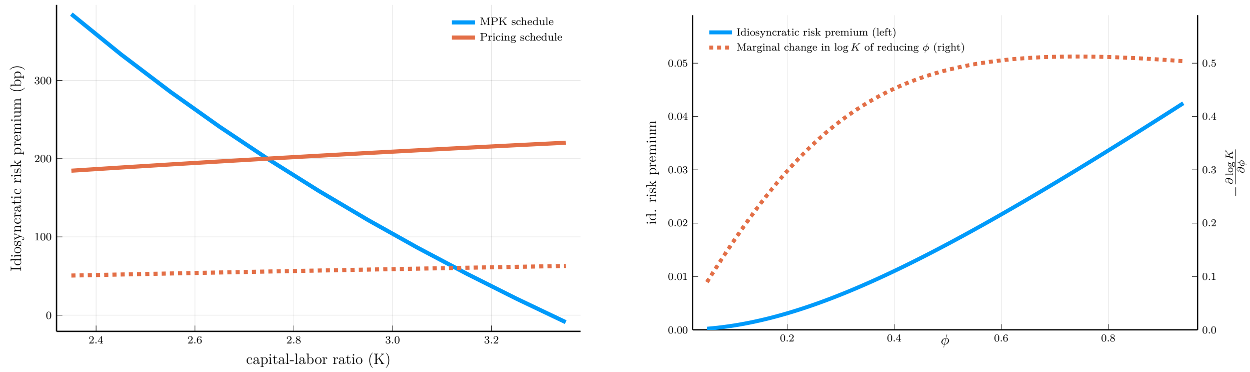
We need another condition relating K and p^{id}. Aggregating the demand for capital (19) across all entrepreneurs, we obtain

$$p^{id} = \underbrace{\gamma}_{\text{risk aversion}} \underbrace{\phi\sigma_{id}}_{\text{effective risk}} \underbrace{\frac{qK}{\chi_e(n_e + h_e)}}_{\text{id. risk exposure}}, \tag{33}$$

where n_e and h_e denote the average financial and human wealth of entrepreneurs.

The price of idiosyncratic risk depends on the product of the risk aversion γ and the idiosyncratic risk (net of insurance) $\phi\sigma_{id}$, analogous to the price of aggregate risk. However, p^{id} depends on an additional term: the *idiosyncratic risk exposure*, that is, the ratio of physical assets to total wealth of entrepreneurs. This term reflects the fact that entrepreneurs require a larger idiosyncratic risk premium when they have a larger fraction of their wealth invested in the business. Using Equation (26) to eliminate $n_e + h_e$, we obtain an implicit relationship between p^{id} and K. The left panel of Figure 9 plots this relationship as the solid upward-sloping curve, which we refer to as the *pricing schedule*. The idiosyncratic risk premium and the capital stock in the economy are determined by the intersection of the MPK and pricing schedules.

Figure 9: Idiosyncratic Risk Premium and Capital Stock



(a) MPK schedule vs. Pricing schedule

(b) Idiosyncratic risk premium and $-\frac{\partial \log K}{\partial \phi}$

Note: In the left panel, the solid (dashed) upward-sloping curve shows the pricing schedule in the initial (new) stationary equilibrium for $\phi_0 = 0.571$ ($\phi_1 = 0.285$). The right panel shows the equilibrium idiosyncratic risk premium and the marginal increase in the capital stock of reducing the moral hazard parameter for different values of ϕ.

5.2 The price of aggregate and idiosyncratic risk in the data

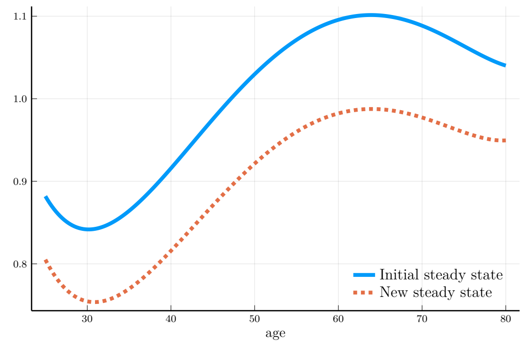
The model replicates the level of risk premium and volatility on aggregate and idiosyncratic risk observed in the data, as given in Table 2. A striking fact is that, despite idiosyncratic volatility being four times larger than aggregate volatility, the idiosyncratic risk premium is slightly smaller than the aggregate risk premium. This leads to a Sharpe ratio more than four times larger for aggregate risk. Equations (30) and (33) help to shed light on this pattern.

The Sharpe ratio of aggregate risk corresponds to $p^{ag} = \gamma\sigma_A$. The Sharpe ratio of idiosyncratic risk is given by $p^{id}\phi$, as the volatility computed in the data does not consider any insurance available to entrepreneurs. If we were to naively price the idiosyncratic risk by analogy with the aggregate risk, the Sharpe ratio would be $\gamma\sigma_{id}$, that is, more than four times larger than the one for aggregate risk. Two factors explain why the price of idiosyncratic risk is actually more than four times *smaller* than the one for aggregate risk: idiosyncratic insurance and risk exposure. The pricing equation shows the role of these two components:

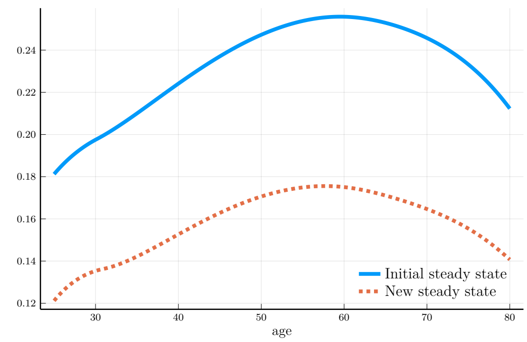
$$p^{id}\phi = \gamma\sigma_{id}\phi^2 \frac{qK/\chi_e}{n_e + h_e}.$$

The moral hazard parameter is $\phi = 0.57$, which reduces the price of idiosyncratic risk by $1 - \phi^2 = 67\%$. The rest of the adjustment comes from the risk exposure factor: $qK/\chi_e(n_e + h_e) \approx 0.16$. Intuitively, the reason for a much smaller price of idiosyncratic risk is that entrepreneurs are proportionally less exposed to idiosyncratic risk due to insurance mechanisms or because only a fraction of their wealth is exposed to this risk. Without human wealth and heterogeneous agents, one would incorrectly attribute the low Sharpe ratio to a high degree of insurance. The economy would appear to have better insurance than it actually has.

Figure 10: Financial development and inequality in the long-run



(a) Between-group inequality



(b) Within-group inequality

Note: Inequality in a stationary equilibrium for $\phi_0 = 0.571$ (initial steady state) and $\phi_1 = 0.286$ (new steady state). Panel (a) shows average financial wealth by age. Panel (b) shows the standard deviation of financial wealth by age. All variables are normalized by entrepreneurs' average wealth in the initial steady state.

5.3 Long-run effects of relaxing insurance constraints

We consider next the aggregate implications of relaxing insurance constraints. High values of ϕ capture situations where access to insurance arrangements, formal or informal, is rather limited. As institutional arrangements improve, e.g. mechanisms to monitor entrepreneurs' activities, such frictions are expected to be reduced and entrepreneurs would bear less risk.[32] Financial development leads to a reduction in the moral hazard parameter ϕ.

The left panel of Figure 9 shows the impact of an intervention that reduces ϕ. The MPK schedule is unchanged, but the pricing schedule is shifted down. In the long run, the reduction in expected returns leads to a reduction in the MPK and higher capital stock. Reducing the moral hazard parameter in half, from $\phi_0 = 0.57$ to $\phi_1 = 0.29$, the idiosyncratic risk premium falls by about 140 basis point, raising the capital stock by roughly 13%.

The right panel of Figure 9 shows that the economy's response to changes in ϕ is highly non-linear. The idiosyncratic risk premium (solid line) is convex in ϕ, so changes in the moral hazard parameter lead to stronger risk premia effects in economies with low financial development. The dashed line shows the marginal change in (log) capital due to a reduction in the moral hazard parameter for different initial values of ϕ. The same reduction in ϕ can lead to an increase in the capital stock that is five times larger in an economy with low financial development (high ϕ) compared to an economy with high financial development (low ϕ).

Financial development also has important implications for inequality. The left panel of Figure 10 shows the average financial wealth for each age group relative to the average across all entrepreneurs on the initial stationary equilibrium. Financial wealth falls for all age groups

[32] To focus on entrepreneurs, we assume that insurance providers can perfectly diversify idiosyncratic risk. If insurance providers also hold under-diversified portfolios, as in Gârleanu et al. (2015), a reduction in the risk premium can be caused by an improvement in their ability to diversify, see e.g. Khorrami (2019).

in the new equilibrium, as it is harder to accumulate wealth with lower expected returns. In the long run, inequality falls after a reduction in ϕ, as shown in the right panel of Figure 10. As entrepreneurs are less exposed to risk, we observe a smaller dispersion of financial wealth.

6 Dynamic effects of relaxing risk constraints

In the previous section, we studied the long-run effects of relaxing risk constraints by comparing two stationary equilibria (or steady states). This approach ignores the transitional dynamics. To compute the welfare implications of relaxing risk constraints, it is important to explicitly take into account what happens during the transition. Moreover, some of the effects may take a long time to materialize. We turn next to the dynamic effects of changes in ϕ.

Computing the transitional dynamics. We consider a "small open economy" version of the model, where the interest rate is kept at the level of the original stationary equilibrium.[33] This allows us to focus on the dynamic implications of fluctuations in the idiosyncratic risk premium on the capital stock. We show in Appendix D that the price of aggregate insurance is given by $p_t^{ag} = \gamma\sigma_A$ during the transition. The next proposition characterizes the evolution of the capital stock and its relative price, K and q, the consumption-wealth ratio of entrepreneurs by age, $\zeta_t(a)$, and the level of total and human wealth, $\omega_t(a)$ and $h_t(a)$.

Proposition 4. *The evolution of $(q_t, K_t, \{\zeta_t(a), h_t(a), \omega_t(a)\})$ is characterized by a pair of ordinary differential equations (ODEs)*

$$\begin{aligned}
\dot{K}_t &= \left[\Phi(\iota(q_t)) - \delta - g\right] K_t \\
\dot{q}_t &= \left[r + \gamma\sigma_A^2 + \gamma\phi^2\sigma_{id}^2 \frac{q_t K_t}{\chi_e \omega_{e,t}} + \delta - \mu_A - \Phi(\iota(q_t)) - \frac{\alpha K_t^{\alpha-1} - \iota(q_t)}{q_t}\right] q_t
\end{aligned}$$

and three partial differential equations (PDEs)

$$\begin{aligned}
\frac{\partial \zeta_t(a)}{\partial t} &= -\frac{\partial \zeta_t(a)}{\partial a} + \zeta_t^2(a) - \bar{r}_t \zeta_t(a) \\
\frac{\partial h_t(a)}{\partial t} &= -\frac{\partial h_t(a)}{\partial a} + (r + \gamma\sigma_A^2 - \mu_A) h_t(a) - (1-\alpha) K_t^\alpha \bar{l}(a) \\
\frac{\partial \omega_t(a)}{\partial t} &= -\frac{\partial \omega_t(a)}{\partial a} + \left[r + \gamma\sigma_A^2 - \mu_A + \gamma\phi^2\sigma_{id}^2 \left(\frac{q_t K_t}{\chi_e \omega_{e,t}}\right)^2 - \zeta_t(a)\right] \omega_t(a),
\end{aligned}$$

subject to the boundary conditions described in the appendix.

[33] We show in Appendix D that the interest rate is constant during the transitional dynamics when wage earners have Epstein-Zin utility with linear intertemporal preferences.

Proof. See Appendix A.5. □

In the expressions above, we have eliminated the price of aggregate insurance using $p_t^{ag} = \gamma\sigma_A$ and the price of idiosyncratic insurance using $p_t^{id} = \gamma\phi\sigma_{id}\frac{q_t K_t}{\chi_e \omega_{e,t}}$, where $\omega_{e,t} = n_{e,t} + h_{e,t}$. We obtain the first ODE by aggregating (3) and the second one by rearranging the time-dependent version of (32). The first PDE comes from the HJB equation for entrepreneurs, the second one corresponds to the time-dependent version of (14), and the last PDE can be obtained by averaging the budget constraint of entrepreneurs.

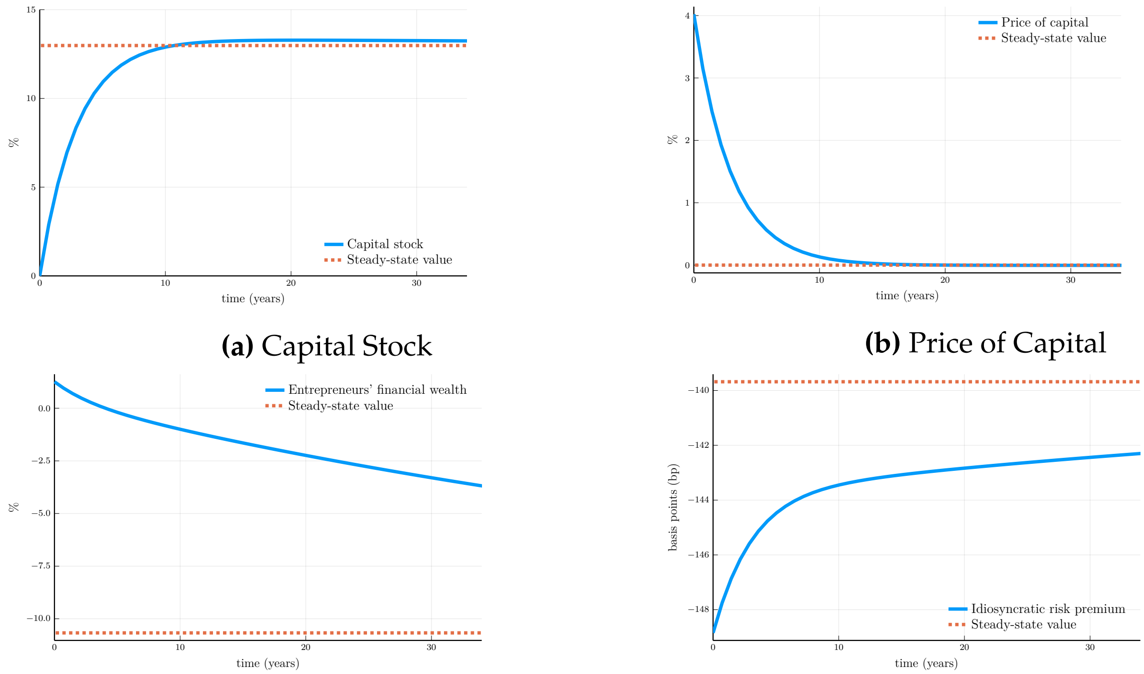
The transitional dynamics for heterogeneous agents models are often computed using a shooting algorithm, as in e.g. Guerrieri and Lorenzoni (2017) or Achdou et al. (2017). Given the large number of forward-looking variables, such an algorithm is impractical in our setting. We adopt instead a combination of perturbation and finite-difference methods. First, we use finite differences to discretize the system of ODE/PDEs. We end up with a (large) non-linear boundary value problem, since $(K_t, \omega_t(a))$ have initial conditions, while $(q_t, \zeta_t(a), h_t(a))$ have terminal conditions. We then linearize the system around the new stationary equilibrium. In contrast to the approach in e.g. Ahn et al. (2018), which linearizes around the economy without aggregate shocks, we do not assume that aggregate or idiosyncratic shocks are small. By avoiding these small-risk approximations, we are able to capture time-varying risk premia and precautionary savings effects using this method.[34] The final step consists of solving the resulting linear rational expectation model, which can be done by standard techniques, as the one by Blanchard and Kahn (1980). See Appendix D for a detailed discussion of the method.

6.1 Short-run dynamics: the overshooting effect

Figure 11 shows the evolution of the capital stock, the relative price of capital, entrepreneurs' financial wealth, and the idiosyncratic risk premium, as deviations from the initial steady state. We consider a reform that reduces the moral hazard parameter by half, from $\phi_0 = 0.571$ under our calibration to $\phi_1 = 0.286$. In response to a relaxation of insurance constraints, there is an investment boom and a sharp increase in the value of businesses that lasts for roughly a decade, as shown in Panels (a) and (b). The short-run response of the price of capital exceeds by a large margin its long-run level, i.e. there is an *overshooting* effect. As entrepreneurs bear less of the risk, they also require a smaller premium. In the long run, this leads to a smaller MPK and larger capital stock. However, capital is fixed in the short run, so the only way expected returns can go down is through expected capital losses. The price of capital then jumps on impact and slowly reverts to its long-run level. This increase in marginal q induces entrepreneurs to invest more, explaining the investment boom. This logic is reminiscent of Dornbusch's (1976) overshooting model, where exchange rates react more strongly to shocks

[34] See Winberry (2018) on perturbation methods to solve heterogeneous agent models with aggregate risk.

Figure 11: Transitional dynamics: aggregate variables



(a) Capital Stock

(b) Price of Capital

(c) Entrepreneurs' Financial Wealth

(d) Idiosyncratic Risk Premium

Note: Transitional dynamics from a stationary equilibrium with $\phi_0 = 0.571$ to a stationary equilibrium with $\phi_1 = 0.286$. Capital stock, the relative price of capital, and entrepreneurs' financial wealth are expressed as percentage deviations from the initial steady state. The idiosyncratic risk premium is expressed as absolute deviation from the initial steady state in basis points.

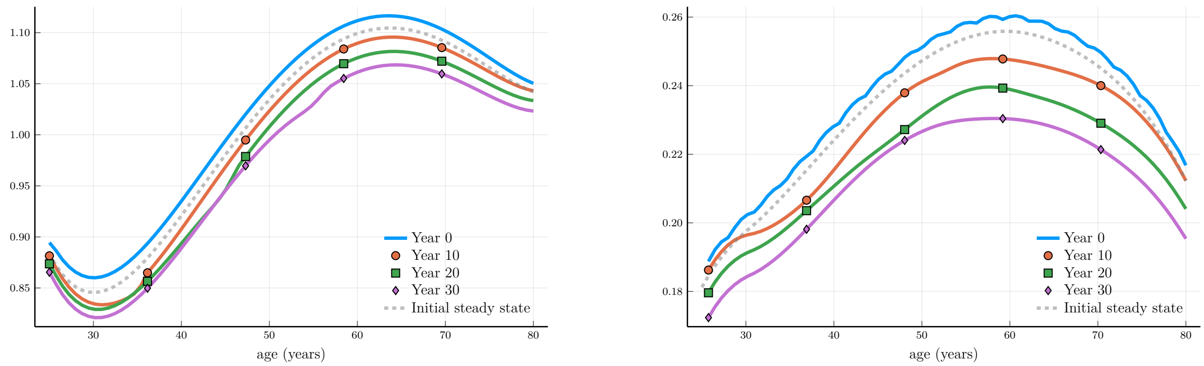
in the short run to create expected capital losses to domestic investors.[35] The overshooting effect has important implications for wealth accumulation and inequality.

As discussed in Subsection 5.3, the financial wealth of entrepreneurs goes down in the long run. In contrast, their financial wealth actually increases in the short run. The reason for the contrast between the short-run and long-run responses is a *revaluation* effect. Since entrepreneurs own the capital stock, their wealth jumps on impact as the relative price of capital goes up in the short run. As the expected return on the business goes down with the reduction in the idiosyncratic risk premium, entrepreneurs accumulate wealth at a slower pace and end up with less wealth in the long run. This long-run effect takes a long time to materialize, as shown in Panel (c) of Figure 11, given it is partially transmitted to future generations through lower bequests. Even thirty years after the shock, the reduction in entrepreneurs' financial wealth is only 25% of the long-run effect.

The slow wealth dynamics affect the behavior of risk premia. The short-run response of the idiosyncratic risk premium exceeds its long-run level by nearly ten basis points, as shown in Panel (d). After a decade, the difference between the risk premium to the new steady state

[35]The presence of adjustment costs is crucial for the overshooting result. Without adjustment costs, capital jumps to the new steady-state level, and there is no movement in prices. Similarly, there is no overshooting in Dornbusch's model when the price level can immediately jump to its steady-state level.

Figure 12: Transitional dynamics: inequality



(a) Between-group inequality

(b) Within-group inequality

Note: Transitional dynamics from a stationary equilibrium with $\phi_0 = 0.73$ to a stationary equilibrium with $\phi_1 = 0.365$. Panel (a) shows the dynamics of average financial wealth by age. Panel (b) shows the dynamics of the standard deviation of financial wealth by age.

level is cut by more than half, and then the risk premium increases slowly as entrepreneurs' wealth declines, in line with the risk exposure effect discussed in Subsection 5.2.

6.2 Kuznets dynamics

Relaxing the risk constraints has important implications for the dynamics of inequality. Figure 12 shows the evolution of between- and within-group wealth inequality. The left panel shows the evolution of average financial wealth for each age group at different points in time, normalized by average wealth across all entrepreneurs in the initial equilibrium. Because of the revaluation effect, financial wealth increases on impact for all age groups, but the effect is stronger for younger entrepreneurs, as they are proportionally more exposed to the business. Over time the financial wealth goes down due to the reduction in expected returns.

The right panel on Figure 12 shows the standard deviation of financial wealth by age. Again the short-run and long-run responses are different. While wealth inequality goes down in the long run, wealth inequality goes up in the short run. Entrepreneurs with initially more wealth also hold more capital, so wealthier entrepreneurs benefit the most from the reform.

The evolution of inequality interacts in interesting ways with the demographic structure. After the initial increase in inequality, wealth dispersion goes down for all age groups, but entrepreneurs starting their professional lives just after the reform are the ones most affected. Ten years after the reform, the drop in inequality is two times larger for 35-year-old entrepreneurs, who lived their entire professional life under the new regime, relative to 80-year-old entrepreneurs, who lived most of their lives under the old regime. Similarly, the drop in inequality is more pronounced for 35-year-old entrepreneurs than for 25-year-old entrepreneurs.

Taking stock. The reform initially led to an investment boom and an increase in wealth inequality. As the economy approaches its new level of output, inequality starts to recede and it reaches a lower level in the long run. The initial increase in inequality as the economy enters a high growth phase and eventual reduction in inequality as the economy reaches a higher level of development is consistent with a Kuznets's (1955) curve.[36]

6.3 Welfare implications

We turn next to the welfare implications of insurance constraints. Remember that the value function of an entrepreneur of age a at time t is given by

$$V_t(n,a) = \zeta_t^{-\frac{1}{\gamma}}(a)\frac{(n+h_t(a))^{1-\gamma}}{1-\gamma}.$$

Hence, the welfare of an entrepreneur depends on financial wealth n, human wealth $h_t(a)$, and the consumption-wealth ratio $\zeta_t(a)$, which captures the path of expected future returns. We evaluate financial wealth at the average level of the age group, $n = n_t(a)$, but one can use the inequality results previously discussed to infer the dispersion in welfare within age groups. Finally, we take a monotonic transformation of the value function to measure welfare in consumption units. Hence, our measure of welfare will be

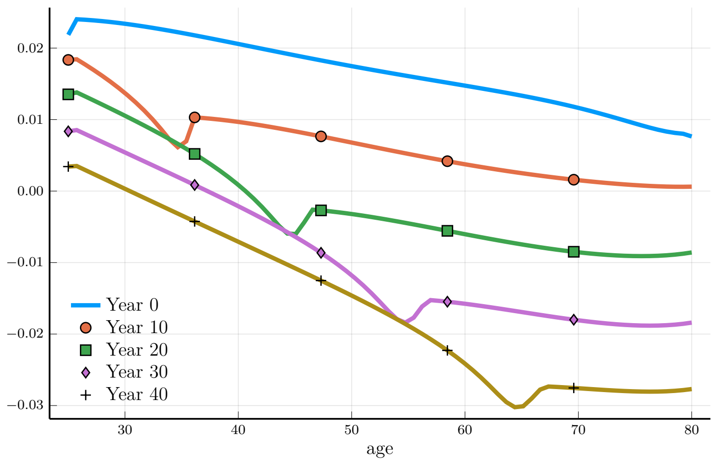
$$\mathcal{W}_t(a) = \log\left[u^{-1}\left(V_t(n_t(a),a)\right)\right] - \log\left[u^{-1}\left(V^*(n^*(a),a)\right)\right] = \frac{\hat{\zeta}_t(a)}{\gamma(\gamma-1)} + \hat{\omega}_t(a),$$

where $u(c) = \frac{c^{1-\gamma}}{1-\gamma}$, and a hat denotes log deviations from the initial steady state.

Figure 13 shows the welfare gains for each age group at different points in time. The generation that is alive at the moment of the reform benefits the most, with gains concentrated on younger entrepreneurs. This is the result of the revaluation effect. However, the negative impact on wealth accumulation affects future generations of entrepreneurs. As they receive smaller bequests and it becomes harder to accumulate wealth, their welfare is adversely impacted. Figure 13 shows how demographics affect welfare gains. For instance, ten years after the intervention, the welfare gains for entrepreneurs who started their professional life after the reform, the ones with age between 25 to 35 years old, have welfare gains that are smaller than the entrepreneurs of the same age at the time of the intervention. Thirty years after the reform, entrepreneurs aged between 35 and 55 are worse off compared to an equilibrium without the reform, with larger welfare losses for the older entrepreneurs. Therefore, the initial generation of entrepreneurs reaps most of the benefits of the reform.

[36] Moll (2012) derived a Kuznet's curve by showing that the steady-state top wealth share is hump-shaped in financial development. In contrast, we focus on the transitional dynamics instead of long-run comparisons.

Figure 13: Transitional dynamics: welfare



7 Conclusion

In this paper, we study the aggregate and distributive implications of entrepreneurial risk. We propose a life-cycle model of entrepreneurship with aggregate and idiosyncratic risk under limited insurance. We show that entrepreneurial returns command a positive idiosyncratic risk premium, which accounts for a large fraction of total returns. The model captures quantitatively the empirical patterns of risk-taking and savings over the life cycle, the inverted-U shape of wealth inequality, and the level of aggregate and idiosyncratic risk premia.

We also study the impact of relaxing insurance constraints. An improvement in idiosyncratic insurance increases output, reduces inequality in the long run, and generates rich transitional dynamics. The price of capital overshoots in the short run, generating a large investment boom and an increase in the value of the business. This overshooting leads to an initial increase in inequality. As the reduction in risk and expected returns have time to play out, inequality goes down in the long run with important intergenerational effects. Finally, most of the welfare gains are concentrated in the generations that are alive at the time of the change, and future generations of entrepreneurs are actually worse off.

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A Proofs

A.1 Proofs of Lemma 1 and Proposition 1

Proof. We start by showing part (b) of Lemma 1, that is, we solve for $h_t(a)$ and its dynamics. Then, we proceed to solve for the entrepreneurs' value function and policy functions, deriving items (a) and (c) of Lemma 1 as well as the results in Proposition 1.

Pricing human wealth. Define the stochastic discount factor (SDF) for this economy as the process π_t satisfying the law of motion

$$\frac{d\pi_t}{\pi_t} = -r_t dt - p_t^{ag} dZ_t. \tag{A.1}$$

Without loss of generality, we assumed that the SDF is not exposed to idiosyncratic risk, as we only use the SDF to price human wealth which is not exposed to idiosyncratic risk. Integrating the process above, we obtain

$$\frac{\pi_z}{\pi_t} = \exp\left(-\int_t^z \left(r_u + \frac{(p_u^{ag})^2}{2}\right) du - \int_t^z p_u^{ag} dZ_u\right). \tag{A.2}$$

Similarly, integrating the process for A_t

$$\frac{A_z}{A_t} = \exp\left(\int_t^z \left(\mu_A - \frac{\sigma_A^2}{2}\right) du + \int_t^z \sigma_A dZ_u\right). \tag{A.3}$$

Hence, we can explicitly compute the following expectation

$$\begin{aligned}
\mathbb{E}_t\left[\frac{\pi_z A_z}{\pi_t A_t}\right] &= \mathbb{E}_t\left[\exp\left(-\int_t^z \left(r_u - \mu_A + \frac{(p_u^{ag})^2 + \sigma_A^2}{2}\right) du - \int_t^z \left(p_u^{ag} - \sigma_A\right) dZ_u\right)\right] \\
&= \exp\left(-\int_t^z \left(r_u - \mu_A + \frac{(p_u^{ag})^2 + \sigma_A^2}{2}\right) du + \frac{1}{2}\int_t^z \left(p_u^{ag} - \sigma_A\right)^2 du\right) \\
&= \exp\left(-\int_t^z \left(r_u + p_u^{ag}\sigma_A - \mu_A\right) du\right),
\end{aligned} \tag{A.4}$$

where we used Ito's isometry in the second equality and the fact that p_t^{ag} is deterministic.

Human wealth is given by

$$h_t(a) = \mathbb{E}_t \left[\int_t^{t+T-a} \frac{\pi_z A_z}{\pi_t A_t} w_z \bar{l}(a+z-t) dz \right]$$
$$= \int_t^{t+T-a} e^{-\int_t^z \left(r_u + p_u^{ag}\sigma_A - \mu_A\right) du} w_z \bar{l}(a+z-t) dz. \tag{A.5}$$

Consider the human wealth for someone born at date s, so $a = t - s$:

$$h_t(t-s) = \int_t^{s+T} e^{-\int_t^z \left(r_u + p_u^{ag}\sigma_A - \mu_A\right) du} w_z \bar{l}(z-s) dz. \tag{A.6}$$

Differentiating the expression above with respect to time yields

$$\frac{\partial h_t(a)}{\partial t} + \frac{\partial h_t(a)}{\partial a} = \left(r_t + p_t^{ag}\sigma_A - \mu_A\right) h_t(a) - w_t \bar{l}(a), \tag{A.7}$$

which gives (14) in a stationary equilibrium.

The HJB equation. The HJB equation for problem (9) is given by

$$\rho \tilde{V}_t(\tilde{n}, t-s; A_t) = \max_{\tilde{c}_t, \tilde{\theta}_t^{ag}, \tilde{\theta}_t^{id}, k_t, l_t, \iota_t} \left\{ \frac{\tilde{c}_t^{1-\gamma}}{1-\gamma} + \frac{\mathbb{E}_t \left[d\tilde{V}_{i,t} \right]}{dt} \right\}, \tag{A.8}$$

subject to (7) as well as the terminal and boundary conditions

$$\tilde{V}_t(\tilde{n}, T) = (1-\psi)^\gamma V^* \frac{\tilde{n}^{1-\gamma}}{1-\gamma}; \qquad \lim_{\tilde{n} \to -\tilde{h}_t(a)} \tilde{V}_t(\tilde{n}, a) = \begin{cases} 0, & \text{if } \gamma < 1 \\ -\infty, & \text{if } \gamma \geq 1 \end{cases}, \tag{A.9}$$

where the terminal condition captures the effect of bequests and the boundary condition captures the fact that consumption is zero if the entrepreneur hits the natural borrowing limit.

Using Ito's lemma, the HJB reduces to a partial differential equation for $\tilde{V}_t(\tilde{n}, a; A_t)$:

$$\rho \tilde{V}_t = \max_{\tilde{c}_t, \tilde{\theta}_t^{ag}, \tilde{\theta}_t^{id}, k_t, l_t, \iota_t} \frac{\tilde{c}_t^{1-\gamma}}{1-\gamma} + \frac{\partial \tilde{V}_t}{\partial t} + \frac{\partial \tilde{V}_t}{\partial a} + \frac{\partial \tilde{V}_t}{\partial \tilde{n}} \mu_{\tilde{n},t} + \frac{\partial \tilde{V}_t}{\partial A_t} \mu_A A_t + \frac{1}{2} \frac{\partial^2 \tilde{V}_t}{\partial \tilde{n}^2} (\sigma_{ag,t}^2 + \sigma_{id,t}^2) + \frac{\partial^2 \tilde{V}_t}{\partial \tilde{n} \partial A_t} \sigma_{ag,t} \sigma_A A_t + \frac{1}{2} \frac{\partial \tilde{V}_i}{\partial A_t^2} \sigma_A^2 A_t^2, \tag{A.10}$$

where $(\mu_{\tilde{n},t}, \sigma_{ag,t}, \sigma_{id,t})$ are the drift and diffusion terms for $\tilde{n}_t$, and the maximization is subject to (7).

First, we verify that the following guess for the value function solves the PDE

$$\tilde{V}_t(\tilde{n}, a; A_t) = \zeta_t(a)^{-\gamma} \frac{(\tilde{n} + A_t h_t(a))^{1-\gamma}}{1-\gamma}. \tag{A.11}$$

Plugging the derivatives of the equation above into the HJB equation, we obtain

$$
\begin{aligned}
\frac{\rho}{1-\gamma} = \max_{c_{i,t},k_{i,t},l_{i,t},\iota_{i,t},\theta^{ag}_{i,t},\theta^{id}_{i,t}} \Bigg\{ & \frac{\zeta_t^{\gamma}(a)}{1-\gamma}\left(\frac{c_{i,t}}{\omega_{i,t}}\right)^{1-\gamma} - \frac{\gamma}{1-\gamma}\frac{1}{\zeta_t(a)}\left(\frac{\partial \zeta_t(a)}{\partial t} + \frac{\partial \zeta_t(a)}{\partial a}\right) + r_t + \frac{q_t k_{i,t}}{\omega_{i,t}}(\mu^R_{i,t} - r_t) \\
& - \frac{p_t^{ag}\theta^{ag}_{i,t}}{\omega_{i,t}} + \frac{h_{i,t}}{\omega_{i,t}}\sigma_A p_t^{ag} - \frac{c_{i,t}}{\omega_{i,t}} - \frac{\gamma}{2}\left[\left(\frac{q_t k_{i,t} + h_{i,t}}{\omega_{i,t}}\sigma_A - \frac{\theta^{ag}_{i,t}}{\omega_{i,t}}\right)^2 + \left(\frac{q_t k_{i,t}}{\omega_{i,t}}\sigma_{id} - \frac{\theta^{id}_{i,t}}{\omega_{i,t}}\right)^2\right] \\
& + p_t^{id}\left[(1-\phi)\frac{q_t k_{i,t}}{\omega_{i,t}}\sigma_{id} - \frac{\theta^{id}_{i,t}}{\omega_{i,t}}\right]\Bigg\}, \qquad \text{(A.12)}
\end{aligned}
$$

where $\omega_{i,t} = n_{i,t} + h_t(a)$ and $p^{id}_{i,t}$ denotes the Lagrange multiplier on the skin-in-the-game constraint.

From the expression above, it is immediate that the optimal value of $(l_{i,t}, \iota_{i,t})$ maximizes the expected return on the business. The first-order conditions for $(l_{i,t}, \iota_{i,t})$ are given in (11) and (12), respectively. The expected return on the business will be equalized, allowing us to write $\mu^R_{i,t} = \mu^R_t$.

The optimal quantity of capital, aggregate insurance, and idiosyncratic insurance solve the problem

$$
\max_{k_{i,t},\theta^{ag}_{i,t},\theta^{id}_{i,t}} \left\{ \frac{q_t k_{i,t}}{\omega_{i,t}}(\mu^R_t - r) - \frac{p^{ag}\theta^{ag}_{i,t}}{\omega_{i,t}} + \frac{h_{i,t}}{\omega_{i,t}}p^{ag}\sigma_A - \frac{\gamma}{2}\left[\left(\frac{q_t k_{i,t} + h_{i,t}}{\omega_{i,t}}\sigma_A - \frac{\theta^{ag}_{i,t}}{\omega_{i,t}}\right)^2 + \left(\frac{q_t k_{i,t}}{\omega_{i,t}}\sigma_{id} - \frac{\theta^{id}_{i,t}}{\omega_{i,t}}\right)^2\right]\right\}, \qquad \text{(A.13)}
$$

subject to (7).

Multiplying the expression above by $\omega_{i,t}/n_{i,t}$ gives (18).

Policy functions. The first order condition for $\theta^{id}_{i,t}$ is given by

$$
\gamma\left[\frac{q_t k_{i,t}}{\omega_{i,t}}\sigma_{id} - \frac{\theta^{id}_{i,t}}{\omega_{i,t}}\right] = p^{id}_{i,t}. \qquad \text{(A.14)}
$$

The equation above implies that the skin-in-the-game constraint is always binding, so $p^{id}_{i,t} > 0$ and $\theta^{id}_t = (1-\phi)q_t k_{i,t}\sigma_{id}$. If this was not the case, i.e. $p^{id}_{i,t} = 0$, then we would have $\theta^{id}_{i,t} = q_t k_{i,t}\sigma_{id}$, which violates the skin-in-the-game constraint.

The first-order conditions for capital and aggregate insurance are given by

$$\mu_t^R - r + p_{i,t}^{id}(1-\phi)\sigma_{id} = \gamma\left[\left(\frac{q_t k_{i,t} + h_{i,t}}{\omega_{i,t}}\sigma_A - \frac{\theta_{i,t}^{ag}}{\omega_{i,t}}\right)\sigma_A + \left(\frac{q_t k_{i,t}}{\omega_{i,t}}\sigma_{id} - \frac{\theta_{i,t}^{id}}{\omega_{i,t}}\right)\sigma_{id}\right]$$
$$p_t^{ag} = \gamma\left(\frac{q_t k_{i,t} + h_{i,t}}{\omega_{i,t}}\sigma_A - \frac{\theta_{i,t}^{ag}}{\omega_{i,t}}\right). \tag{A.15}$$

Combining the expressions above, we obtain

$$p_{i,t}^{id} = \frac{\mu_t^R - r_t - p_t^{ag}\sigma_A}{\phi\sigma_{id}}, \tag{A.16}$$

which coincides with expression (20) after we write $p_{i,t}^{id} = p_t^{id}$.

The demand for capital can be written as

$$\frac{q_t k_{i,t}}{\omega_{i,t}} = \frac{p_t^{id}}{\gamma\phi\sigma_{id}}. \tag{A.17}$$

Multiplying by $\omega_{i,t}/n_{i,t}$, we obtain expression (19). Solving for $\theta_{i,t}^{ag}$ in the optimality condition for aggregate insurance we obtain (21).

The first-order condition for consumption gives

$$\frac{c_{i,t}}{\omega_{i,t}} = \zeta_t(a). \tag{A.18}$$

Plugging the expressions above back into the HJB, we obtain a PDE for $\zeta_t(a)$

$$\frac{\partial \zeta_t(a)}{\partial t} + \frac{\partial \zeta_t(a)}{\partial a} = \zeta_t^2(a) - \bar{r}_t\zeta_t(a), \tag{A.19}$$

where

$$\bar{r}_t \equiv \frac{1}{\gamma}\rho + \left(1 - \frac{1}{\gamma}\right)\left[r + \frac{(p_t^{id})^2 + (p_t^{ag})^2}{2\gamma}\right]. \tag{A.20}$$

Define $z_{s,t} \equiv \zeta_t^{-1}(t-s)$ as the wealth-consumption ratio for an entrepreneur born at date s. Differentiating with respect to t, we obtain

$$\dot{z}_{s,t} = -\frac{1}{\zeta_t^2(a)}\left[\frac{\partial \zeta_t}{\partial t} + \frac{\partial \zeta_t}{\partial a}\right] = \bar{r}_t z_{s,t} - 1. \tag{A.21}$$

Solving the above differential equation, we get

$$z_{s,t} = \int_t^{s+T} e^{-\int_t^u \bar{r}_z dz} du + e^{-\int_t^{s+T} \bar{r}_z dz} z_{s,s+T}, \tag{A.22}$$

or in terms of $\zeta_t(a)$, we have

$$\zeta_t(a) = \frac{1}{\int_t^{t+T-a} e^{-\int_t^u \bar{r}_z dz} du + e^{-\int_t^{t+T-a} \bar{r}_z dz} (1-\psi)(V^*)^{\frac{1}{\gamma}}}, \tag{A.23}$$

where we used the boundary condition $\zeta_t^{-1}(T) = (1-\psi)\,(V^*)^{\frac{1}{\gamma}}$.

Assuming $(V^*)^{\frac{1}{\gamma}} = \frac{1}{\bar{r}}$ and a stationary equilibrium, where $\bar{r}_t = \bar{r}$, we obtain

$$\zeta_t(a) = \frac{\bar{r}}{1 - \psi e^{-\bar{r}(T-a)}}, \tag{A.24}$$

which coincides with (22).

Notice that the assumption $(V^*)^{\frac{1}{\gamma}} = \frac{1}{\bar{r}}$ guarantees that the consumption-wealth ratio for $\psi = 0$ is the same as in the infinite horizon economy.

□

A.2 Proof of Proposition 2

Proof. We start by deriving the law of motion of financial wealth for an entrepreneur of a given age. Using the value of capital, $k_{i,t}$, aggregate and idiosyncratic insurance, $(\theta_{i,t}^{ag}, \theta_{i,t}^{id})$, and the definition of the price of idiosyncratic risk, p_t^{id}, given in Proposition 1, we can write the law of motion of financial wealth as follows

$$d\tilde{n}_{i,t} = \left[r_t \tilde{\omega}_{i,t} + \frac{(p_t^{id})^2}{\gamma} \tilde{\omega}_{i,t} + \frac{(p_t^{ag})^2}{\gamma} \tilde{\omega}_{i,t} - \tilde{h}_{i,t}(r_t + p_t^{ag}\sigma_A) + \tilde{w}_t \bar{l}_{i,t} - \tilde{c}_{i,t} \right] dt + \left(\tilde{\omega}_{i,t} \frac{p_t^{ag}}{\gamma} - \tilde{h}_{i,t}\sigma_A \right) dZ_t + \frac{p_t^{id}}{\gamma} \tilde{\omega}_{i,t} dZ_{i,t}, \tag{A.25}$$

where $\tilde{\omega}_{i,t} = \tilde{n}_{i,t} + \tilde{h}_{i,t}$.

Using the fact that $p_t^{ag} = \gamma \sigma_A$ in equilibrium, we find that the aggregate risk exposure of entrepreneurs is given $\tilde{n}_{i,t}\sigma_A$. Hence, scaled financial wealth, $n_{i,t} = \tilde{n}_{i,t}/A_t$, does not respond to aggregate shocks. The evolution of $n_{i,t}$ can then be written as

$$dn_{i,t} = \mu_{n,t}(n_{i,t}, a)dt + \sigma_{n,t}(n_{i,t}, a)dZ_{i,t}, \tag{A.26}$$

where

$$\mu_{n,t}(n,a) = \left[r_t + \frac{(p_t^{id})^2}{\gamma} + \frac{(p_t^{ag})^2}{\gamma} - \mu_A - \zeta_t(a)\right](n + h_t(a)) - \mu_{h,t}(a) \tag{A.27}$$

$$\sigma_{n,t}(n,a) = \frac{p_t^{id}}{\gamma}(n + h_t(a)), \tag{A.28}$$

and $\mu_{h,t}(a)$ is the drift of $h_t(a)$.

Derivation of Equation (24). Notice that total wealth evolves according to

$$\frac{d\omega_{i,t}}{\omega_{i,t}} = \left[r_t + \frac{(p_t^{ag})^2}{\gamma} + \frac{(p_t^{id})^2}{\gamma} - \mu_A - \zeta_t(t - s_i)\right]dt + \frac{p_t^{id}}{\gamma}dZ_{i,t}, \tag{A.29}$$

where s_i is the birthdate of entrepreneur i.

Let $\overline{\omega}_{s,t} \equiv \frac{\int_{\mathcal{E}_t} \mathbb{1}_{s_i=s}\omega_{i,t}di}{\int_{\mathcal{E}_t} \mathbb{1}_{s_i=s}di}$ denote the average total wealth of entrepreneurs born at date s. The law of motion of $\overline{\omega}_{s,t}$ is given by

$$d\overline{\omega}_{s,t} = \left[r_t + \frac{(p_t^{ag})^2}{\gamma} + \frac{(p_t^{id})^2}{\gamma} - \mu_A - \zeta_t(t - s)\right]\overline{\omega}_{s,t}dt \tag{A.30}$$

where the idiosyncratic risk is diversified by averaging out across entrepreneurs of a given cohort.

It is convenient to express total wealth as a function of age instead of the entrepreneurs' birthdate. Let $\omega_t(a)$ denote the average total wealth of investors with age a at period t. Using the fact that $\overline{\omega}_{s,t} = \omega_t(t - s)$, we obtain the following PDE for $\omega_t(a)$:

$$\frac{\partial\omega_t(a)}{\partial t} + \frac{\partial\omega_t(a)}{\partial a} = \left[r_t + \frac{(p_t^{ag})^2}{\gamma} + \frac{(p_t^{id})^2}{\gamma} - \mu_A - \zeta_t(a)\right]\omega_t(a). \tag{A.31}$$

In a stationary equilibrium, $\omega_t(a)$ does not depend on calendar time t, which allow us to write

$$\frac{d\log\omega(a)}{da} = r + \frac{(p^{ag})^2}{\gamma} + \frac{(p^{id})^2}{\gamma} - \mu_A - \zeta(a). \tag{A.32}$$

Integrating the expression above, we obtain

$$\log\omega(a) = \log\omega(0) + \left[r + \frac{(p^{ag})^2}{\gamma} + \frac{(p^{id})^2}{\gamma} - \mu_A\right]a - \int_0^a \zeta(u)du. \tag{A.33}$$

Using the fact $\log\frac{\omega(a)}{\omega(0)} = \log\frac{f(a)\omega(a)}{f(0)\omega(0)} + ga$ and the identity $\omega(a) = n(a)\left(1 + \frac{h(a)}{n(a)}\right)$, we obtain expression (24) after some rearrangement.

Derivation of Equation (25). The expression for $\omega(a)$ in levels can be written as

$$\omega(a) = \omega(0) e^{\left(r + \frac{(p^{ag})^2}{\gamma} + \frac{(p^{id})^2}{\gamma} - \mu_A\right) a} \frac{e^{-\bar{r}a} - \psi e^{-\bar{r}T}}{1 - \psi e^{-\bar{r}T}}. \tag{A.34}$$

Evaluating at $a = T$ gives

$$\omega(T) = \omega(0) e^{\left(r + \frac{(p^{ag})^2}{\gamma} + \frac{(p^{id})^2}{\gamma} - \mu_A - mpc_e\right) T}, \tag{A.35}$$

where $mpc_e = \frac{1}{T}\int_0^T \zeta(a)da$.

The boundary condition at age T implies $\omega(0) = e^{-gT}\omega(T) + h(0)$, then

$$\omega(0) = \frac{h(0)}{1 - e^{\left(r + \frac{(p^{ag})^2}{\gamma} + \frac{(p^{id})^2}{\gamma} - \mu_A - g - mpc_e\right) T}}. \tag{A.36}$$

Using $\omega(0) = n(0) + h(0)$ and rearranging the resulting expression, we obtain (25).

Derivation of Equation (26). Multiplying Equation (A.34) by $f(a)$, integrating over age, and using the fact that $f(a) = e^{-ga} f(0)$, we obtain

$$n_e + h_e = f(0)\omega(0) \int_0^T e^{\left(r + \frac{(p^{ag})^2}{\gamma} + \frac{(p^{id})^2}{\gamma} - (g + \mu_A)\right) a} \frac{e^{-\bar{r}a} - \psi e^{-\bar{r}T}}{1 - \psi e^{-\bar{r}T}} da, \tag{A.37}$$

which gives Equation (26) after some rearrangement .

□

A.3 Proof of Lemma 2

Proof. We derive the Kolmogorov Forward Equation as the limit of a discrete-time economy. The discrete-time approximation goes as follows. Time takes values on the discrete set $\{t^1, \ldots, t^L\}$, where $\Delta t = t^{l+1} - t^l$ is the constant time step. Scaled financial wealth $n_{i,t}$ takes values on a discrete grid, $n_{i,t} \in \{n^1, n^2, \ldots, n^J\}$ with a constant step size $\Delta n = n^{j+1} - n^j$. Age is also assumed to take values in a discrete grid $\{a^1, \ldots, a^K\}$, where $\Delta a = a^{k+1} - a^k$, $a^1 = 0$, and $a^K = T$. For simplicity, assume $\Delta a = \Delta t$. The probability of moving up, down, or staying at the same point of the grid are chosen to approximate (A.26) and are given, respectively, by

$$p_u(n^j, a^k) = \frac{1}{2}\left[\frac{\sigma_n(n^j, a^k)^2}{\overline{\sigma}^2} + \frac{\mu_n(n^j, a^k)}{\overline{\sigma}^2}\Delta n\right] \tag{A.38}$$

$$p_d(n^j, a^k) = \frac{1}{2}\left[\frac{\sigma_n(n^j, a^k)^2}{\overline{\sigma}^2} - \frac{\mu_n(n^j, a^k)}{\overline{\sigma}^2}\Delta n\right] \tag{A.39}$$

$$p_s(n^j, a^k) = 1 - \frac{\sigma_n(n^j, a^k)^2}{\overline{\sigma}^2}. \tag{A.40}$$

where $\overline{\sigma} = \max_{1\leq j\leq J, 1\leq k\leq K} \sigma_n(n^j, a^k)$, $\Delta n = \overline{\sigma}\sqrt{\Delta t}$, and $\Delta a = \Delta t$.

Notice that the expected change in $n_{i,t}$, where $n_{i,t} = n^j$ and $a_i = a^k$, is given by

$$\mathbb{E}\left[n_{i,t+1} - n_{i,t}\right] = p_u(n^j, a^k)\Delta n + p_d(n^j, a^k)(-\Delta n) = \mu_n(n^j, a^k)\Delta t, \tag{A.41}$$

and

$$\mathbb{E}\left[(n_{i,t+1} - n_{i,t})^2\right] = p_u(n^j, a^k)\Delta n^2 + p_d(n^j, a^k)(-\Delta n)^2 = \sigma_n(n^j, a^k)^2\Delta t. \tag{A.42}$$

Let $m(n^j, a^k, t^l)$ denote the mass of agents with financial wealth n^j, age a^k, at period t^l. Summing over n^j, we obtain the mass of agents with age a^k, $M_{k,l} \equiv \sum_{j=1}^{J} m(n^j, a^k, t^l) = e^{g(t^l-(k-1)\Delta t)}$. Summing over (n^j, a^k), we obtain the total population $M_l = \sum_{k=1}^{K} M_{k,l}$, so $M_{l+1} = e^{g\Delta t} M_l$. The law of motion of m, for $k > 1$ and $1 < j < J$, is given by

$$\begin{aligned} m(n^j, a^k, t^l + \Delta t) = {} & p_u(n^j - \Delta n, a^k - \Delta a)m(n^j - \Delta n, a^k - \Delta a, t^l) + p_s(n^j, a^k - \Delta a)m(n^j, a^k - \Delta a, t^l) \\ & + p_d(n^j + \Delta n, a^k - \Delta a)m(n^j + \Delta n, a^k - \Delta a, t^l). \end{aligned} \tag{A.43}$$

The boundary conditions are defined as follows. For $j = 1$ and $j = J$, we will assume a reflecting boundary, that is, if n moves up from n_J or down from n_1, it is immediately reflected back to its initial position

$$\begin{aligned} m(n^J, a^k, t^l + \Delta t) = {} & p_u(n^J - \Delta n, a^k - \Delta a)m(n^J - \Delta n, a^k - \Delta a, t^l) + p_s(n^J, a^k - \Delta a)m(n^J, a^k - \Delta a, t^l) \\ & + p_u(n^j, a^k - \Delta a)m(n^J, a^k - \Delta a, t^l), \end{aligned} \tag{A.44}$$

and analogously for $j = 1$.

Finally, for $k = 1$, we have

$$\begin{aligned} m(e^{-gT}n^j, a^1, t^l + \Delta t) = e^{gT} & \left[p_u\left(n^j - \Delta n, a^K\right) m\left(n^j - \Delta n, a^K, t^l\right) + p_s\left(n^j, a^k\right) m\left(n^j, a^K, t^l\right)\right. \\ & \left. + p_d\left(n^j + \Delta n, a^K\right) m\left(n^j + \Delta n, a^K, t^l\right)\right]. \end{aligned} \tag{A.45}$$

since each one of the e^{gT} heirs inherit $e^{-gT}n^j$, where we assumed $e^{-gT}n^j$ belongs to the grid.

Let $f(n^j, a^k, t^l) \equiv \frac{m(n^j,a^k,t^l)}{M_l}$ denote the share of agents in state (n^j, a^k) in period t^l. Dividing both sides of (A.43) by M_l and taking a Taylor expansion, we obtain

$$\begin{aligned}(1+g\Delta t)(f+f_t\Delta t) = {} & \frac{1}{2}\left(\frac{\sigma_n^2 - (\sigma_n^2)_n\Delta n + 0.5(\sigma_n^2)_{nn}\Delta n^2 - (\sigma_n^2)_a\Delta t}{\overline{\sigma}^2} + \frac{\mu_n - (\mu_n)_n\Delta n}{\overline{\sigma}^2}\Delta n\right)\left(f - f_a\Delta t - f_n\Delta n + 0.5 f_{nn}\Delta n^2\right) \\ & + \frac{1}{2}\left(\frac{\sigma_n^2 + (\sigma_n^2)_n\Delta n + 0.5(\sigma_n^2)_{nn}\Delta n^2 - (\sigma_n^2)_a\Delta t}{\overline{\sigma}^2} - \frac{\mu_n + (\mu_n)_n\Delta n}{\overline{\sigma}^2}\Delta n\right)\left(f - f_a\Delta t + f_n\Delta n + 0.5 f_{nn}\Delta n^2\right) \\ & + \left(1 - \frac{\sigma_n^2 - (\sigma_n^2)_a\Delta t}{\overline{\sigma}^2}\right)(f - f_a\Delta t) + o(\Delta t). \end{aligned} \tag{A.46}$$

Simplifying the expression above and taking the limit $\Delta t \to 0$, we obtain

$$f_t + f_a + gf = \frac{1}{2}(\sigma_n^2)_{nn} f - (\mu_n)_n f + (\sigma_n^2)_n f_n - \mu_n f_n + \frac{1}{2}\sigma_n^2 f_{nn}, \tag{A.47}$$

or, more explicitly, we can write the expression as follows

$$\frac{\partial f(n,a,t)}{\partial t} + \frac{\partial f(n,a,t)}{\partial a} + gf(n,a,t) = -\frac{\partial[f(n,a,t)\mu_n(n,a)]}{\partial n} + \frac{1}{2}\frac{\partial\left[f(n,a,t)\sigma_n^2(n,a)\right]}{\partial n^2}. \tag{A.48}$$

Let $f_t(n|a)$ denote the conditional density at date t, so $f_t(n,a) = f_t(n|a)f(a)$. We can write the Kolmogorov Forward Equation in terms of the conditional density:

$$f(a)\frac{\partial f_t(n|a)}{\partial t} + f(a)\frac{\partial f_t(n|a)}{\partial a} + f_t(n|a)f'(a) = -f(a)\frac{\partial[f_t(n|a)\mu_n(n,a)]}{\partial n} + f(a)\frac{1}{2}\frac{\partial\left[f_t(n|a)\sigma_n^2(n,a)\right]}{\partial n^2} - gf_t(n,a). \tag{A.49}$$

Dividing by $f(a)$ and using the fact that $f'(a) = -gf(a)$, we obtain

$$\frac{\partial f_t(n|a)}{\partial t} + \frac{\partial f_t(n|a)}{\partial a} = -\frac{\partial[f_t(n|a)\mu_n(n,a)]}{\partial n} + \frac{1}{2}\frac{\partial\left[f_t(n|a)\sigma_n^2(n,a)\right]}{\partial n^2}. \tag{A.50}$$

In a stationary equilibrium, we can ignore the dependence on calendar time to obtain

$$\frac{\partial f(n|a)}{\partial a} = -\frac{\partial[f(n|a)\mu_n(n,a)]}{\partial n} + \frac{1}{2}\frac{\partial\left[f(n|a)\sigma_n^2(n,a)\right]}{\partial n^2}. \tag{A.51}$$

□

A.4 Proof of Proposition 3

Proof. The law of motion of (log) total wealth is

$$d\log\omega_{i,t} = \left[r + \frac{(p^{ag})^2}{\gamma} + \frac{(p^{id})^2}{\gamma} - \frac{\overline{r}}{1 - e^{-\overline{r}(T-(t-s))}} - \mu_A - \frac{1}{2}\left(\frac{p^{id}}{\gamma}\right)^2\right]dt + \frac{p^{id}}{\gamma}dZ_{i,t}, \tag{A.52}$$

where s denotes the birth date of entrepreneur i.

Integrating the expression above, we obtain

$$\log \omega_{i,t} = \log \omega_{i,s} + \int_s^t \left[r + \frac{(p^{ag})^2}{\gamma} + \frac{(p^{id})^2}{\gamma} - \frac{\bar{r}}{1 - e^{-\bar{r}(T-(t'-s))}} - \mu_A - \frac{1}{2}\left(\frac{p^{id}}{\gamma}\right)^2 \right] dt' + \frac{p^{id}}{\gamma}(Z_{i,t} - Z_{i,s}), \tag{A.53}$$

where $Z_{i,t} - Z_{i,s} \sim \mathcal{N}(0, a)$ and $a = t - s$.

Hence, $\log \omega_{i,t} \sim \mathcal{N}(m(a), v(a))$, where the mean and variance are given by

$$m(a) = \log h(0) + \left[r + \frac{(p^{ag})^2}{\gamma} + \frac{(p^{id})^2}{\gamma} - \mu_A - \frac{1}{2}\left(\frac{p^{id}}{\gamma}\right)^2 - \bar{r} \right] a + \log \frac{1 - e^{-\bar{r}(T-a)}}{1 - e^{-\bar{r}T}} \tag{A.54}$$

$$v(a) = \left(\frac{p^{id}}{\gamma}\right)^2 a, \tag{A.55}$$

using the fact that $\omega_{i,s_i} = h(0)$ when $\psi = 1$.

Note that, as the ratio of consumption to total wealth is the same for all entrepreneurs with the same age, the variance of log consumption is then given by

$$\mathbb{V}[\log c_{i,t}|a] = \mathbb{V}[\log \omega_{i,t}|a] = \left(\frac{p^{id}}{\gamma}\right)^2 a. \tag{A.56}$$

Normalized financial wealth $n_{i,t} = \omega_{i,t} - h_{i,t}$ has a shifted log-normal distribution conditional on $s_i = s$, with support $(-h(a), \infty)$. The expected value and variance of $n_{i,t}$ is given by

$$\mathbb{E}[n|a] = h(0) e^{\left(r + \frac{(p^{ag})^2}{\gamma} + \frac{(p^{id})^2}{\gamma} - \mu_A - \bar{r}\right) a} \frac{1 - e^{-\bar{r}(T-a)}}{1 - e^{-\bar{r}T}} - h(a) \tag{A.57}$$

$$\mathbb{V}[n|a] = \left[e^{\left(\frac{p^{id}}{\gamma}\right)^2 a} - 1 \right] \left[h(0) e^{\left(r + \frac{(p^{ag})^2}{\gamma} + \frac{(p^{id})^2}{\gamma} - \mu_A\right) a} \frac{e^{-\bar{r}a} - e^{-\bar{r}T}}{1 - e^{-\bar{r}T}} \right]^2. \tag{A.58}$$

We show next that $\mathbb{V}[n|a]$ has an inverted U shape. Define the following functions:

$$v_1(a) = \left[e^{\left(\frac{p^{id}}{\gamma}\right)^2 a} - 1 \right]^{\frac{1}{2}} e^{\left(r + \frac{(p^{ag})^2}{\gamma} + \frac{(p^{id})^2}{\gamma} - \mu_A\right) a}; \qquad v_2(a) = \frac{e^{-\bar{r}a} - e^{-\bar{r}T}}{1 - e^{-\bar{r}T}}. \tag{A.59}$$

The derivative of the product of $v_1(a)$ and $v_2(a)$ will be positive if

$$v_1'(a) v_2(a) + v_1(a) v_2'(a) > 0 \iff \frac{v_1'(a)}{v_1(a)} > -\frac{v_2'(a)}{v_2(a)}, \tag{A.60}$$

for $a \neq 0$ and $a \neq T$.

Notice that $-v_2'(a)/v_2(a)$ is positive, monotonically increasing, and approaches ∞ as a approaches T:

$$-\frac{v_2'(a)}{v_2(a)} = \bar{r}\frac{1}{1-e^{-\bar{r}(T-a)}}. \tag{A.61}$$

The term $v_1'(a)/v_1(a)$ is positive, monotonically decreasing, and approaches $+\infty$ as $a \to 0$:

$$\frac{v_1'(a)}{v_1(a)} = \frac{1}{2}\left(\frac{p^{id}}{\gamma}\right)^2 \frac{e^{\left(\frac{p^{id}}{\gamma}\right)^2 a}}{e^{\left(\frac{p^{id}}{\gamma}\right)^2 a}-1} + r + \frac{(p^{ag})^2}{\gamma} + \frac{(p^{id})^2}{\gamma} - \mu_A. \tag{A.62}$$

Hence, there exists a unique $0 < \hat{a} < T$ such that $v_1'(a)v_2(a) + v_1(a)v_2'(a) > 0$ for all $a < \hat{a}$ and $v_1'(a)v_2(a) + v_1(a)v_2'(a) < 0$ for all $a > \hat{a}$. Hence, $\mathbb{V}[n|a]$ is equal to zero at $a = 0$, it increases monotonically for $a < \hat{a}$, where it achieves the maximum, and it decreases towards zero for $\hat{a} < a \leq T$.

□

A.5 Proof of Proposition 4

Proof. Aggregating Equation (3) and using the fact that labor supply grows at rate g, we obtain

$$\dot{K}_t = \left[\Phi(\iota(q_t)) - \delta - g\right] K_t, \tag{A.63}$$

given the initial condition $K_0 = K^*$.

From (A.16) and (5), we obtain the expression

$$r + p_t^{ag}\sigma_A + p_t^{id}\phi\sigma_{id} = \frac{\alpha K_t^{\alpha-1} - \iota(q_t)}{q_t} + \frac{\dot{q}_t}{q_t} + \Phi(\iota(q_t)) - \delta + \mu_A. \tag{A.64}$$

Using $p_t^{ag} = \gamma\sigma_A$ and $p^{id} = \gamma\phi\sigma_{id}\frac{q_t K_t}{\chi_e \omega_{e,t}}$ and solving for $\dot{q}_t$, we obtain

$$\dot{q}_t = \left[r + \gamma\sigma_A^2 + \gamma\phi^2\sigma_{id}^2\frac{q_t K_t}{\chi_e \omega_{e,t}} + \delta - \mu_A - \Phi(\iota(q_t)) - \frac{\alpha K_t^{\alpha-1} - \iota(q_t)}{q_t}\right] q_t, \tag{A.65}$$

where $\omega_{e,t} = n_{e,t} + h_{e,t}$.

The ODE above is subject to the terminal condition

$$\lim_{t\to\infty} q_t = q, \tag{A.66}$$

where q is the value of q_t in the new stationary equilibrium.

The first PDE was derived in the proof of Lemma 1 and it was given in (A.19). The boundary conditions are $\zeta_t(T) = (1-\psi)^{-1} (V^*)^{-\frac{1}{\gamma}}$ and

$$\lim_{t\to\infty} \zeta_t(a) = \zeta(a), \tag{A.67}$$

where $\zeta(a)$ is the value in the new stationary equilibrium.

The PDE for the human wealth was given in (A.7). The boundary conditions are $h_t(T) = 0$ and

$$\lim_{t\to\infty} h_t(a) = h(a), \tag{A.68}$$

where $h(a)$ is the value in the new stationary equilibrium.

The PDE for total wealth was derived in the proof of Proposition 2 and it was given in (A.31). The first boundary condition is $\omega_t(0) = e^{-gT}\omega_t(T) + h_t(0)$. The initial condition for $\omega_t(a)$ is given by

$$\omega_0(a) = n^*(a) + (q_0 - q^*)k^*(a) + h_0(a), \tag{A.69}$$

where variables with an asterisk denote values before the change in ϕ.

The initial condition captures two types of revaluation effects. First, the value of the capital stock changes, since the price of capital jumps on impact. Second, the value of human wealth also changes since expected future wages respond on impact.

□

A.6 Proof of Lemma 3

Proof. With a slight abuse of notation, we denote the entrepreneur's value function as a function of total wealth, age, and aggregate productivity: $\tilde{V}_t(\tilde{\omega}_i, a_i) = \tilde{V}(\tilde{\omega}_i, a_i; A_t)$. The HJB for the entrepreneur's problem is given by

$$\rho\tilde{V}_t = \max_{\tilde{c}_i, \tilde{\theta}_i^{ag}, \tilde{\theta}_t^{id}, k_i, l_i, \iota_i} \frac{\tilde{c}_t^{1-\gamma}}{1-\gamma} + \frac{\mathbb{E}_t\left[d\tilde{V}_t\right]}{dt} \tag{A.70}$$

subject to the insurance and borrowing constraints (E.11).

We guess and verify that the value function can be written as

$$\tilde{V}(\tilde{\omega}, a; A_t) = A_t^{1-\gamma} V\left(\frac{\tilde{\omega}}{A_t}, a\right), \qquad \tilde{V}^d(\tilde{\omega}, a; A_t) = A_t^{1-\gamma} V^d\left(\frac{\tilde{\omega}}{A_t}, a\right), \tag{A.71}$$

where $V(\omega, a)$ and $V^d(\omega, a)$ are independent of A_t.

Let $\omega_{i,t} \equiv \tilde{\omega}_{i,t}/A_t$ denote scaled total wealth. Using an argument analogous to the one used in Lemma 1, we can derive the law of motion $h_{i,t} = \tilde{h}_{i,t}/A_t$, which then gives the law of motion of $\omega_{i,t}$:

$$d\omega_{i,t} = \mu_{\omega_i,t}dt + \sigma_{i,t}^{ag}dZ_t + \sigma_{i,t}^{id}dZ_{i,t}, \tag{A.72}$$

where $\sigma^{ag}_{i,t} \equiv (qk_{i,t} + (1-\xi_d)h_{i,t} - \omega_{i,t})\,\sigma_A - \theta^{ag}_{i,t}$, $\sigma^{id}_{i,t} \equiv qk_{i,t}\sigma_{id} - \theta^{id}_{i,t}$, and

$$\mu_{\omega_i,t} \equiv (r + p^{ag}\sigma_A - \mu_A)\omega_{i,t} + qk_{i,t}(\mu^R_{i,t} - r - p^{ag}\sigma_A) + (p^{ag} - \sigma_A)\sigma^{ag}_{i,t} + \xi_d w_t \bar{l}_{i,t} - c_{i,t}. \quad \text{(A.73)}$$

The HJB equation for the scaled value function can be written as

$$\hat{\rho} V = \max_{c_i,\theta^{ag}_i,\theta^{id}_t,k_i,l_i,\iota_i} \frac{c_t^{1-\gamma}}{1-\gamma} + V_a + V_\omega \left[\hat{r}\omega_{i,t} + qk_{i,t}(\mu^R_{i,t} - r - p^{ag}\sigma_A) + \hat{p}^{ag}\sigma^{ag}_{i,t} + \xi_d w_t \bar{l}_{i,t} - c_{i,t}\right] +$$
$$+ \frac{1}{2} V_{\omega\omega}\left((\sigma^{ag}_{i,t})^2 + (\sigma^{id}_{i,t})^2\right) + \lambda_d\left(V^d - V\right) \quad \text{(A.74)}$$

subject to $\theta^{id}_{i,t} \leq (1-\phi)qk_{i,t}\sigma_{id}$ and $\omega_{i,t} \geq 0$, where

$$\hat{\rho} \equiv \rho - (1-\gamma)\left(\mu_A - \frac{\gamma\sigma_A^2}{2}\right), \qquad \hat{r} \equiv r + p^{ag}\sigma_A - \mu_A, \qquad \hat{p}^{ag} \equiv p^{ag} - \gamma\sigma_A. \quad \text{(A.75)}$$

As the HJB equation above is independent of A_t, we conclude that $V(\omega, a)$ is also independent of A_t, confirming our initial guess.

As $l_{i,t}$ and $\iota_{i,t}$ only enter the problem through $\mu^R_{i,t}$, it is optimal to choose them to maximize expected returns. Expected returns is then constant and equalized across entrepreneurs, so we drop the dependence on entrepreneur and time: $\mu^R_{i,t} = \mu^R$. The first-order condition with respect to $\theta^{ag}_{i,t}$ is given by

$$\hat{p}^{ag} = -\frac{V_{\omega\omega}}{V_\omega}\sigma^{ag}_{i,t} \Rightarrow \theta^{ag}_{i,t} = (qk_{i,t} + (1-\xi_d)h_{i,t} - \omega_{i,t})\,\sigma_A + \frac{V_\omega}{V_{\omega\omega}}\hat{p}^{ag}. \quad \text{(A.76)}$$

An argument analogous to the one used for the model without labor income risk establishes that $p^{ag} = \gamma\sigma_A$ in a stationary equilibrium. This implies that $\hat{p}^{ag} = 0$ and $\sigma^{ag}_{i,t} = 0$, so entrepreneurs choose the same exposure to aggregate risk in equilibrium.

Let $V_\omega p^{id}$ denote the Lagrange multiplier on the insurance constraint. The first-order conditions with respect to $k_{i,t}$ and $\theta^{id}_{i,t}$ are given by

$$\mu^R_{i,t} - r - \gamma\sigma_A^2 + p^{id}(1-\phi)\sigma_{id} = -\frac{V_{\omega\omega}}{V_\omega}\left[\sigma^{ag}_{i,t}\sigma_A + \sigma^{id}_{i,t}\sigma_{id}\right], \qquad p^{id} = -\frac{V_{\omega\omega}}{V_\omega}\sigma^{id}_{i,t}. \quad \text{(A.77)}$$

Given that $\sigma^{id}_{i,t} > 0$ by the insurance constraint and given the concavity of the value function, $V_{\omega\omega} < 0$, we have that $p^{id} > 0$. Therefore, the insurance constraint is always binding, that is, $\theta^{id}_{i,t} = (1-\phi)qk_{i,t}\sigma_{id}$. Rearranging the expressions above, we obtain

$$qk_{i,t} = -\frac{V_\omega}{V_{\omega\omega}}\frac{p^{id}}{\phi\sigma_{id}}, \qquad p^{id} = \frac{\mu^R - r - p^{ag}\sigma_A}{\phi\sigma_{id}}. \quad \text{(A.78)}$$

Finally, the optimality condition for consumption is given by

$$c_{i,t}^{-\gamma} = V_\omega \Rightarrow c(\omega_{i,t}, a_{i,t}) = V_\omega^{-\frac{1}{\gamma}}(\omega_{i,t}, a_{i,t}). \tag{A.79}$$

□

A.7 Proof of Proposition 5

Proof. We provide next a complete characterization of the first-order approximation of the entrepreneurs' problem. We proceed in four steps. First, we derive the law of motion of $\hat{\omega}_{i,t}$. Second, we solve for the demand for capital. Third, we will solve for the consumption function. Fourth, we derive the conditions that determine the approximation point $\overline{\omega}$.

Law of motion of the state. The log of total wealth for entrepreneur i evolves according to

$$d\log\omega_{i,t} = \left[\hat{r} + \frac{qk_{i,t}}{\omega_{i,t}}p^{id}\phi\sigma_{id} - \frac{1}{2}\left(\frac{qk_{i,t}}{\omega_{i,t}}\phi\sigma_{id}\right)^2 + \xi_d\frac{w\bar{l}_{i,t}}{\omega_{i,t}} - \frac{c_{i,t}}{\omega_{i,t}}\right]dt + \frac{qk_{i,t}}{\omega_{i,t}}\phi\sigma_{id}dZ_{i,t}. \tag{A.80}$$

Log-linearizing the law of motion of $\omega_{i,t}$, we obtain

$$\begin{aligned} d\log\omega_{i,t} = \Bigg[&\hat{r} + \frac{q\bar{k}}{\overline{\omega}}p^{id}\phi\sigma_{id}(1+\hat{k}_{i,t}-\hat{\omega}_{i,t}) - \frac{1}{2}\left(\frac{q\bar{k}}{\overline{\omega}}\phi\sigma_{id}\right)^2\left(1+2(\hat{k}_{i,t}-\hat{\omega}_{i,t})\right) \\ &+\xi_d\frac{w\bar{l}(\bar{a})}{\overline{\omega}}\left(1+\frac{\bar{l}'(\bar{a})}{\bar{l}(\bar{a})}\hat{a}_{i,t}-\hat{\omega}_{i,t}\right) - \frac{\bar{c}}{\overline{\omega}}(1+\hat{c}_{i,t}-\hat{\omega}_{i,t})\Bigg]dt + \frac{q\bar{k}}{\overline{\omega}}\phi\sigma_{id}\left(1+\hat{k}_{i,t}-\hat{\omega}_{i,t}\right)dZ_{i,t}. \end{aligned} \tag{A.81}$$

Rearranging the expression above, we get

$$\begin{aligned} d\hat{\omega}_{i,t} = &\left[\left(\frac{q\bar{k}}{\overline{\omega}}p^{id}\phi\sigma_{id} - \left(\frac{q\bar{k}}{\overline{\omega}}\phi\sigma_{id}\right)^2\right)\left(\hat{k}_{i,t}-\hat{\omega}_{i,t}\right) + \xi_d\frac{w\bar{l}(\bar{a})}{\overline{\omega}}\left(\frac{\bar{l}'(\bar{a})}{\bar{l}(\bar{a})}\hat{a}_{i,t}-\hat{\omega}_{i,t}\right) - \frac{\bar{c}}{\overline{\omega}}(\hat{c}_{i,t}-\hat{\omega}_{i,t})\right]dt \\ &+\mathbb{E}[d\log\omega_{i,t}|a_{i,t}=\bar{a}] + \frac{q\bar{k}}{\overline{\omega}}\phi\sigma_{id}\left(1+\hat{k}_{i,t}-\hat{\omega}_{i,t}\right)dZ_{i,t}. \end{aligned} \tag{A.82}$$

We can write the expression above in more compact form:

$$d\hat{\omega}_{i,t} = [\psi_{\omega,0} + \psi_{\omega,a}\hat{a}_{i,t} + \psi_{\omega,\omega}\hat{\omega}_{i,t}]\,dt + \left[\psi^{\sigma}_{\omega,0} + \psi^{\sigma}_{\omega,a}\hat{a}_{i,t} + \psi^{\sigma}_{\omega,\omega}\hat{\omega}_{i,t}\right]dZ_{i,t}, \tag{A.83}$$

where

$$\psi^{\sigma}_{\omega,0} \equiv \frac{q\bar{k}}{\overline{\omega}}\phi\sigma_{id}, \qquad \psi^{\sigma}_{\omega,\omega} \equiv \psi^{\sigma}_{\omega,0}(\psi_{k,\omega}-1), \qquad \psi^{\sigma}_{\omega,a} \equiv \psi^{\sigma}_{\omega,0}\psi_{k,a}. \tag{A.84}$$

and

$$\psi_{\omega,0} = \hat{r} + \frac{q\bar{k}}{\overline{\omega}}p^{id}\phi\sigma_{id} - \frac{1}{2}\left(\frac{q\bar{k}}{\overline{\omega}}\phi\sigma_{id}\right)^2 + \xi_d\frac{w\bar{l}(\bar{a})}{\overline{\omega}} - \frac{\bar{c}}{\overline{\omega}} \tag{A.85}$$

$$\psi_{\omega,\omega} = \left(\frac{q\bar{k}}{\overline{\omega}}p^{id}\phi\sigma_{id} - \left(\frac{q\bar{k}}{\overline{\omega}}\phi\sigma_{id}\right)^2\right)(\psi_{k,\omega}-1) - \xi_d\frac{w\bar{l}(\bar{a})}{\overline{\omega}} - \frac{\bar{c}}{\overline{\omega}}(\psi_{c,\omega}-1), \tag{A.86}$$

$$\psi_{\omega,a} = \left(\frac{q\bar{k}}{\overline{\omega}}p^{id}\phi\sigma_{id} - \left(\frac{q\bar{k}}{\overline{\omega}}\phi\sigma_{id}\right)^2\right)\psi_{k,a} + \xi_d\frac{w\bar{l}'(\bar{a})}{\overline{\omega}} - \frac{\bar{c}}{\overline{\omega}}\psi_{c,a}, \tag{A.87}$$

Demand for capital. The first-order condition for capital and consumption are given by

$$-V_{\omega\omega}\sigma^{id}_{i,t} = V_\omega p^{id}, \qquad V_\omega = c^{-\gamma}_{i,t}. \tag{A.88}$$

Using the fact that $dV_\omega dZ_{i,t} = V_{\omega\omega}\sigma^{id}_{i,t}dt$ and the expressions above, we can express the optimality condition for capital as follows:

$$p^{id}dt = \gamma\frac{dc_{i,t}}{c_{i,t}}dZ_{i,t}. \tag{A.89}$$

Up to first order, we have that $\frac{dc_{i,t}}{c_{i,t}}dZ_{i,t} = d\hat{c}_{i,t}dZ_{i,t} = \psi_{c,\omega}d\hat{\omega}_{i,t}dZ_{i,t}$. We can then write the expression above as follows:

$$p^{id} = \gamma\psi_{c,\omega}\left[\psi^{\sigma}_{\omega,0} + \psi^{\sigma}_{\omega,a}\hat{a}_{i,t} + \psi^{\sigma}_{\omega,\omega}\hat{\omega}_{i,t}\right]. \tag{A.90}$$

As the expression above must hold for all $\hat{a}$ and $\hat{\omega}_{i,t}$, then we must have $\psi^{\sigma}_{\omega,\omega} = \psi^{\sigma}_{\omega,a} = 0$. This implies that coefficients in the expansion for capital are given by

$$\psi_{k,\omega} = 1, \qquad \psi_{k,a} = 0. \tag{A.91}$$

The exposure to the business relative to total wealth is then the same for all entrepreneurs, in line with the results in Section 3:

$$\frac{qk_{i,t}}{\omega_{i,t}} = \frac{q\bar{k}}{\overline{\omega}}. \tag{A.92}$$

Using the fact that $V_{\omega\omega} = -\gamma c^{-\gamma-1}c_\omega$ and evaluating the first-order condition for capital at $(\overline{\omega}, \overline{a})$, we obtain

$$\gamma \frac{c_\omega(\overline{\omega}, \overline{a})}{c(\overline{\omega}, \overline{a})} qk(\overline{\omega}, \overline{a}) \phi \sigma_{id} = p^{id} \Rightarrow \frac{qk(\overline{\omega}, \overline{a})}{\overline{\omega}} = \frac{1}{\gamma \psi_{c,\omega}} \frac{p^{id}}{\phi \sigma_{id}}, \tag{A.93}$$

where we used that $\psi_{c,\omega} = \frac{c_\omega(\overline{\omega}, \overline{a})\overline{\omega}}{c(\overline{\omega}, \overline{a})}$.

The demand for capital can then be written as

$$\frac{qk_{i,t}}{n_{i,t}} = \frac{1 + (1 - \xi_d)\frac{h_{i,t}}{n_{i,t}}}{\gamma \psi_{c,\omega}} \frac{p^{id}}{\phi \sigma_{id}}. \tag{A.94}$$

Consumption. The envelope condition with respect to ω for problem (E.12) is given by

$$\hat{\rho} V_\omega = V_\omega \hat{r} + \frac{\mathbb{E}\left[dV_\omega\right]}{dt}. \tag{A.95}$$

Using Ito's lemma, we can write the expression above as follows

$$r = \rho + \gamma \left(\mu_A + \frac{1}{dt} \mathbb{E}[d\hat{c}_{i,t}] \right) - \frac{\gamma(\gamma+1)}{2} \sigma_A^2 - \frac{\gamma^2}{2} \mathbb{E}[d\hat{c}_{i,t}^2] - \lambda_d \left(\left(\frac{c_{i,t}^d}{c_{i,t}} \right)^{-\gamma} - 1 \right). \tag{A.96}$$

Up to first order, the expression above can be written as

$$\gamma \psi_{c,\omega} \left(\psi_{\omega,\omega} \hat{\omega}_{i,t} + \psi_{\omega,a} \hat{a}_{i,t} \right) + \gamma \lambda_d \left(\frac{\zeta(\overline{a})\overline{\omega}}{\overline{c}} \right)^{-\gamma} \left(\hat{\omega}_{i,t} (1 - \psi_{c,w}) + \left(\frac{\zeta'(\overline{a})}{\zeta(\overline{a})} - \psi_{c,a} \right) \hat{a} \right) = constant. \tag{A.97}$$

As the expression above must hold for all values of $\hat{\omega}_{i,t}$ and $\hat{a}_{i,t}$, we obtain $\psi_{\omega,\omega} = \psi_{\omega,a} = 0$. This implies the following conditions must hold:

$$\psi_{c,\omega} \psi_{\omega,\omega} + \lambda_d \left(\frac{\zeta(\overline{a})\overline{\omega}}{\overline{c}} \right)^{-\gamma} (1 - \psi_{c,\omega}) = 0, \qquad \psi_{c,\omega} \psi_{\omega,a} + \lambda_d \left(\frac{\zeta(\overline{a})\overline{\omega}}{\overline{c}} \right)^{-\gamma} \left(\frac{\zeta'(\overline{a})}{\zeta(\overline{a})} - \psi_{c,a} \right) = 0. \tag{A.98}$$

Using the expression for $\psi_{\omega,\omega}$, we obtain a quadratic equation for $\psi_{c,\omega}$:

$$\psi_{c,\omega}^2 - \psi_{c,\omega} \left[1 - \lambda_d \frac{\overline{\omega}}{\overline{c}} \left(\frac{\zeta(\overline{a})\overline{\omega}}{\overline{c}} \right)^{-\gamma} - \xi_d \frac{w\bar{l}(\overline{a})}{\overline{c}} \right] - \lambda_d \frac{\overline{\omega}}{\overline{c}} \left(\frac{\zeta(\overline{a})\overline{\omega}}{\overline{c}} \right)^{-\gamma} = 0. \tag{A.99}$$

The equation above has a positive and a negative solution, where the economically relevant solution is the positive one:

$$\psi_{c,\omega} = \frac{1}{2}\left[1 - \lambda_d \frac{\overline{\omega}}{\overline{c}}\left(\frac{\zeta(\overline{a})\overline{\omega}}{\overline{c}}\right)^{-\gamma} - \xi_d \frac{w\bar{l}(\overline{a})}{\overline{c}} + \sqrt{\left(1 - \lambda_d \frac{\overline{\omega}}{\overline{c}}\left(\frac{\zeta(\overline{a})\overline{\omega}}{\overline{c}}\right)^{-\gamma} - \xi_d \frac{w\bar{l}(\overline{a})}{\overline{c}}\right)^2 + 4\lambda_d \frac{\overline{\omega}}{\overline{c}}\left(\frac{\zeta(\overline{a})\overline{\omega}}{\overline{c}}\right)^{-\gamma}}\right]. \tag{A.100}$$

We will show next that $\psi_{c,\omega} < 1$. Assuming that $\psi_{c,\omega} > 1$, the expression above implies that

$$\sqrt{\left(1 - \lambda_d \frac{\overline{\omega}}{\overline{c}}\left(\frac{\zeta(\overline{a})\overline{\omega}}{\overline{c}}\right)^{-\gamma} - \xi_d \frac{w\bar{l}(\overline{a})}{\overline{c}}\right)^2 + 4\lambda_d \frac{\overline{\omega}}{\overline{c}}\left(\frac{\zeta(\overline{a})\overline{\omega}}{\overline{c}}\right)^{-\gamma}} > 1 + \lambda_d \frac{\overline{\omega}}{\overline{c}}\left(\frac{\zeta(\overline{a})\overline{\omega}}{\overline{c}}\right)^{-\gamma} + \xi_d \frac{w\bar{l}(\overline{a})}{\overline{c}}. \tag{A.101}$$

Squaring both sides of the inequality above, we obtain

$$\left(1 - \lambda_d \frac{\overline{\omega}}{\overline{c}}\left(\frac{\zeta(\overline{a})\overline{\omega}}{\overline{c}}\right)^{-\gamma} - \xi_d \frac{w\bar{l}(\overline{a})}{\overline{c}}\right)^2 + 4\lambda_d \frac{\overline{\omega}}{\overline{c}}\left(\frac{\zeta(\overline{a})\overline{\omega}}{\overline{c}}\right)^{-\gamma} > \left(1 + \lambda_d \frac{\overline{\omega}}{\overline{c}}\left(\frac{\zeta(\overline{a})\overline{\omega}}{\overline{c}}\right)^{-\gamma} + \xi_d \frac{w\bar{l}(\overline{a})}{\overline{c}}\right)^2. \tag{A.102}$$

Rearranging the expression above, we obtain

$$4\lambda_d \frac{\overline{\omega}}{\overline{c}}\left(\frac{\zeta(\overline{a})\overline{\omega}}{\overline{c}}\right)^{-\gamma} > 4\left(\lambda_d \frac{\overline{\omega}}{\overline{c}}\left(\frac{\zeta(\overline{a})\overline{\omega}}{\overline{c}}\right)^{-\gamma} + \xi_d \frac{w\bar{l}(\overline{a})}{\overline{c}}\right), \tag{A.103}$$

which is a contradiction. Therefore, we must have $0 < \psi_{c,\omega} < 1$.

We solve next for $\psi_{c,a}$. The coefficient $\psi_{c,a}$ satisfies the equation:

$$\psi_{c,\omega}\left(\xi_d \frac{w\bar{l}'(\overline{a})}{\overline{\omega}} - \frac{\overline{c}}{\overline{\omega}}\psi_{c,a}\right) + \lambda_d\left(\frac{\zeta(\overline{a})\overline{\omega}}{\overline{c}}\right)^{-\gamma}\left(\frac{\zeta'(\overline{a})}{\zeta(\overline{a})} - \psi_{c,a}\right) = 0. \tag{A.104}$$

Rearranging the expression above, we obtain

$$\psi_{c,a} = \frac{\psi_{c,\omega}\frac{\overline{c}}{\overline{\omega}}\xi_d \frac{w\bar{l}'(\overline{a})}{\overline{c}} + \lambda_d\left(\frac{\zeta(\overline{a})\overline{\omega}}{\overline{c}}\right)^{-\gamma}\frac{\zeta'(\overline{a})}{\zeta(\overline{a})}}{\psi_{c,\omega}\frac{\overline{c}}{\overline{\omega}} + \lambda_d\left(\frac{\zeta(\overline{a})\overline{\omega}}{\overline{c}}\right)^{-\gamma}}. \tag{A.105}$$

For ξ_d sufficiently small, the expression above is positive, as $\zeta'(a)$, the consumption-wealth ratio in the absence of labor income risk, is positive.

We can then write consumption as follows

$$c_{i,t} = c(\overline{\omega}, \overline{a})e^{\psi_{c,a}(a_{i,t}-\overline{a})}\left(\frac{\omega_{i,t}}{\overline{\omega}}\right)^{\psi_{c,\omega}} \equiv f_c(a_{i,t})\omega_{i,t}^{\psi_{c,\omega}}, \tag{A.106}$$

where the second equality defines the age-dependent function $f_c(a)$. Note that $f_c(a)$ is increasing in a for ξ_d sufficiently small.

It remains to solve for the consumption-wealth ratio at $(\overline{\omega}, \overline{a})$. From the envelope condition, we obtain

$$r = \rho + \gamma\left(\mu_A + \psi_{c,\omega}\psi_{\omega,0} + \psi_{c,a}\right) - \frac{\gamma(\gamma+1)}{2}\sigma_A^2 - \frac{\gamma^2}{2}(\psi_{c,\omega}\psi_{\omega,0}^{\sigma})^2 - \lambda_d\left(\left(\frac{\zeta(\overline{a})\overline{\omega}}{\overline{c}}\right)^{-\gamma} - 1\right). \tag{A.107}$$

where $\psi_{\omega,0} = r + \gamma\sigma_A^2 - \mu_A + \frac{q\bar{k}}{\overline{\omega}}p^{id}\phi\sigma_{id} - \frac{1}{2}\left(\frac{q\bar{k}}{\overline{\omega}}\phi\sigma_{id}\right)^2 + \xi_d\frac{w\bar{l}(\overline{a})}{\overline{\omega}} - \frac{\overline{c}}{\overline{\omega}}$.

Rearranging the expression above, we obtain

$$\psi_{c,a} = \psi_{c,\omega}\left(\frac{\overline{c}}{\overline{\omega}} - \xi_d\frac{w\bar{l}(\overline{a})}{\overline{\omega}}\right) - \overline{r}_d - (1-\psi_{c,\omega})\left(\mu_A - \frac{\gamma\sigma_A^2}{2}\right) + \frac{\lambda_d}{\gamma}\left(\left(\frac{\zeta(\overline{a})\overline{\omega}}{\overline{c}}\right)^{-\gamma} - 1\right), \tag{A.108}$$

where

$$\overline{r}_d \equiv \frac{1}{\gamma}\rho + \left(\psi_{c,\omega} - \frac{1}{\gamma}\right)\left(r + \frac{(p^{ag})^2}{2\gamma} + \frac{(p^{id})^2}{2\gamma\psi_{c,\omega}}\right). \tag{A.109}$$

Using the expressions for $\psi_{c,\omega}$ and $\psi_{c,a}$, we obtain a non-linear equation for $\overline{c}$ given $\overline{\omega}$. Note that if $\xi_d = \lambda_d = 0$, we recover a linearized version of the equation determining $\frac{\zeta'(a)}{\zeta(a)}$, Equation (A.19).

Wealth dynamics and the approximation point. The law of motion of $\hat{\omega}_{i,t}$ can be written as

$$d\hat{\omega}_{i,t} = \left[\psi_{\omega,0} + \psi_{\omega,a}\hat{a}_{i,t} + \psi_{\omega,\omega}\hat{\omega}_{i,t}\right]dt + \frac{p^{id}}{\gamma\psi_{c,\omega}}dZ_{i,t}, \tag{A.110}$$

where

$$\psi_{\omega,\omega} = \frac{\overline{c}}{\overline{\omega}}\left[1 - \psi_{c,\omega} - \xi_d\frac{w\bar{l}(\overline{a})}{\overline{c}}\right], \qquad \psi_{\omega,a} = \lambda_d\left(\frac{\zeta(\overline{a})\overline{\omega}}{\overline{c}}\right)^{-\gamma}\frac{\overline{c}}{\overline{\omega}}\frac{\xi_d\frac{w\bar{l}'(\overline{a})}{\overline{c}} - \frac{\zeta'(\overline{a})}{\zeta(\overline{a})}}{\psi_{c,\omega}\frac{\overline{c}}{\overline{\omega}} + \lambda_d\left(\frac{\zeta(\overline{a})\overline{\omega}}{\overline{c}}\right)^{-\gamma}}. \tag{A.111}$$

We will show next that $\psi_{\omega,\omega} < 0$. For the sake of reaching a contradiction, assume that $\psi_{\omega,\omega} \geq 0$. This implies that the following condition is satisfied:

$$1 + \lambda_d\frac{\overline{\omega}}{\overline{c}}\left(\frac{\zeta(\overline{a})\overline{\omega}}{\overline{c}}\right)^{-\gamma} - \xi_d\frac{w\bar{l}(\overline{a})}{\overline{c}} \geq \sqrt{\left(1 - \lambda_d\frac{\overline{\omega}}{\overline{c}}\left(\frac{\zeta(\overline{a})\overline{\omega}}{\overline{c}}\right)^{-\gamma} - \xi_d\frac{w\bar{l}(\overline{a})}{\overline{c}}\right)^2 + 4\lambda_d\frac{\overline{\omega}}{\overline{c}}\left(\frac{\zeta(\overline{a})\overline{\omega}}{\overline{c}}\right)^{-\gamma}} \tag{A.112}$$

Squaring both sides of the inequality and rearranging, we obtain

$$4\lambda_d \frac{\overline{\omega}}{\overline{c}} \left(\frac{\zeta(\overline{a})\overline{\omega}}{\overline{c}} \right)^{-\gamma} \left(1 - \xi_d \frac{w\bar{l}(\overline{a})}{\overline{c}} \right) \geq 4\lambda_d \frac{\overline{\omega}}{\overline{c}} \left(\frac{\zeta(\overline{a})\overline{\omega}}{\overline{c}} \right)^{-\gamma}, \tag{A.113}$$

which is a contradiction. Thus, we must have $\psi_{\omega,\omega} < 0$.

We solve next for $\overline{\omega}$. Let $\overline{\omega}(a) \equiv \exp\left(\mathbb{E}[\log \omega_{i,t} | a_{i,t} = a]\right)$, then

$$\frac{d \log \overline{\omega}(a)}{da} = \psi_{\omega,0} + \psi_{\omega,a}(a - \overline{a}) + \psi_{\omega,\omega}(\log \overline{\omega}(a) - \log \overline{\omega}). \tag{A.114}$$

Solving the differential equation above, we obtain

$$\log \overline{\omega}(a) = e^{\psi_{\omega,\omega} a} \log \overline{\omega}(0) + \int_0^a \left[\tilde{\psi}_{\omega,0} + \psi_{\omega,a} a' \right] e^{\psi_{\omega,\omega}(a-a')} da'. \tag{A.115}$$

Using the fact that the wealth of entrepreneurs at date T is left as a bequest to the next generation, we can pin down $\overline{\omega}(0)$. Evaluating the expression above at $a = \overline{a}$, for a given reference age $\overline{a}$, we obtain an equation for $\overline{\omega}$. □

B Data

In this appendix, we discuss our empirical measures in more detail. For an extensive discussion of the Townsend Thai Monthly Survey and the derivation of entrepreneurs' balance sheet information from the survey questionnaire, see Samphantharak and Townsend (2010).

B.1 Sample selection and variable definition

The dataset includes information on both economic and demographic variables. Economic variables include households' assets and liabilities, financial wealth, business and labor income, and consumption. Demographic and geographic variables consist of the age of the household's head, year, number of household members, number of children in the household, and province in which the household is located. We focus on a sample of households from age 25 to 80. We drop observations for households without information on age or financial wealth, resulting in an unbalanced panel of 796 households over 14 years, from 1999 to 2012.

Business exposure. The Townsend Thai Monthly Survey contains detailed information on entrepreneurs' assets, including fixed assets, inventories, and financial assets. We classify these assets into business assets and non-business assets, which we treat as safe. The value of the business includes inventories, livestock, agricultural assets, business assets, and household assets. We follow Samphantharak and Townsend (2018) and include the value of household assets (cars, pick-up trucks, fishing boats, and so on) as part of the business. The motivation for this choice is that many of these assets are also used by households in their production activities. The value of safe assets includes cash in hand, account receivables, deposits at financial institutions, ROSCA, other lending, prepaid insurance, and land. Given this breakdown, we can compute the fraction of financial wealth (total asset net of liabilities) invested in the business, which we use as our measure of risk-taking.

Return on business activity. Given estimates for the value of entrepreneurs' businesses and the flow of business income generated in a given period, we can compute the return on assets (ROA) as the ratio of business income over the value of the business. ROA is a common accounting measure used to capture the profitability of business activities.

Human wealth. We compute our empirical measure of human wealth in a way analogous to its counterpart in the model as the present discounted value of future expected labor income. This requires us to specify the discount rate and the expected value of future labor. We use the same discount rate used in the model, as shown in Equation (15). The expected value of future labor income is age-dependent and computed as the average labor income for households of that age. As in the construction of life-cycle profiles discussed below, we use trimmed means to limit the influence of outliers. Consistent with the model's assumption, human wealth has no idiosyncratic component, as it corresponds to the present discounted value of future labor income across different households conditional on age.

B.2 Standard-error adjusted for generated regressors

To take into account the effect of generated regressors, we embed our two-stage procedure into a GMM framework, where all parameters are estimated simultaneously, and the standard errors properly account for the uncertainty in the coefficients.

GMM formulas. We will use the general GMM formulas to compute the standard errors. The moment conditions are given by

$$\mathbb{E}[f(z_t, \theta)] = 0, \tag{B.1}$$

where $f(\cdot)$ is a M-vector of moments, z_t is a Z-vector of data, and θ denotes a K-vector of parameters, where $M \geq K$. The GMM estimate of θ, $\hat{\theta}$, satisfies the condition

$$a_T g_T(\hat{\theta}) = 0, \tag{B.2}$$

for a given $K \times M$ matrix a_T and $g_T(\theta) \equiv \frac{1}{T}\sum_{t=1}^{T} f(z_t, \theta)$.

The asymptotic distribution of the GMM estimate is:

$$\sqrt{T}(\hat{\theta} - \theta) \to \mathcal{N}[0, (ad)^{-1} aSa'(ad)^{-1'}], \tag{B.3}$$

where $d = \mathbb{E}[\frac{\partial f(z_t,\theta)}{\partial \theta'}]$ and S is given by

$$S = \sum_{j=-\infty}^{\infty} \mathbb{E}[f(z_t, \theta) f(x_{t-j}, \theta)']. \tag{B.4}$$

Two-pass regressions. Our two-step procedure consists of first estimating a time-series regression for each entrepreneur:

$$R_{it} = a_i + \beta_i f_{it} + \epsilon_{it}, \tag{B.5}$$

for $i = 1, \ldots, N$, where $t = 1, \ldots, T$. The variance of ϵ_t is denoted by σ_i^2. We assume that $\mathbb{E}[\epsilon_t \epsilon_{t+j}'] = \Sigma$ for $j = 0$ and equal to zero for $j \neq 0$. Therefore, we assume that errors are not autocorrelated, but we allow them to be correlated across entrepreneurs, standard assumptions in the cross-sectional asset pricing literature (see, e.g., Shanken 1992). In the second stage, we regress average returns on betas and idiosyncratic variance:

$$\overline{R}_i = \lambda_0 + \lambda_1 \hat{\beta}_i + \lambda_2 \hat{\sigma}_i^2 + u_i. \tag{B.6}$$

Mapping to the GMM framework. Consider the following $M = 4N$-moment vector:

$$f(z_t, \theta) = \begin{bmatrix} R_t - a - \beta \cdot f_t \\ (R_t - a - \beta \cdot f_t) \cdot f_t \\ (R_t - a - \beta \cdot f_t) \cdot (R_t - a - \beta \cdot f_t) - \sigma^2 \\ R_t - \lambda_0 \mathbf{1}_N - \lambda_1 \beta - \lambda_2 \sigma^2 \end{bmatrix}, \tag{B.7}$$

where $R_t = [R_{1t}, \ldots, R_{Nt}]'$, $a = [a_1, \ldots, a_N]'$, $\beta = [\beta_1, \ldots, \beta_N]'$, $f_t = [f_{1t}, \ldots, f_{Nt}]'$, and $\sigma^2 = [\sigma_1^2, \ldots, \sigma_N^2]$. The operator $\cdot$ denotes element-wise multiplication.

The $3(N+1) \times 4N$ matrix a_T is given by

$$a_T = \begin{bmatrix} I_{3N} & \mathbf{0}_{3N\times N} \\ \mathbf{0}_{1\times 3N} & -\mathbf{1}'_N \\ \mathbf{0}_{1\times 3N} & -\beta' \\ \mathbf{0}_{1\times 3N} & -\sigma^{2'} \end{bmatrix}. \tag{B.8}$$

The $4N \times 3(N+1)$ matrix d is given by

$$d = \begin{bmatrix} -I_N & -diag(\mathbb{E}[f_t]) & \mathbf{0}_{N\times N} & \mathbf{0}_{N\times 1} & \mathbf{0}_{N\times 1} & \mathbf{0}_{N\times 1} \\ -diag(\mathbb{E}[f_t]) & -diag(\mathbb{E}[f_t \cdot f_t]) & \mathbf{0}_{N\times N} & \mathbf{0}_{N\times 1} & \mathbf{0}_{N\times 1} & \mathbf{0}_{N\times 1} \\ \mathbf{0}_{N\times N} & \mathbf{0}_{N\times N} & -I_N & \mathbf{0}_{N\times 1} & \mathbf{0}_{N\times 1} & \mathbf{0}_{N\times 1} \\ 0_{N\times N} & -\lambda_1 I_N & -\lambda_2 I_N & -\mathbf{1}_N & -\beta & -\sigma^2 \end{bmatrix}. \tag{B.9}$$

The matrix S is given by

$$S = \mathbb{E}\left[\begin{bmatrix} \epsilon_t \\ \epsilon_t \cdot f_t \\ \epsilon_t \cdot \epsilon_t - \sigma^2 \\ \beta \cdot (f_t - \mathbb{E}[f_t]) + \epsilon_t \end{bmatrix} \begin{bmatrix} \epsilon_t \\ \epsilon_t \cdot f_t \\ \epsilon_t \cdot \epsilon_t - \sigma^2 \\ \beta \cdot (f_t - \mathbb{E}[f_t]) + \epsilon_t \end{bmatrix}'\right], \tag{B.10}$$

where we used the fact that errors are not autocorrelated and $\lambda_0 + \lambda_1\beta_i + \lambda_2\sigma_i^2 = a_i + \beta_i\mathbb{E}[f_{it}]$.

We can write the matrix above as follows:

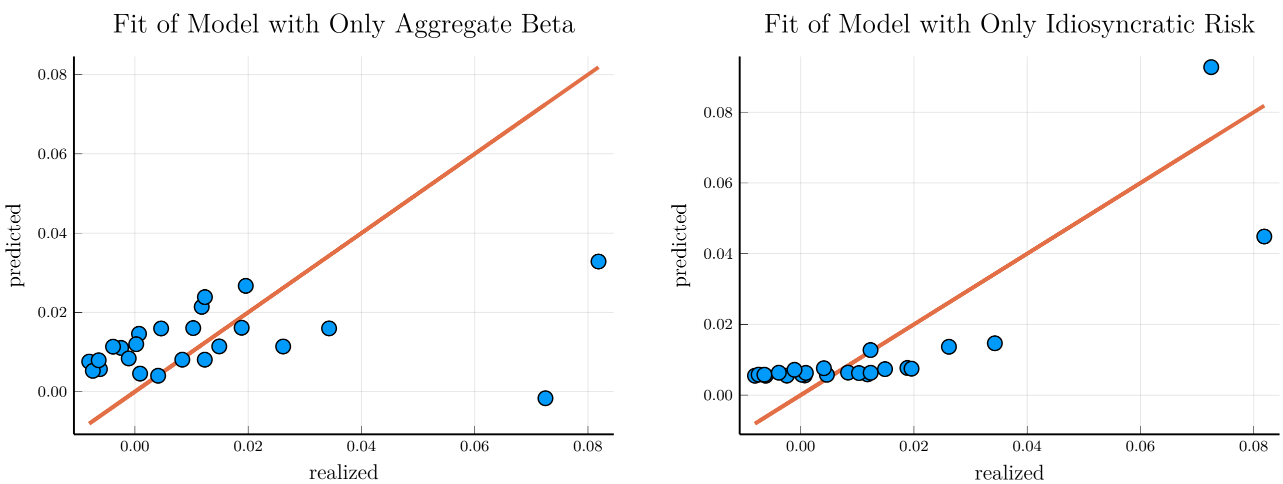
$$S = \begin{bmatrix} \Sigma & \Sigma diag(\mathbb{E}[f_t]) & \mathbb{E}[\epsilon_t(\epsilon_t \cdot \epsilon_t)'] & \Sigma \\ \Sigma diag(\mathbb{E}[f_t]) & \Sigma \cdot \mathbb{E}[f_t f_t'] & diag(\mathbb{E}[f_t])\mathbb{E}[\epsilon_t(\epsilon_t \cdot \epsilon_t)'] & diag(\mathbb{E}[f_t])\Sigma \\ \mathbb{E}[(\epsilon_t \cdot \epsilon_t)\epsilon_t'] & \mathbb{E}[(\epsilon_t \cdot \epsilon_t)\epsilon_t']diag(\mathbb{E}[f_t]) & \mathbb{E}[(\epsilon_t \cdot \epsilon_t - \sigma^2) \cdot (\epsilon_t \cdot \epsilon_t - \sigma^2)] & \mathbb{E}[(\epsilon_t \cdot \epsilon_t)\epsilon_t'] \\ \Sigma & \Sigma diag(\mathbb{E}[f_t]) & \mathbb{E}[\epsilon_t(\epsilon_t \cdot \epsilon_t)'] & \beta\beta' \cdot \Sigma_f + \Sigma \end{bmatrix},$$

where $\Sigma_f \equiv \mathbb{E}[(f_t - \mathbb{E}[f_t])(f_t - \mathbb{E}[f_t])']$. We can then compute the variance-covariance matrix of $\hat{\theta}$ as follows:

$$var(\hat{\theta}) = \frac{1}{T}(ad)^{-1}aSa'(ad)^{-1'}, \tag{B.11}$$

where the variance of $[\lambda_0, \lambda_1, \lambda_2]'$ is given by the 3×3 matrix in the lowest right corner of the matrix above. An analogous derivation applies to the case where only β_i or σ_i^2 are included in the cross-sectional regression.

Figure B.1: Realized vs. predicted returns: one-factor models



Note: The left (right) panel shows a scatter plot of predicted returns of the single factor with aggregate beta (idiosyncratic variance) as a factor against the realized returns for the portfolio-level analysis.

B.3 The fit of the single-factor models

In Section 2, Figure 2 shows the fit of the two-factor model. We consider here the performance of the one-factor models using either aggregate beta or idiosyncratic variance as the factor. Figure B.1 shows that both models struggle to properly account for the dispersion in entrepreneurial returns, which shows the importance of considering a two-factor model.

B.4 Life-cycle profiles

When computing life-cycle profiles for a given variable, we aggregate households into 15 age groups. The thresholds determining each group are chosen such that groups have roughly the same number of households. To limit the influence of outliers, we compute trimmed means with a trimming parameter of 7.5% in each side. We trim the data in a similar manner before running the regressions.

The life-cycle profiles presented in the main text are computed without any controls, which we denote by *raw* moments. We show next that controlling for year fixed-effects or demographics variables maintains our results essentially unchanged.

Let $z_{i,k,t}$ denote variable z for household i in age group k at year t. Consider the following process for $z_{i,k,t}$:

$$z_{i,k,t} = \alpha_t + age_k + \delta' x_{i,k,t} + u_{i,k,t}, \tag{B.12}$$

where α_t represents the year fixed effect, age_k is the age-group effect, and $x_{i,k,t}$ is a vector of demographic and geographic controls, which includes the size of the household, the number of children in the household, and a set of province dummies.

The raw age-group effect is given by

$$age_k^{raw} = \mathbb{E}\left[z_{i,k',t}|k' = k\right]. \tag{B.13}$$

The raw age-group effect can be estimated by taking averages by age or by regressing $z_{i,k,t}$ on a set of dummies for age groups.

We follow Kaplan (2012) and define the age-group effect controlling for year fixed-effects as follows:

$$age_k^{year-FE} = \frac{1}{\overline{T}}\sum_{t=1}^{\overline{T}} \mathbb{E}\left[z_{i,k',t'}|k' = k, t' = t\right]. \tag{B.14}$$

We can estimate $age_k^{year-FE}$ by running a regression on a set of dummy of age-group and year fixed-effects and computing the predicted value of the regression evaluated at age k and at an "average" year. Similarly, we define the age-group effect controlling for year fixed-effects and demographic/geographic controls as follows:

$$age_k^{year-FE+dem} = \frac{1}{\overline{T}}\sum_{t=1}^{\overline{T}} \mathbb{E}\left[z_{i,k',t'}|k' = k, t' = t, x_{i,k',t'} = \overline{x}_{k,t}\right], \tag{B.15}$$

where $\overline{x}_{k,t}$ denotes the average value of the controls $x_{i,k,t}$ conditional on age group k and year t. We can estimate $age_k^{year-FE+dem}$ from the full regression with year and age-group fixed-effects and demographic controls.

Figure **??** shows our estimates of the different age-group effects for the variables of interest. The grey dashed lines show the 95% confidence interval for the raw estimates. We cluster standard errors by household to allow shocks to be correlated over time. The life-cycle patterns obtained under the raw measure are essentially identical to the ones obtained after controlling for year fixed-effects and demographic variables.

B.5 Life-cycle profiles and average business returns

We have seen that an entrepreneur's business exposure, or the fraction of financial wealth invested in the business, varies substantially over the life cycle. We consider next the effect of average return on the business on risk-taking and savings decisions.

Table 3 shows regressions of business exposure and the consumption-wealth ratio on a set of age dummies and the average return on the business. We also consider specifications where we control for year fixed-effects and demographic controls. Standard errors are clustered at the household level as before. The table shows that the age effects are strongly significant and account for a large fraction of the variation in both risk-taking and savings decisions, with the regression R^2 ranging from 60% to 70%.

The effect of the average return on risk-taking is positive and significant. Note that adding the average return on the business affects only marginally the regression R^2. This is consistent with the approach in the model where cross-sectional differences are generated by variables with a strong life-cycle component instead of differences in expected returns.

The expected return's effect on the consumption-wealth ratio is positive and strongly significant in the case with year fixed-effects and demographic controls and marginally insignificant in the other cases. The positive effect is consistent with the income effect dominating the substitution effect in the savings decision, in line with the calibrated model.

C Derivations

C.1 The Optimal Contract

In this appendix, we consider the contracting problem in more detail. In particular, we show that the market structure assumed in Section 3, where entrepreneurs have access to a riskless bond and both aggregate and idiosyncratic insurance, corresponds to a specific implementation of the optimal contract allocation. The derivation closely follows the work of Di Tella (2017), and it is provided for completeness.

C.1.1 Moral hazard

We assume that the aggregate productivity shock Z_t and the individual cumulative return $R_{i,t}$ are publicly observable, but the idiosyncratic investment shock $Z_{i,t}$ is privately observed by entrepreneur i. Moreover, the entrepreneur may secretly divert capital at rate $\varsigma_{i,t}$. The return on the business is then given by

$$dR_{i,t} = \left[\frac{y_{i,t} - w_t l_{i,t} - \iota_{i,t} k_{i,t}}{q_t k_{i,t}} + \frac{\dot{q}_t}{q_t} + \mu_A + \Phi(\iota_{i,t}) - \delta - \varsigma_{i,t}\right] dt + \sigma_A dZ_t + \sigma_{id} dZ_{i,t}. \tag{C.1}$$

Because $Z_{i,t}$ and $\varsigma_{i,t}$ are not publicly observable, a principal contracting with the entrepreneur cannot determine whether a low return results from a negative investment shock or positive stealing. The optimal contract ensures that it is incentive-compatible for the entrepreneur to choose $\varsigma_{i,t} = 0$ at all times.[37] Note that the expected return coincides with the one in condition (5) in the case of no stealing.

Diverted capital can be sold in the market, but a fraction of $1 - \phi$ is lost in the process. The sale proceedings are invested in a hidden account, which is remunerated at the risk-free rate

[37]This is typical of cash-flow diversion models, see, e.g., DeMarzo and Sannikov (2006) and DeMarzo and Fishman (2007).

r_t. The entrepreneur's hidden savings $S_{i,t}$ evolve as follows:

$$dS_{i,t} = r_t S_{i,t} dt + \phi \tilde{q}_t k_{i,t} \varsigma_{i,t} dt. \tag{C.2}$$

C.1.2 The optimal contract problem

Consumption, investment, and factor demands are contractible. A contract between a principal and an entrepreneur is then given by $(\tilde{c}_i, \iota_i, k_i, l_i, \tilde{F}_i)$, where all variables are adapted to the filtration generated by (Z, R_i), and $\tilde{F}_{i,t}$ denotes the transfer to the principal. Entrepreneurs cannot commit to long-term contracts. At any point, an entrepreneur can settle her promises with the principal, transfer the funds from the hidden account to her bank account, and offer the principal a new contract. Therefore, contracts are effectively short-term and the contract can be redefined at every period.

The continuation value to the principal is given by

$$\tilde{J}_{i,t} = \mathbb{E}_t \left[\int_t^{s_i+T} \frac{\pi_z}{\pi_t} \tilde{F}_{i,z} dz \right], \tag{C.3}$$

where the expectation is taken under no stealing, $\varsigma_i = 0$ and π_t corresponds to the principal's SDF, which evolves according to $d\pi_t = -r_t \pi_t dt - p_t^{ag} \pi_t dZ_t$, given the processes for r_t and p_t^{ag}.[38]

To compute the law of motion of $\tilde{J}_{i,t}$, let $G_{i,t}$ denote a martingale defined as follows

$$G_{i,t} = \int_{s_i}^t \pi_z d\tilde{F}_{i,z} + \mathbb{E}_t \left[\int_t^{s_i+T} \pi_z d\tilde{F}_{i,z} \right], \tag{C.4}$$

where $G_{i,s_i} = \mathbb{E}_{s_i}[G_{i,t}]$. By the martingale representation theorem, there exists $\sigma^Z_{G_i,t}$ and $\sigma^R_{G_i,t}$ such that

$$\pi_t \tilde{F}_{i,t} dt + d(\pi_t \tilde{J}_{i,t}) = \pi_t \sigma^Z_{G_i,t} dZ_t + \pi_t \sigma^R_{G_i,t} (dR_{i,t} - \mathbb{E}_t[dR_{i,t}]). \tag{C.5}$$

Applying Ito's lemma on $\pi_t \tilde{J}_{i,t}$ and combining with the expression above, we obtain

$$d\tilde{J}_{i,t} = \left[r_t \tilde{J}_{i,t} + p_t^{ag} (\sigma^Z_{J_i,t} + \sigma^R_{J_i,t} \sigma_A) - \tilde{F}_{i,t} \right] dt + \sigma^Z_{J_i,t} dZ_t + \sigma^R_{J_i,t} (dR_{i,t} - \mathbb{E}_t[dR_{i,t}]), \tag{C.6}$$

where $\sigma^Z_{J_i,t} = \sigma^Z_{G_i,t} + p_t^{ag} \tilde{J}_{i,t}$ and $\sigma^R_{J_i,t} = \sigma^R_{G_i,t}$.

[38] When contracting with a wage earner, the relevant SDF is $\pi_t = e^{-\rho_w t} \tilde{c}_{j,t}^{-\gamma}$. In a stationary equilibrium, consumption follows the process $d\tilde{c}_{j,t} = \mu_A \tilde{c}_{j,t} dt + \sigma_A \tilde{c}_{j,t} dZ_t$, then $d\pi_t = -\left[\rho_w + \gamma \mu_A - \gamma(\gamma+1) \frac{\sigma_A^2}{2} \right] dt - \gamma \sigma_A dZ_t$, where the drift corresponds to the interest rate r_t and the diffusion term corresponds to the the price of aggregate risk p_t^{ag}.

The financial wealth of entrepreneur i is defined as $\tilde{n}_{i,t} \equiv \tilde{b}_{i,t} + \tilde{q}_t k_{i,t} - \tilde{J}_{i,t}$, which corresponds to the sum of holdings of the risk-free asset $\tilde{b}_{i,t}$ and the value of the business $\tilde{q}_t k_{i,t}$, net of the promised payments to the principal $\tilde{J}_{i,t}$. The law of motion of financial wealth is given by

$$d\tilde{n}_{i,t} = \left[r_t \tilde{b}_{i,t} + \tilde{w}_t \bar{l}_{i,t} - \tilde{F}_{i,t} - \tilde{c}_{i,t} \right] dt + \tilde{q}_t k_{i,t} dR_{i,t} - d\tilde{J}_{i,t}. \tag{C.7}$$

Combining (C.6) and (C.7), and assuming $\varsigma_{i,t} = 0$, we obtain

$$d\tilde{n}_{i,t} = \left[r_t(\tilde{n}_{i,t} - \tilde{q}k_{i,t}) + \tilde{q}_t k_{i,t} \mu^R_{i,t} - p^{ag}_t (\sigma^Z_{J_i,t} + \sigma^R_{J_i,t}\sigma_A) + \tilde{w}_t \bar{l}_{i,t} - \tilde{c}_{i,t} + \left(-\tilde{q}_t k_{i,t} + \phi \tilde{q}_t k_{i,t} + \frac{\tilde{\theta}^{id}_t}{\sigma_{id}} \right) \varsigma_{i,t} \right] dt$$
$$+ \left(\tilde{q}_t k_{i,t} \sigma_A - \left(\sigma^Z_{J_i,t} + \sigma^R_{J_i,t}\sigma_A \right) \right) dZ_t + \left(\tilde{q}_t k_{i,t} \sigma_{id} - \sigma^R_{J_i,t}\sigma_{id} \right) dZ_{i,t}. \tag{C.8}$$

By imposing $\varsigma_{i,t} = 0$ and defining $\tilde{\theta}^{ag}_{i,t} \equiv \sigma^Z_{J_i,t} + \sigma^R_{J_i,t}\sigma_A$ and $\tilde{\theta}^{id}_{i,t} \equiv \sigma^R_{J_i,t}\sigma_{id}$, we obtain the law of motion of financial wealth presented in Section 3. Note that the transfers to the principal $\tilde{F}_{i,t}$ only affect the law of motion of $\tilde{n}_{i,t}$ through the diffusion terms of the principal's continuation value $\tilde{J}_i$. Therefore, we can write the contract in terms of $(\tilde{\theta}^{ag}_{i,t}, \tilde{\theta}^{id}_{i,t})$ instead of $\tilde{F}_{i,t}$.[39] The entrepreneur's problem can then be written as

$$\rho \tilde{V}_t = \max_{\tilde{c}_i, \tilde{\theta}^{ag}_t, \tilde{\theta}^{id}_t, k_i, l_i, \iota_i} \frac{c^{1-\gamma}_{i,t}}{1-\gamma} + \mathbb{E}_t \left[dV_t \right], \tag{C.9}$$

subject to the law of motion of financial wealth, $\tilde{n}_{i,t} \geq -\tilde{h}_{i,t}$, and the incentive-compatibility (IC) constraint

$$0 \in \arg\max_{\varsigma_{i,t} \geq 0} \left\{ \frac{c^{1-\gamma}_{i,t}}{1-\gamma} + \mathbb{E}_t \left[dV_t \right] \right\}. \tag{C.10}$$

Applying Ito's lemma to the value function, we can write the IC constraint as follows

$$- \tilde{q}_t k_{i,t} + \phi \tilde{q}_t k_{i,t} + \frac{\tilde{\theta}^{id}_t}{\sigma_{id}} \leq 0 \Rightarrow \tilde{\theta}^{id}_{i,t} \leq (1-\phi)\tilde{q}_t k_{i,t} \sigma_{id}, \tag{C.11}$$

where we used the fact that $V_{n,t}$ is positive.

Therefore, the optimal contract problem, where entrepreneurs choose transfers to a principal, is equivalent to problem (9), where entrepreneurs have access to aggregate and idiosyncratic insurance subject to the skin-in-the-game constraint (7).

[39] This reformulation also avoids the issue that the path of transfers $\tilde{F}_{i,t}$ is not uniquely determined, as an entrepreneur can, for instance, borrow from the principal and invest in the risk-free asset without affecting her utility.

C.2 Numerical solution of the KFE

We compute the KFE solution using a finite-difference scheme. We consider first the case of a stationary equilibrium and then discuss the solution in the case of a time-dependent KFE.

C.2.1 Stationary KFE

Consider the case of a stationary solution to the KFE. We solve the PDE (27) using a finite-differences method. As in the proof of Lemma 2, we assume that n takes values in the grid $\{n_1, n_2, \ldots, n_I\}$ and age a takes values in the grid $\{a_1, a_2, \ldots, a_K\}$, with $a_1 = 0$ and $a_K = T$. We adopt the following (upwind) difference scheme for $1 < i < I$ and $k < K$

$$\begin{aligned}\frac{f^{i,k+1} - f^{i,k}}{\Delta a} = -\frac{(\mu_n^{i,k})^+ f^{i,k} - (\mu_n^{i-1,k})^+ f^{i-1,k}}{\Delta n} - \frac{(\mu_n^{i+1,k})^- f^{i+1,k} - (\mu_n^{i,k})^- f^{i,k}}{\Delta n} \\ + \frac{(\sigma_n^{i+1})^2 f^{i+1,k} - 2(\sigma_n^i)^2 f^{i,k} + (\sigma_n^{i-1})^2 f^{i-1,k}}{2\Delta n^2},\end{aligned} \tag{C.12}$$

where $f^{i,k} \equiv f(n_i, a_k)$, $\mu_n^{i,k} \equiv \mu_n(n^i, a^k)$, $\sigma_n^i \equiv \sigma_n(n^i)$, $(x)^+ = \max\{x, 0\}$, and $(x)^- = \min\{x, 0\}$.

Rearranging the expression above and collecting terms, we obtain

$$f^{i,k+1} = \pi_u^{i-1,k} f^{i-1,k} + \pi_s^{i,k} f^{i,k} + \pi_d^{i+1,k} f^{i+1,k}, \tag{C.13}$$

where

$$\pi_u^{i,k} = (\mu_n^{i,k})^+ \frac{\Delta a}{\Delta n} + \frac{(\sigma_n^i)^2}{2} \frac{\Delta a}{\Delta n^2}, \qquad \pi_d^{i,k} = -(\mu_n^{i,k})^- \frac{\Delta a}{\Delta n} + \frac{(\sigma_n^i)^2}{2} \frac{\Delta a}{\Delta n^2}, \qquad \pi_s^{i,k} = 1 - \left(|\mu_n^{i,k}| \frac{\Delta a}{\Delta n} + (\sigma_n^i)^2 \frac{\Delta a}{\Delta n^2}\right).$$

The above scheme converges if, for all (i, k), the following variant of the Courant-Friedrichs-Lewy (CFL) condition holds

$$|\mu_n^{i,k}| \frac{\Delta a}{\Delta n} + (\sigma_n^i)^2 \frac{\Delta a}{\Delta n^2} \leq 1. \tag{C.14}$$

We adopt a reflecting boundary at n_1 and n_I

$$f^{1,k+1} = (\pi_d^{1,k} + \pi_s^{1,k}) f^{1,k} + \pi_d^{2,k} f^{2,k} \tag{C.15}$$

$$f^{I,k+1} = \pi_u^{I,k} f^{I-1,k} + (\pi_s^{I,k} + \pi_u^{I,k}) f^{I,k}. \tag{C.16}$$

In matrix form, we can write

$$\underbrace{f^{k+1} = \Pi^k f^k,}_{\text{explicit method}} \qquad \underbrace{f^{k+1} = [2I - \Pi^k]^{-1} f^k,}_{\text{implicit method}} \tag{C.17}$$

where

$$\Pi^k = \begin{bmatrix} \pi_d^{1,k} + \pi_s^{1,k} & \pi_d^{2,k} & 0 & 0 & \cdots & 0 & 0 & 0 \\ \pi_u^{1,k} & \pi_s^{2,k} & \pi_d^{3,k} & 0 & \cdots & 0 & 0 & 0 \\ 0 & \pi_u^{2,k} & \pi_s^{2,k} & \pi_d^{3,k} & \cdots & 0 & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \cdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \pi_s^{I-2,k} & \pi_d^{I-1,k} & 0 \\ 0 & 0 & 0 & 0 & \cdots & \pi_u^{I-2,k} & \pi_s^{I-1,k} & \pi_d^{I,k} \\ 0 & 0 & 0 & 0 & \cdots & 0 & \pi_u^{I-1,k} & \pi_s^{I,k} + \pi_u^{I,k} \end{bmatrix}. \tag{C.18}$$

Notice that $(\Pi^k)'$ is a stochastic matrix, so we can interpret the coefficients as probabilities. It remains to specify the boundary condition in the age dimension. An entrepreneur with age T and financial wealth n leaves as bequest $e^{-gT}n$ for each child. The quantity $e^{-gT}n$ may not be in the grid, so we linearly interpolate between the points in the grid. For any point n_i in the grid, there exists coefficients (i', b_i) such that

$$e^{-gT} n_i = n_{i'} + b_i \Delta n = (1 - b_i) n_{i'} + b_i n_{i'+1}, \tag{C.19}$$

where $0 \leq b_i < 1$.

We can interpret b_i as the probability of receiving a bequest of size $n_{i'+1}$ and $1 - b_i$ as the probability of receiving a bequest of size $n_{i'}$. The boundary condition can be written as follows:

$$f(n_{i'}, 0) = (1 - b_i) f(n_i, T), \qquad f(n_{i'+1}, 0) = b_i f(n_i, T). \tag{C.20}$$

Collecting the coefficients in a matrix B, we obtain the following condition in matrix form

$$f^1 = B f^K. \tag{C.21}$$

Combining the expressions above with the difference scheme for the f, we obtain

$$f^K = \overline{\Pi} B f^K \iff [I - \overline{\Pi} B] f^K = 0, \tag{C.22}$$

where

$$\overline{\Pi} \equiv \Pi^1 \times \Pi^2 \times \ldots \Pi^{K-1}. \tag{C.23}$$

Since B' and $(\Pi^k)'$ are stochastic matrices, we have that $B'(\overline{\Pi})'$ is also a stochastic matrix. Hence, the matrix has a unit eigenvalue, and a solution to the system of equations above exists.

C.2.2 Time-dependent KFE.

The discretized time-dependent KFE can be written as

$$f_{t+1}^{i,k} = \frac{\Delta t}{\Delta a} f_t^{i,k-1} + \pi_{u,t}^{i-1,k} f_t^{i-1,k} + \pi_{s,t}^{i,k} f_t^{i,k} + \pi_{d,t}^{i+1,k} f_t^{i+1,k}, \tag{C.24}$$

for $1 < i \leq I$ and $1 < k \leq K$, where

$$\pi_{u,t}^{i,k} = (\mu_n^{i,k})^+ \frac{\Delta t}{\Delta n} + \frac{(\sigma_{n,t}^i)^2}{2} \frac{\Delta t}{\Delta n^2} \tag{C.25}$$

$$\pi_{d,t}^{i,k} = -(\mu_n^{i,k})^- \frac{\Delta t}{\Delta n} + \frac{(\sigma_{n,t}^i)^2}{2} \frac{\Delta t}{\Delta n^2} \tag{C.26}$$

$$\pi_{s,t}^{i,k} = 1 - \frac{\Delta t}{\Delta a} - \left(|\mu_{n,t}^{i,k}| \frac{\Delta t}{\Delta n} + (\sigma_{n,t}^i)^2 \frac{\Delta t}{\Delta n^2} \right). \tag{C.27}$$

Note that the difference equation above corresponds to the implicit scheme for the stationary KFE if $f_{t+1}^{i,k} = f_t^{i,k}$. The boundary conditions are given by

$$f_{t+1}^{i,1} = (1 - b_i) f_{t+1}^{i',K} + b_i f_{t+1}^{i'+1,K} \tag{C.28}$$

$$f_{t+1}^{1,k} = \frac{\Delta t}{\Delta a} f_t^{1,k-1} + (\pi_{d,t}^{1,k} + \pi_{s,t}^{1,k}) f_t^{1,k} + \pi_{d,t}^{2,k} f_t^{2,k}, \tag{C.29}$$

$$f_{t+1}^{I,k} = \frac{\Delta t}{\Delta a} f_t^{I,k-1} + \pi_{u,t}^{I-1,k} f_t^{I-1,k} + (\pi_{s,t}^{I,k} + \pi_{u,t}^{I,k}) f_t^{I,k}. \tag{C.30}$$

Let $f_t^k \equiv [f_t^{1,k}, f_t^{2,k}, \ldots, f_t^{I,k}]'$ and $f_t \equiv [f_t^{2'}, f_t^{3'}, \ldots, f_t^{I'}]'$. The recursion for f_t can be written as

$$\begin{bmatrix} f_{t+1}^2 \\ f_{t+1}^3 \\ \vdots \\ f_{t+1}^{K-1} \\ f_{t+1}^K \end{bmatrix} = \begin{bmatrix} \Pi_t^2 & 0 & \cdots & 0 & \frac{\Delta t}{\Delta a} B \\ \frac{\Delta t}{\Delta a} I_I & \Pi_t^3 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \Pi_t^{K-1} & 0 \\ 0 & 0 & \cdots & \frac{\Delta t}{\Delta a} I_I & \Pi_t^K \end{bmatrix} \begin{bmatrix} f_t^2 \\ f_t^3 \\ \vdots \\ f_t^{K-1} \\ f_t^K \end{bmatrix}. \tag{C.31}$$

In matrix form, we can write the recursion for both the explicit scheme above and an implicit scheme:

$$f_{t+1} = A_t f_t, \qquad f_{t+1} = [2I - A_t]^{-1} f_t. \tag{C.32}$$

C.3 Equilibrium prices and capital stock in a stationary equilibrium

We derive next the equilibrium prices and the capital stock in a stationary equilibrium.

C.3.1 Price of aggregate insurance, interest rate, and the relative price of capital

Price of aggregate insurance. The demand for aggregate insurance for wage earners is given by

$$\theta^{ag}_{j,t} = h_{j,t}\sigma_A - (n_{j,t} + h_{j,t})\frac{p^{ag}}{\gamma}, \tag{C.33}$$

which is analogous to the expression for entrepreneurs (21).

Combining the demand for aggregate insurance for entrepreneurs and wage earners with the corresponding market-clearing condition, we obtain

$$\int_{\mathcal{E}_t} \left[(qk_{i,t} + h_{i,t})\sigma_A - (n_{i,t} + h_{i,t})\frac{p^{ag}}{\gamma}\right] di + \int_{\mathcal{W}_t} \left[h_{j,t}\sigma_A - (n_{j,t} + h_{j,t})\frac{p^{ag}}{\gamma}\right] di = 0. \tag{C.34}$$

Rearranging the expression above, we can solve for the price of aggregate insurance p^{ag}

$$p^{ag} = \frac{\int_{\mathcal{E}_t}(qk_{i,t} + h_{i,t})di + \int_{\mathcal{W}_t} h_{j,t}dj}{\int_{\mathcal{E}_t}(n_{i,t} + h_{i,t})di + \int_{\mathcal{W}_t}(n_{j,t} + h_{j,t})dj}\gamma\sigma_A = \gamma\sigma_A, \tag{C.35}$$

using the fact that $\int_{\mathcal{E}_t} n_{i,t}di + \int_{\mathcal{W}_t} n_{j,t}dj = \int_{\mathcal{E}_t} qk_{i,t}di$. Notice that this result does not rely on the assumption of a stationary equilibrium.

Interest rate. The financial wealth of wage earners evolves according to

$$d\tilde{n}_{j,t} = \left[(r + \gamma\sigma_A^2)\tilde{n}_{j,t} + \tilde{w}_t\bar{l}_{j,t} - \tilde{c}_{j,t}\right] dt + \tilde{n}_{j,t}\sigma_A dZ_t. \tag{C.36}$$

using the fact that the demand for aggregate insurance is given by $\theta_{j,t} = -n_{j,t}\sigma_A$ in equilibrium.

Combining the expression above with the law of motion for human wealth, we obtain the law of motion of total wealth:

$$\frac{d\tilde{\omega}_{i,t}}{\tilde{\omega}_{i,t}} = \left[r + \gamma\sigma_A^2 - \frac{c_{j,t}}{n_{j,t} + h_{j,t}}\right] dt + \sigma_A dZ_t. \tag{C.37}$$

In a stationary equilibrium, wage earners' scaled total wealth, $\omega_{j,t} = \tilde{\omega}_{j,t}/A_t$, is constant. Therefore, the drift of $\omega_{j,t}$ must be zero. We can compute the drift of $\omega_{j,t}$ using Ito's lemma:

$$\frac{d\omega_{i,t}}{\omega_{i,t}} = \frac{d\tilde{\omega}_{i,t}}{\tilde{\omega}_{i,t}} - \frac{dA_t}{A_t} + \left(\frac{dA_t}{A_t}\right)^2 - \frac{dA_t}{A_t}\frac{d\tilde{\omega}_{i,t}}{\tilde{\omega}_{i,t}} \tag{C.38}$$

$$= \left[r + \gamma\sigma_A^2 - \frac{c_{j,t}}{n_{j,t} + h_{j,t}} - \mu_A\right] dt. \tag{C.39}$$

The interest rate must then satisfy the condition

$$r + \gamma\sigma_A^2 - \left[\frac{1}{\gamma}\rho_w + \left(1 - \frac{1}{\gamma}\right)\left(r + \frac{\gamma\sigma_A^2}{2}\right)\right] - \mu_A = 0, \tag{C.40}$$

using the fact that the consumption-wealth ratio is given by

$$\frac{c_{j,t}}{n_{j,t} + h_{j,t}} = \frac{1}{\gamma}\rho_w + \left(1 - \frac{1}{\gamma}\right)\left(r + \frac{(p^{ag})^2}{2\gamma}\right), \tag{C.41}$$

which is a special case of (22), as we set $p_t^{id} = 0$ and $T \to \infty$.

Rearranging expression (C.40), we obtain

$$r = \rho_w + \gamma\mu_A - \gamma(\gamma + 1)\frac{\sigma_A^2}{2}. \tag{C.42}$$

Relative price of capital. Plugging the expression for $\Phi'(\iota)$ into the first-order condition for ι in equation (12), we obtain

$$\frac{1}{\sqrt{\Phi_0^2 + 2\Phi_1\iota}} = \frac{1}{q} \Rightarrow \iota = \frac{q^2 - \Phi_0^2}{2\Phi_1}. \tag{C.43}$$

In a stationary equilibrium, the capital-labor ratio is constant. Thus, capital grows at the population rate g, which gives us the condition

$$\Phi(\iota) - \delta = g \Rightarrow q = \Phi_0 + \Phi_1(g + \delta), \tag{C.44}$$

where we used the fact that $\Phi(\iota) = \frac{q - \Phi_0}{\Phi_1}$, obtained by plugging the expression for ι into the functional form for $\Phi(\iota)$.

C.3.2 Entrepreneurs' human and total wealth

The value of human wealth for an entrepreneur with age a is given by

$$\begin{aligned} h(a) &= \int_0^{T-a} e^{-(r + p^{ag}\sigma_A - \mu_A)z} \sum_{l=1}^{L} \Gamma_l e^{\varphi_l(z+a)} dz w \bar{l}_e \\ &= (1-\alpha)K^\alpha \bar{l}_e \sum_{l=1}^{L} \Gamma_l e^{\varphi_l a} \frac{1 - e^{-(r + p^{ag}\sigma_A - \mu_A - \varphi_l)(T-a)}}{r + p^{ag}\sigma_A - \mu_A - \varphi_l}. \end{aligned} \tag{C.45}$$

The average human wealth for entrepreneurs is given by

$$h_e = \int_0^T \frac{ge^{-ga}}{1-e^{-gT}} h(a)da = (1-\alpha)K^\alpha \bar{l}_e \sum_{l=1}^{L} \Gamma_l \frac{ge^{(\varphi_l - g)a}}{1-e^{-gT}} \frac{1-e^{-(r+p^{ag}\sigma_A - \mu_A - \varphi_l)(T-a)}}{r + p^{ag}\sigma_A - \mu_A - \varphi_l}. \quad \text{(C.46)}$$

Expressing condition (24) in levels, and after some rearrangement, we obtain entrepreneurs' total wealth by age

$$\omega(a) = \omega(0)e^{\left(r + \frac{(p^{ag})^2}{\gamma} + \frac{(p^{id})^2}{\gamma} - \mu_A\right)a} \frac{e^{-\bar{r}a} - \psi e^{-\bar{r}T}}{1-\psi e^{-\bar{r}T}}, \quad \text{(C.47)}$$

where $\omega(0)$ is given by

$$\omega(0) = \frac{h(0)}{1 - e^{\left(r + \frac{(p^{ag})^2}{\gamma} + \frac{(p^{id})^2}{\gamma} - g - \mu_A - \bar{r}\right)T} \frac{1-\psi}{1-\psi e^{-\bar{r}T}}}. \quad \text{(C.48)}$$

The average total wealth for entrepreneurs is given by

$$\omega_e = \int_0^T \frac{ge^{-ga}}{1-e^{-gT}} \omega(a)da = \omega(0)f(0) \int_0^T \frac{e^{\hat{r}_1 a} - \psi e^{-\overline{T}} e^{\hat{r}_2 a}}{1-\psi e^{-\bar{r}T}} da, \quad \text{(C.49)}$$

where $\hat{r}_1 = r + \gamma\sigma_A^2 + \frac{(p^{id})^2}{\gamma} - \mu_A - g - \bar{r}$ and $\hat{r}_2 = \hat{r}_1 + \bar{r}$.

C.3.3 Capital stock and idiosyncratic risk premium

Rearranging the expression for the shadow price of idiosyncratic risk (20), and using the definition of expected returns (13), we obtain the MPK schedule, as discussed in Section 5:

$$r + p^{ag}\sigma_A + p^{id}\phi\sigma_{id} = \frac{\alpha K^{\alpha-1} - \iota(q)}{q} + \mu_A + \Phi(\iota(q)) - \delta. \quad \text{(C.50)}$$

where r, p^{ag}, and q are functions of parameters, as derived above.

Integrating condition (19) across all entrepreneurs, we obtain

$$p^{id} = \gamma\phi\sigma_{id} \frac{qK}{\chi_e \omega_e}, \quad \text{(C.51)}$$

where χ_e is the fraction of entrepreneurs in the population.

Note that ω_e is a function of p^{id} and K. Plugging the expression for (C.49) into the equation above, we obtain p^{id} implicitly as a function of K. This relationship between p^{id} and K corresponds to the pricing schedule discussed in Section 5. Therefore, we obtain the equilibrium capital stock and equilibrium price of idiosyncratic risk by finding the pair (K, p^{id}) that simultaneously satisfies the MPK schedule and the pricing schedule.

D Transitional dynamics

In this section, we describe the computation of the transitional dynamics. We focus on the case where the interest rate is fixed during the transition. This simplifies the numerical solution while allow us to focus on the response of the idiosyncratic risk premium. To ensure that that the interest rate is constant, we assume that wage earners have Epstein-Zin preferences and take the limit as the elasticity of intertemporal substitution goes to infinity, that is, workers have linear intertemporal preferences.[40]

D.1 Wage earners with Epstein-Zin preferences

Wage earners have the continuous-time analog of Epstein-Zin preferences with EIS ψ_w and risk aversion γ. The wage earner's problem is given by

$$\tilde{V}_t^w(\tilde{n}_j) = \max_{\tilde{c}_j, \tilde{\theta}_j^{ag}} \mathbb{E}_t \left[\int_t^\infty f_w(\tilde{c}_{j,z}, \tilde{V}_z) dz \right], \tag{D.1}$$

subject to $\tilde{n}_{j,t} \geq \tilde{h}_{j,t}$, where $\tilde{h}_{j,t}$ denotes wage earner j's human wealth, non-negativity constraint $\tilde{c}_{j,t} \geq 0$, and the law of motion of financial wealth $\tilde{n}_{j,t}$

$$d\tilde{n}_{j,t} = \left[\tilde{n}_{j,t} r_t - p_t^{ag} \tilde{\theta}_{j,t}^{ag} + \tilde{w}_t \bar{l}_{j,t} - \tilde{c}_{j,t} \right] dt - \tilde{\theta}_{j,t}^{ag} dZ_t,$$

where $f_w(\tilde{c}, V)$ is the aggregator given by

$$f_w(\tilde{c}, \tilde{V}) = \rho_w \frac{(1-\gamma)\tilde{V}}{1-\psi_w^{-1}} \left\{ \left(\frac{\tilde{c}}{\left((1-\gamma)\tilde{V}\right)^{\frac{1}{1-\gamma}}} \right)^{1-\psi_w^{-1}} - 1 \right\}. \tag{D.2}$$

It is convenient to work with the scaled value function $V_t^w(n)$, which satisfies the condition $\tilde{V}_t^w(\tilde{n}) = A_t^{1-\gamma} V_t^w\left(\frac{\tilde{n}_t}{A_t}\right)$, where $V_t^w(\cdot)$ is independent of A_t. The HJB equation in terms of the

[40] This assumption is meant to capture, in an extreme form, the essence of the macro-finance literature which assumes a high EIS (see e.g. Bansal and Yaron 2004 and Barro 2009). In these models, the high EIS dampens movements in interest rates, so risk premia accounts for most of the variation in discount rates.

scaled value function is given by

$$\tilde{\rho}_w \frac{(1-\gamma)V^w}{1-\psi_w^{-1}} = \max_{\tilde{c},\tilde{\theta}^{ag}} \rho_w \frac{(1-\gamma)V^w}{1-\psi_w^{-1}} \left(\frac{c}{((1-\gamma)V^w)^{\frac{1}{1-\gamma}}} \right)^{1-\psi_w^{-1}} + V_t^w + V_n^w \left[\tilde{r}n + (\gamma\sigma_A - p^{ag})\theta^{ag} + w\bar{l} - c \right]$$
$$+ \frac{1}{2} V_{nn}^w \left(\theta^{ag} + n\sigma_A\right)^2. \tag{D.3}$$

where $\tilde{\rho}_w \equiv \rho_w - (1-\psi_w^{-1}) \left(\mu_A - \frac{\gamma\sigma_A^2}{2} \right)$ and $\tilde{r}_t \equiv r_t + \gamma\sigma_A^2 - \mu_A$.

Policy functions. The first-order conditions for this problem are given by

$$\rho_w \left((1-\gamma)V^w\right)^{\frac{\psi_w^{-1}-\gamma}{1-\gamma}} c^{-\psi_w^{-1}} = V_n^w, \qquad \gamma\sigma_A - p^{ag} = -\frac{V_{nn}^w}{V_n^w}\left(\theta^{ag} + n\sigma_A\right). \tag{D.4}$$

We will guess and verify that the scaled value function takes the form

$$V_t^w(n_{j,t}) = \left(\frac{\zeta_{w,t}}{\rho_w^{\psi_w}} \right)^{\frac{1-\gamma}{1-\psi_w}} \frac{(n_{j,t} + h_{j,t})^{1-\gamma}}{1-\gamma}, \tag{D.5}$$

where $\zeta_{w,t}$ and $h_{j,t}$ are potentially time-varying, but they are non-stochastic.

Using the expression for the value function above, we obtain the policy functions:

$$\frac{c_{j,t}}{n_{j,t} + h_{j,t}} = \zeta_{w,t}, \qquad \theta_{j,t}^{ag} = \sigma_A h_{j,t} - \frac{p_t^{ag}}{\gamma}(n_{j,t} + h_{j,t}). \tag{D.6}$$

Inserting the policy functions derived above back into the HJB equation, we obtain

$$\frac{\tilde{\rho}_w}{1-\psi_w^{-1}} = \frac{\psi_w^{-1}}{1-\psi_w^{-1}}\zeta_{w,t} + \frac{1}{1-\psi_w}\frac{\dot{\zeta}_{w,t}}{\zeta_{w,t}} + \tilde{r}_t. \tag{D.7}$$

Rearranging the expression above for the case of a stationary equilibrium, so $\dot{\zeta}_{w,t} = 0$, we obtain the consumption-wealth ratio:

$$\zeta_{w,t} = \psi_w\rho_w + (1-\psi_w)\left(r + \frac{\gamma\sigma_A^2}{2} \right), \tag{D.8}$$

which coincides with the expression for the consumption-wealth ratio for wage earners given in (C.41) in the special case of CRRA preferences, i.e., $\psi_w^{-1} = \gamma$.

Price of aggregate risk and interest rate. The demand for aggregate insurance derived above coincides with the expression for aggregate insurance in the CRRA case (see Equation

C.33). Therefore, the same argument used in Section C.3 to solve for the price of aggregate risk can be applied in the case with Epstein-Zin preferences in a non-stationary setting. This implies that the price of aggregate risk is constant and given by $p_t^{ag} = \gamma\sigma_A$ during the transitional dynamics.

We consider next the behavior of the interest rate. A derivation analogous to the one in Section C.3 shows that the interest rate in a stationary equilibrium is given by

$$r = \rho_w + \psi_w^{-1}\mu_A - (1 + \psi_w^{-1})\frac{\gamma\sigma_A^2}{2}, \tag{D.9}$$

which coincides with (31) when $\psi_w^{-1} = \gamma$.

Taking the limit of (D.7) as $\psi_w \to \infty$, we obtain r_t in the case of a non-stationary equilibrium:

$$r_t + \gamma\sigma_A^2 - \mu_A = \rho_w + \frac{\gamma\sigma_A^2}{2} - \mu_A \Rightarrow r_t = \rho_w - \frac{\gamma\sigma_A^2}{2}. \tag{D.10}$$

Therefore, the interest rate is constant when wage earners have linear intertemporal preferences. Moreover, the expression above coincides with the one for the interest rate in the stationary equilibrium (D.9) when specialized to $\psi_w^{-1} = 0$.

D.2 Computation of the transition dynamics

Let $\hat{x}_t \equiv \log \frac{x_t}{x}$ for any variable x_t, where variables without a time subscript indicate the value in the new stationary equilibrium. The system of differential equations can then be written as

$$\dot{\hat{K}}_t = \frac{q}{\Phi_1}(e^{\hat{q}_t} - 1) \tag{D.11}$$

$$\dot{\hat{q}}_t = \gamma\phi^2\sigma_{id}^2 \frac{qK}{\chi_e\omega_e}\left(e^{\hat{q}_t + \hat{K}_t - \hat{\omega}_{e,t}} - 1\right) - \frac{q}{\Phi_1}(e^{\hat{q}_t} - 1) - \left[\frac{\alpha K^{\alpha-1}e^{(\alpha-1)\hat{K}_t} - \iota(qe^{\hat{q}_t})}{qe^{\hat{q}_t}} - \frac{\alpha K^{\alpha-1} - \iota(q)}{q}\right] \tag{D.12}$$

$$\frac{\partial \hat{h}_t(a)}{\partial t} = -\frac{\partial \hat{h}_t(a)}{\partial a} - (1-\alpha)\frac{K^\alpha \bar{l}(a)}{h(a)}\left(e^{\alpha\hat{K}_t - \hat{h}_t(a)} - 1\right) \tag{D.13}$$

$$\frac{\partial \hat{\omega}_t(a)}{\partial t} = -\frac{\partial \hat{\omega}_t(a)}{\partial a} + \gamma(\phi\sigma_{id})^2\left(\frac{qK}{\chi_e\omega_e}\right)^2\left(e^{2(\hat{q}_t + \hat{K}_t - \hat{\omega}_{e,t})} - 1\right) - \zeta(a)\left(e^{\hat{\zeta}_t(a)} - 1\right) \tag{D.14}$$

$$\frac{\partial \hat{\zeta}_t(a)}{\partial t} = -\frac{\partial \hat{\zeta}_t(a)}{\partial a} + \zeta(a)\left(e^{\hat{\zeta}_t(a)} - 1\right) - (\gamma - 1)\frac{(\phi\sigma_{id})^2}{2}\left(\frac{qK}{\chi_e\omega_e}\right)^2\left(e^{2(\hat{q}_t + \hat{K}_t - \hat{\omega}_{e,t})} - 1\right). \tag{D.15}$$

Linearizing the system above, we obtain

$$\dot{\hat{K}}_t = \frac{q}{\Phi_1}\hat{q}_t \tag{D.16}$$

$$\dot{\hat{q}}_t = \gamma\phi^2\sigma_{id}^2\frac{qK}{\chi_e\omega_e}\left(\hat{q}_t + \hat{K}_t - \hat{\omega}_{e,t}\right) + \frac{\alpha K^{\alpha-1} - \iota(q)}{q}\hat{q}_t + (1-\alpha)\frac{\alpha K^{\alpha-1}}{q}\hat{K}_t \tag{D.17}$$

$$\frac{\partial \hat{h}_t(a)}{\partial t} = -\frac{\partial \hat{h}_t(a)}{\partial a} - (1-\alpha)\frac{K^\alpha \bar{l}(a)}{h(a)}\left(\alpha\hat{K}_t - \hat{h}_t(a)\right) \tag{D.18}$$

$$\frac{\partial \hat{\omega}_t(a)}{\partial t} = -\frac{\partial \hat{\omega}_t(a)}{\partial a} + 2\gamma(\phi\sigma_{id})^2\left(\frac{qK}{\chi_e\omega_e}\right)^2\left(\hat{q}_t + \hat{K}_t - \hat{\omega}_{e,t}\right) - \zeta(a)\hat{\zeta}_t(a) \tag{D.19}$$

$$\frac{\partial \hat{\zeta}_t(a)}{\partial t} = -\frac{\partial \hat{\zeta}_t(a)}{\partial a} + \zeta(a)\hat{\zeta}_t(a) - (\gamma-1)(\phi\sigma_{id})^2\left(\frac{qK}{\chi_e\omega_e}\right)^2\left(\hat{q}_t + \hat{K}_t - \hat{\omega}_{e,t}\right), \tag{D.20}$$

where

$$\hat{\omega}_{e,t} = \int_0^T \frac{\omega(a)f(a)}{\omega_e}\hat{\omega}_t(a)da. \tag{D.21}$$

We now discretize the system using a finite-differences method. The time variable t and age a will take values in the equally spaced grid $\{t_1, t_2, \dots, t_N\}$ and $\{a_1, a_2, \dots, a_K\}$, respectively. We adopt the following notation: $\zeta_n^k = \zeta_{t_n}(a_k)$ denotes the consumption-wealth ratio at time t_n and age a_k, and an analogous notation holds for the remaining variables. The time and age steps are denoted by $\Delta t = t_{n+1} - t_n$ and $\Delta a = a_{k+1} - a_k$. The discretized version of the ODEs is given by

$$\frac{\hat{K}_{n+1} - \hat{K}_n}{\Delta t} = \frac{q}{\Phi_1}\hat{q}_n \tag{D.22}$$

$$\frac{\hat{q}_{n+1} - \hat{q}_n}{\Delta t} = \gamma\phi^2\sigma_{id}^2\frac{qK}{\omega_e}\left(\hat{q}_n + \hat{K}_n - \hat{\omega}_{e,n}\right) + \frac{\alpha K^{\alpha-1} - \iota(q)}{q}\hat{q}_n + \alpha(1-\alpha)\frac{K^{\alpha-1}}{q}\hat{K}_n. \tag{D.23}$$

Discretizing the PDEs we obtain at the interior points

$$\frac{\hat{h}_{n+1}^k - \hat{h}_n^k}{\Delta t} = -\frac{\hat{h}_n^{k+1} - \hat{h}_n^k}{\Delta a} - (1-\alpha)\frac{K^\alpha \bar{l}^k}{h^k}\left(\alpha\hat{K}_n - \hat{h}_n^k\right) \tag{D.24}$$

$$\frac{\hat{\omega}_{n+1}^k - \hat{\omega}_n^k}{\Delta t} = -\frac{\hat{\omega}_n^k - \hat{\omega}_n^{k-1}}{\Delta a} + 2\gamma(\phi\sigma_{id})^2\left(\frac{qK}{\omega_e}\right)^2\left(\hat{q}_n + \hat{K}_n - \hat{\omega}_{e,n}\right) - \zeta^k\hat{\zeta}_n^k \tag{D.25}$$

$$\frac{\hat{\zeta}_{n+1}^k - \hat{\zeta}_n^k}{\Delta t} = -\frac{\hat{\zeta}_n^{k+1} - \hat{\zeta}_n^k}{\Delta a} + \zeta^k\hat{\zeta}_n^k - (\gamma-1)(\phi\sigma_{id})^2\left(\frac{qK}{\omega_e}\right)^2\left(\hat{q}_n + \hat{K}_n - \hat{\omega}_{e,n}\right), \tag{D.26}$$

where, using the Trapezoidal rule, $\hat{\omega}_{e,n}$ is given by

$$\hat{\omega}_{e,n} = \left[\sum_{k=2}^{K-1}\hat{\omega}_n^k\frac{f(a_k)\omega(a_k)}{\omega_e} + \hat{\omega}_n^1\frac{f(a_1)\omega(a_1)}{2\omega_e} + \hat{\omega}_n^K\frac{f(a_K)\omega(a_K)}{2\omega_e}\right]\Delta a. \tag{D.27}$$

It remains to specify the boundary conditions. We have initial conditions for state variables and terminal conditions for jump variables. For the first two equations, we have that capital starts at the old steady state and the relative price of capital converges to the new one.

$$\hat{k}_1 = \hat{k}^*; \qquad \lim_{n\to\infty} \hat{q}_n = 0, \tag{D.28}$$

where $\hat{k}^*$ is the log deviation of the capital stock at the old steady state relative to the new one.

The boundary conditions associated with $\hat{\zeta}_t(a)$ and $\hat{h}_t(a)$ are the following

$$\hat{\zeta}_n^K = 0; \qquad \lim_{n\to\infty} \hat{\zeta}_n^k = 0, \forall k \tag{D.29}$$

$$\hat{h}_n^K = 0; \qquad \lim_{n\to\infty} \hat{h}_n^k = 0, \forall k. \tag{D.30}$$

The consumption-wealth ratio and human wealth are forward-looking variables, so they have terminal conditions instead of initial conditions. Notice that $h_t(T) = 0$, while $\zeta_t(T)$ is determined by the bequest motive, so deviations from the new steady state are equal to zero. We then only need to solve for the vectors $\hat{\zeta}_n = \left[\hat{\zeta}_n^1, \hat{\zeta}_n^2, \ldots, \hat{\zeta}_n^{K-1}\right]$ and $\hat{h}_n = \left[\hat{h}_n^1, \hat{h}_n^2, \ldots, \hat{h}_n^{K-1}\right]$, since the value at the final age is pinned down by the boundary condition.

The boundary condition for total wealth is given by

$$\hat{\omega}_1^k = \hat{\omega}_1^{k,*}; \qquad \hat{\omega}_n^1 = \frac{f^K \omega^K}{f^1 \omega^1} \hat{\omega}_n^K + \frac{h^1}{\omega^1} \hat{h}_n^1. \tag{D.31}$$

Note that the value of $\hat{\omega}_n^1$ is pinned down by the boundary condition, given $\hat{\omega}_n^K$ and $\hat{h}_n^1$. In this case, we have to solve for $\hat{\omega}_n = [\hat{\omega}_n^2, \ldots, \hat{\omega}_n^K]'$, a $K-1$-dimensional vector. The determination of $\hat{\omega}_1^{k,*}$ will be discussed below.

Given the boundary conditions, we can assemble the system in matrix form. First, we can write the difference equations for the state variables in matrix form

$$\begin{bmatrix} \hat{K}_{n+1} \\ \hat{\omega}_{n+1} \end{bmatrix} = \begin{bmatrix} 1 & \mathbf{0}'_{K-1} & \frac{q}{\Phi_1}\Delta t & \mathbf{0}_{K-1} & \mathbf{0}'_{K-1} \\ a_\omega \mathbf{1}_{K-1} & A_{\omega\omega} & a_\omega \mathbf{1}_{K-1} & A_{\omega h} & A_{\omega\zeta} \end{bmatrix} \begin{bmatrix} \hat{K}_n \\ \hat{\omega}_n \\ \hat{q}_n \\ \hat{h}_n \\ \hat{\zeta}_n \end{bmatrix}, \tag{D.32}$$

where $a_\omega = 2\gamma(\phi\sigma_{id})^2 \left(\frac{qK}{\chi_e \omega_e}\right)^2 \Delta t$, and $A_{\omega\omega}$, $A_{\omega h}$, and $A_{\omega\zeta}$ are $K-1 \times K-1$ matrices given by

$$A_{\omega\omega} = \begin{bmatrix} 1-a_\omega \frac{f^2\omega^2}{\omega_e}\Delta a - \frac{\Delta t}{\Delta a} & -a_\omega \frac{f^3\omega^3}{\omega_e}\Delta a & \cdots & -a_\omega \frac{f^{K-1}\omega^{K-1}}{\omega_e}\Delta a & -a_\omega \frac{f^K\omega^K}{\omega_e}\Delta a + \frac{f^K\omega^K}{f^1\omega^1}\frac{\Delta t}{\Delta a} \\ -a_\omega \frac{f^2\omega^2}{\omega_e}\Delta a + \frac{\Delta t}{\Delta a} & 1-a_\omega \frac{f^3\omega^3}{\omega_e}\Delta a - \frac{\Delta t}{\Delta a} & \cdots & -a_\omega \frac{f^{K-1}\omega^{K-1}}{\omega_e}\Delta a & -a_\omega \frac{f^K\omega^K}{\omega_e}\Delta a \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -a_\omega \frac{f^2\omega^2}{\omega_e}\Delta a & -a_\omega \frac{f^3\omega^3}{\omega_e}\Delta a & \cdots & 1-a_\omega \frac{f^{K-1}\omega^{K-1}}{\omega_e}\Delta a - \frac{\Delta t}{\Delta a} & -a_\omega \frac{f^K\omega^K}{\omega_e}\Delta a \\ -a_\omega \frac{f^2\omega^2}{\omega_e}\Delta a & -a_\omega \frac{f^3\omega^3}{\omega_e}\Delta a & \cdots & -a_\omega \frac{f^{K-1}\omega^{K-1}}{\omega_e}\Delta a + \frac{\Delta t}{\Delta a} & 1-a_\omega \frac{f^K\omega^K}{\omega_e}\Delta a - \frac{\Delta t}{\Delta a} \end{bmatrix}, \tag{D.33}$$

$$A_{\omega h} = \begin{bmatrix} -a_\omega \frac{f^1h^1}{2\omega_e}\Delta a + \frac{h^1}{\omega^1}\frac{\Delta t}{\Delta a} & 0 & \cdots & 0 & 0 \\ -a_\omega \frac{f^1h^1}{2\omega_e}\Delta a & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -a_\omega \frac{f^1h^1}{2\omega_e}\Delta a & 0 & \cdots & 0 & 0 \\ -a_\omega \frac{f^1h^1}{2\omega_e}\Delta a & 0 & \cdots & 0 & 0 \end{bmatrix}, \quad A_{\omega\zeta} = \begin{bmatrix} 0 & -\zeta^2\Delta t & 0 & \cdots & 0 & 0 \\ 0 & 0 & -\zeta^3\Delta t & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & -\zeta^{K-1}\Delta t \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}. \tag{D.34}$$

The difference equations for the jump variables can be written as

$$\begin{bmatrix} \hat{q}_{n+1} \\ \hat{h}_{n+1} \\ \hat{\zeta}_{n+1} \end{bmatrix} = \begin{bmatrix} A_{qK} & A_{q\omega}\overline{\omega}' & A_{qq} & A_{qh} & \mathbf{0}'_{K-1} \\ A_{hK} & \mathbf{0}_{K-1,K-1} & \mathbf{0}_{K-1} & A_{hh} & \mathbf{0}_{K-1,K-1} \\ a_\zeta \mathbf{1}_{K-1} & -a_\zeta \mathbf{1}_{K-1}\overline{\omega}' & a_\zeta \mathbf{1}_{K-1} & A_{\zeta h} & A_{\zeta\zeta} \end{bmatrix} \begin{bmatrix} \hat{K}_n \\ \hat{\omega}_n \\ \hat{q}_n \\ \hat{h}_n \\ \hat{\zeta}_n \end{bmatrix}, \tag{D.35}$$

where

$$A_{qK} = \left[\gamma\phi^2\sigma_{id}^2 \frac{qK}{\chi_e\omega_e} + \alpha(1-\alpha)\frac{K^{\alpha-1}}{q}\right]\Delta t, \qquad A_{q\omega} = -\gamma\phi^2\sigma_{id}^2\frac{qK}{\omega_e}\Delta t, \qquad \overline{\omega} = \left[\frac{f^2\omega^2}{\omega_e}, \frac{f^3\omega^3}{\omega_e}, \dots, \frac{f^K\omega^K}{\omega_e}\right]'\Delta a, \tag{D.36}$$

$$A_{qq} = 1 + \left[\gamma\phi^2\sigma_{id}^2\frac{qK}{\chi_e\omega_e} + \frac{\alpha K^{\alpha-1} - \iota(q)}{q}\right]\Delta t, \qquad A_{qh} = -\gamma\phi^2\sigma_{id}^2\frac{qK}{\chi_e\omega_e}\Delta t \frac{f^1h^1}{2\omega_e}\Delta a \mathbf{e}'_{1,K-1}, \tag{D.37}$$

$$A_{hK} = -(1-\alpha)\alpha K^\alpha \left[\frac{\bar{l}(a^1)}{h(a^1)}, \cdots \frac{\bar{l}(a^{K-1})}{h(a^{K-1})}\right]'\Delta t, \qquad a_\zeta = -(\gamma-1)(\phi\sigma_{id})^2\left(\frac{qK}{\chi_e\omega_e}\right)^2\Delta t, \tag{D.38}$$

$$A_{\zeta h} = -a_\zeta \frac{f^1h^1}{2\omega_e}\Delta a \mathbf{1}_{K-1}\mathbf{e}'_{1,K-1} \tag{D.39}$$

$$A_{hh} = \begin{bmatrix} 1+w\frac{\bar{l}(a^1)}{h(a^1)}\Delta t + \frac{\Delta t}{\Delta a} & -\frac{\Delta t}{\Delta a} & 0 & \cdots & 0 & 0 \\ 0 & 1+w\frac{\bar{l}(a^2)}{h(a^2)}\Delta t + \frac{\Delta t}{\Delta a} & -\frac{\Delta t}{\Delta a} & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \cdots & \ddots & \ddots \\ 0 & 0 & 0 & \cdots & 1+w\frac{\bar{l}(a^{K-2})}{h(a^{K-2})}\Delta t + \frac{\Delta t}{\Delta a} & -\frac{\Delta t}{\Delta a} \\ 0 & 0 & 0 & \cdots & 0 & 1+w\frac{\bar{l}(a^{K-1})}{h(a^{K-1})}\Delta t + \frac{\Delta t}{\Delta a} \end{bmatrix} \tag{D.40}$$

$$A_{\zeta\zeta} = \begin{bmatrix} 1+\hat{\zeta}^1\Delta t + \frac{\Delta t}{\Delta a} & -\frac{\Delta t}{\Delta a} & 0 & \cdots & 0 & 0 \\ 0 & 1+\hat{\zeta}^2\Delta t + \frac{\Delta t}{\Delta a} & -\frac{\Delta t}{\Delta a} & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \cdots & \ddots & \ddots \\ 0 & 0 & 0 & \cdots & 1+\hat{\zeta}^{K-2}\Delta t + \frac{\Delta t}{\Delta a} & -\frac{\Delta t}{\Delta a} \\ 0 & 0 & 0 & \cdots & 0 & 1+\hat{\zeta}^{K-1}\Delta t + \frac{\Delta t}{\Delta a} \end{bmatrix}. \tag{D.41}$$

Let $x_n = [\hat{K}_n, \hat{\omega}_n']'$ denote the vector of predetermined variables, $y_n = [\hat{q}_n, \hat{\zeta}_n', \hat{h}_n']'$ the vector of jump variables, and $z_n = [x_n', y_n']'$ a vector containing the state and jump variables. We can then write the system in matrix form

$$z_{n+1} = Az_n, \tag{D.42}$$

for a matrix of coefficients A.

Provided the Blanchard and Kahn (1980) conditions are satisfied, there exists a unique pair of matrices (P, H) such that

$$x_{n+1} = Px_n; \qquad y_n = Hx_n. \tag{D.43}$$

The initial value of entrepreneurs' total wealth by age is given by

$$\omega_0(a) = \omega^*(a) + \underbrace{(q_0 - q^*)k^*(a) + h_0(a) - h^*(a)}_{\text{revaluation effect}}, \tag{D.44}$$

where $\omega^*(a)$ and $h^*(a)$ denote total wealth and human wealth at the initial steady state, respectively, and $k^*(a)$ denotes the amount of capital held by entrepreneurs of age a in the old steady state, which is given by $k^*(a) = \frac{\omega^*(a)}{\omega_e}K^*$.

Log-linearizing the expression above around the new steady state, we obtain

$$\hat{\omega}_0(a) = \hat{\omega}^*(a) + \frac{qK}{\omega_e}(\hat{q}_0 - \hat{q}^*) + \frac{h(a)}{\omega(a)}(\hat{h}_0(a) - \hat{h}^*(a)). \tag{D.45}$$

We can write the initial condition for x_1 as follows

$$x_1 = x^* + G(y_1 - y^*) \Rightarrow x_1 = [I - GH]^{-1}(x^* - Gy^*), \tag{D.46}$$

where $x^* = [\hat{k}^*, (\hat{\omega}^*)']'$, $y^* = [\hat{q}^*, (\hat{h}^*)', (\hat{\zeta}^*)']'$, and G is a $K \times 2K-1$ matrix given by

$$G = \begin{bmatrix} 0 & \mathbf{0}'_{K-1} & \mathbf{0}'_{K-1} \\ \frac{qK}{\omega_e}\mathbf{1}_{K-1} & G_{\omega h} & \mathbf{0}_{K-1,K-1} \end{bmatrix}, \tag{D.47}$$

where $G_{\omega h}$ has entries $(k, k+1)$ equal to $\frac{h(a_{k+1})}{\omega(a_{k+1})}$ for $k = 1, \ldots, K-1$, and zero otherwise.

E Extensions

E.1 Limited pledgeability and heterogeneous expected returns

In this subsection, we extend the basic environment in three dimensions: i) limited pleadgeability of physical assets, ii) heterogeneous idiosyncratic volatility, and iii) decreasing returns to scale. We focus on the case of a stationary equilibrium.

Limited pleadgeability of capital. Let $b_{i,t} \equiv n_{i,t} - q_t k_t$ denote the amount of safe assets held by entrepreneur i. The natural borrowing limit can be written as

$$-b_{i,t} \leq h_{i,t} + qk_{i,t}. \tag{E.1}$$

This allows the entrepreneur to borrow freely against physical assets or human wealth. Let's now assume that there is limited pledgeability of physical assets, that is, entrepreneurs can only borrow a fraction of $1-\lambda^{-1}$ of the value of physical assets, a form of collateral constraint:

$$-b_{i,t} \geq h_{i,t} + (1-\lambda^{-1})qk_{i,t} \Rightarrow qk_{i,t} \leq \lambda\omega_{i,t}, \tag{E.2}$$

where $\lambda \geq 1$. Hence, the entrepreneur faces a portfolio problem subject to leverage constraints. Moreover, we assume idiosyncratic volatility $\sigma_{i,id}$ is heterogeneous across entrepreneurs. The leverage constraint is more likely to be binding for entrepreneurs with less risky projects.

Given that the leverage constraint is linear in total wealth, we can obtain a closed-form solution for the portfolio problem with leverage constraints.[41] The HJB for an entrepreneur can be written as

$$\frac{\rho}{1-\gamma} = \max_{c_{i,t},k_{i,t},l_{i,t},\iota_{i,t},\theta^{ag}_{i,t},\theta^{id}_{i,t}} \left\{ \frac{\zeta_t^{\gamma}(a)}{1-\gamma}\left(\frac{c_{i,t}}{\omega_{i,t}}\right)^{1-\gamma} - \frac{\gamma}{1-\gamma}\frac{1}{\zeta_t(a)}\left(\frac{\partial \zeta_t(a)}{\partial t} + \frac{\partial \zeta_t(a)}{\partial a}\right) + r_t + \frac{q_t k_{i,t}}{\omega_{i,t}}(\mu^R_{i,t} - r_t) \right.$$
$$\left. - \frac{p_t^{ag}\theta^{ag}_{i,t}}{\omega_{i,t}} + \frac{h_{i,t}}{\omega_{i,t}}\sigma_A p_t^{ag} - \frac{c_{i,t}}{\omega_{i,t}} - \frac{\gamma}{2}\left[\left(\frac{q_t k_{i,t} + h_{i,t}}{\omega_{i,t}}\sigma_A - \frac{\theta^{ag}_{i,t}}{\omega_{i,t}}\right)^2 + \left(\frac{q_t k_{i,t}}{\omega_{i,t}}\sigma_{i,id} - \frac{\theta^{id}_{i,t}}{\omega_{i,t}}\right)^2\right]\right\},$$

[41] A similar result on the optimal portfolio share with leverage constraints can be found for investors without labor income in, for instance, Grossman and Vila (1992) and Detemple and Murthy (1997).

subject to leverage and insurance constraints: $q_t k_{i,t} \leq \lambda \omega_{i,t}$ and $\theta^{id}_{i,t} \leq (1-\phi) q_t k_{i,t} \sigma_{i,id}$.

The optimal capital demand is given by

$$\frac{qk_{i,t}}{n_{i,t}} = \frac{p^{id}_i}{\gamma \phi \sigma_{i,id}} \left(1 + \frac{h_{i,t}}{n_{i,t}}\right), \tag{E.3}$$

where the price of idiosyncratic risk $p^{id}_{i,t}$ satisfies the condition:

$$p^{id}_i = \min \left\{ \frac{\mu^R_i - r - p^{ag}\sigma_A}{\phi \sigma_{i,id}}, \gamma \phi \sigma_{i,id} \lambda \right\}.$$

Risk-taking. The price of idiosyncratic risk is now heterogeneous across entrepreneurs. For entrepreneurs with less risky projects, the leverage constraint may be binding. In this case, the price of idiosyncratic risk is given $p^{id} = \gamma \phi \sigma_{i,id} \lambda$, which implies that $q_t k_{i,t} = \lambda \omega_{i,t}$. To aggregate Equation (E.3) across types, it is convenient to first take logs, average for a given age group, and then convert the expression back to levels. Define $k(a) = \exp\left(\mathbb{E}_i[\log k_{i,t} | a_i = a]\right)$ and $\omega(a) = \exp\left(\mathbb{E}_i[\log(n_{i,t} + h_{i,t}) | a_i = a]\right)$ as the relevant cross-sectional average of capital and total wealth conditional on age. The aggregate exposure to the business for entrepreneurs of age a is then given by

$$\frac{qk(a)}{n(a)} = \frac{1 + \frac{h(a)}{n(a)}}{\gamma} \frac{p^{id}}{\phi \sigma_{id}}, \tag{E.4}$$

where $n(a) = \omega(a) - h(a)$ and

$$\sigma_{id} = \exp\left(\mathbb{E}_i[\log \sigma_{i,id}]\right), \qquad p^{id} = \exp\left(\mathbb{E}_i[\log p^{id}_i]\right). \tag{E.5}$$

Therefore, we obtain the same expression for the business exposure after aggregation as in the baseline model, showing that our results extend to the case with limited pledgeability and heterogeneity in productivity and risk. An analogous derivation shows that our results for the consumption-wealth ratio extend to this case as well.

Introducing decreasing returns to scale. Consider next a span-of-control (Lucas 1978) version of the model. Production depends not only on the amount of capital and labor but also on entrepreneurial ability e_i, which is fixed for the entrepreneur's lifetime:

$$\tilde{y}_{i,t} = A_t k^{\alpha}_{i,t} l^{\beta}_{i,t} e^{1-\alpha-\beta}_i. \tag{E.6}$$

For simplicity, we focus on the case e_i is common across entrepreneurs and normalized to $e_i = 1$. It is straightforward to extend the analysis to the case of heterogeneous ability.

Labor is chosen to maximize expected returns, so labor demand is given by

$$w = \beta k_{i,t}^{\alpha} l_{i,t}^{\beta-1}. \tag{E.7}$$

The expected return for entrepreneur i is given by

$$\mu_{i,t} = \frac{(1-\beta)\left(\frac{\beta}{w}\right)^{\frac{\beta}{1-\beta}} k_{i,t}^{\frac{\alpha+\beta-1}{1-\beta}} - \iota(q)}{q} + \mu_A + \Phi(\iota(q)) - \delta. \tag{E.8}$$

If $\beta = 1 - \alpha$, so entrepreneurial ability does not enter the production technology, we recover the formula in the baseline model. When $\alpha + \beta < 1$, expected returns potentially vary across entrepreneurs.

Expected returns and assets. In the span-of-control version of the model, the demand for capital is still given by (E.3), but the price of idiosyncratic risk is given by

$$p_i^{id} = \min\left\{\frac{\tilde{\mu}_i^R - r - p^{ag}\sigma_A}{\phi\sigma_{i,id}}, \tilde{\gamma}\phi\sigma_{i,id}\lambda\right\},$$

where $\tilde{\mu}_{i,t}^R \equiv \frac{\alpha\left(\frac{\beta}{w}\right)^{\frac{\beta}{1-\beta}} k_{i,t}^{\frac{\alpha+\beta-1}{1-\beta}} - \iota(q)}{q} + \mu_A + \Phi(\iota(q)) - \delta = \mu_i^R + \frac{\partial \mu_i^R}{\partial k_i} k_i$ captures returns on the marginal unit of capital, and $\tilde{\gamma}$ is the effective risk aversion with respect to total wealth bets.

Suppose first that $\lambda \to \infty$, so there is no leverage constraint. In this case, we can write the expected excess return on the business as follows:

$$\tilde{\mu}_{i,t}^R - r = p^{ag}\sigma_A + \frac{qk_{i,t}}{\omega_{i,t}}\tilde{\gamma}(\phi\sigma_{i,id})^2. \tag{E.9}$$

Using the expression for $\tilde{\mu}_i^R$, we obtain that the left-hand side is strictly decreasing in $k_{i,t}$, and it approaches ∞ as $k_{i,t}$ goes to zero when $\alpha + \beta < 1$. The right-hand side is strictly increasing in $k_{i,t}$. Hence, there is a unique value of $k_{i,t}$ that satisfies the equation above. Households with riskier projects have higher expected returns and hold a smaller capital stock, in line with the evidence in Section 2. Moreover, expected returns are higher for households with less total wealth, everything else constant, as they hold less capital.

When λ is finite, the entrepreneur will be constrained if $\omega_{i,t}$ is sufficiently low. Capital demand, $qk_{i,t} = \lambda\omega_{i,t}$, and expected returns are driven by variations in total wealth. For these entrepreneurs, expected returns are independent of the exposure to aggregate or idiosyncratic risk.

E.2 Uninsurable labor income risk and borrowing constraints

In this subsection, we introduce uninsurable labor income risk into the entrepreneur's problem. This will enable us to study the implications of both insurance and borrowing constraints on entrepreneurial behavior. In particular, we focus on how the inability to borrow against future income affects the entrepreneur's risk-taking decision.

E.2.1 The entrepreneurs' problem with labor income risk

Entrepreneurs receive labor income $\bar{l}_{i,t}$. With Poisson intensity λ_d, entrepreneurs suffer a "disability" shock that reduces their labor income by a factor $1-\xi_d$, that is, labor income is given by $\bar{l}_{i,t} = \bar{l}(a_i)$ in the no-disability state and $\bar{l}_{i,t} = (1-\xi_d)\bar{l}(a_i)$ in the disability state. The disability shock happens only once in an entrepreneur's lifetime, and it is permanent. When either $\lambda_d = 0$ or $\xi_d = 0$, we recover the model with no labor income risk discussed in Section 3.

We assume that households are subject to a natural borrowing limit, given by $\tilde{n}_{i,t} \geq -(1-\xi_d)\tilde{h}_{i,t}$.[42] As discussed in Aiyagari (1994), the natural borrowing limit under incomplete markets corresponds to the worst-case scenario of the realization of idiosyncratic shocks, which in our setting corresponds to the disability shock happening immediately. Therefore, under the natural borrowing limit, entrepreneurs can borrow at most a fraction $1-\xi_d$ of the human wealth $\tilde{h}_{i,t}$. Note that, even for an arbitrarily small value of λ_d, the borrowing limit is tighter in the presence of uninsurable income risk. In this case, the parameter ξ_d effectively controls the pledgeability of human wealth.

The entrepreneur's problem in the no-disability state is given by

$$\tilde{V}_t(\tilde{n}_i, a_i) = \max_{\tilde{c}_i, \tilde{\theta}_i^{ag}, \tilde{\theta}_t^{id}, k_i, l_i, \iota_i} \mathbb{E}_t \left[\int_t^{t+T_d} e^{-\rho(z-t)} \frac{\tilde{c}_{i,z}^{1-\gamma}}{1-\gamma} dz + e^{-\rho T_d} \tilde{V}_{t+T_d}^d (\tilde{n}_{i,t+T_d}, a_i + T_d) \right], \tag{E.10}$$

subject to non-negativity constraints $c_{i,t}, k_{i,t} \geq 0$, the law of motion of $\tilde{n}_{i,t}$

$$d\tilde{n}_{i,t} = \left[\tilde{n}_{i,t} r + \tilde{q}_t k_{i,t} (\mu_{i,t}^R - r) - p^{ag} \tilde{\theta}_{i,t}^{ag} + \tilde{w}_t \bar{l}_{i,t} - \tilde{c}_{i,t} \right] dt + \left(\tilde{q}_t k_{i,t} \sigma_A - \tilde{\theta}_{i,t}^{ag} \right) dZ_t + \left(\tilde{q}_t k_{i,t} \sigma_{id} - \tilde{\theta}_{i,t}^{id} \right) dZ_{i,t},$$

and insurance and borrowing constraints

$$\tilde{\theta}_{i,t}^{id} \leq (1-\phi) \tilde{q}_t k_{i,t} \sigma_{id}, \qquad \tilde{n}_{i,t} \geq -(1-\xi_d) \tilde{h}_{i,t}, \tag{E.11}$$

where $\tilde{h}_{i,t} \equiv \mathbb{E}_t \left[\int_t^{t+T-a_i} \frac{\pi_z}{\pi_t} \tilde{w}_z \bar{l}_i dz \right]$, T_d is the minimum of the (random) arrival time of a Pois-

[42] Kaplan and Violante (2010) argue that the standard incomplete markets model with a natural borrowing limit better captures the degree of partial insurance observed in the data than versions of the model with tighter borrowing limits.

son process with intensity $\lambda_d > 0$ and the entrepreneur's life horizon $T - a_i$. If $T_d < T - a_i$, then $\tilde{V}^d(\tilde{n}_i, a_i)$ corresponds to the value function in the disability state. As there is no labor income risk in the disability state, this is equal to the value function derived in Section 3. The value function evaluated at $a_i = T$, $\tilde{V}_t^d(\tilde{n}_i, T)$, is given by the bequest function.

E.2.2 The stationary problem

It is convenient to adopt a change of variables and to detrend the problem. First, we follow Aiyagari (1994) and define total (pledgeable) wealth $\tilde{\omega}_{i,t} = \tilde{n}_{i,t} + (1 - \xi_d)\tilde{h}_{i,t}$. This corresponds to the total amount of funds available to an entrepreneur, that is, the sum of the financial wealth and the borrowing limit. In the special case where $\xi_d = 0$, we recover the definition of total wealth given in Section 3. With a slight abuse of notation, we denote the entrepreneur's value function as $\tilde{V}_t(\tilde{\omega}_{i,t}, a_{i,t}) = \tilde{V}(\tilde{\omega}_{i,t}, a_{i,t}; A_t)$. Therefore, the entrepreneur's problem depends on the level of aggregate productivity. The next lemma shows that, despite the presence of uninsurable labor income risk and aggregate shocks, it is possible to detrend the problem and to define a stationary equilibrium in a way analogous to Lemma 1.

Lemma 3. *Suppose the economy is in a stationary equilibrium. Then,*

i. *Scaled variables are independent of aggregate productivity, that is, the scaled value function $V(\omega_t, a_t) = \frac{\tilde{V}_t(\omega_t A_t, a_t)}{A_t^{1-\gamma}}$ and the scaled policy functions $c_{i,t} = \frac{\tilde{c}_{i,t}}{A_t}$, $\theta_{i,t}^{id} = \frac{\tilde{\theta}_{i,t}^{id}}{A_t}$, and $\theta_{i,t}^{ag} = \frac{\tilde{\theta}_{i,t}^{ag}}{A_t}$ do not depend on A_t.*

ii. *The optimal value of $l_{i,t}$ and $\iota_{i,t}$ are given by (11) and (12), respectively. The insurance constraint is binding, and the shadow price of idiosyncratic insurance is given by (20). The price of aggregate insurance is given by (30) and the demand for aggregate insurance is given by $\theta_{i,t}^{ag} = (qk_{i,t} - n_{i,t})\,\sigma_A$.*

iii. *The scaled value function satisfies the HJB equation*

$$\hat{\rho} V = \max_{c_i, \sigma_{i,t}^{id}} \frac{c_t^{1-\gamma}}{1-\gamma} + V_a + V_\omega \left[\hat{r}\omega_{i,t} + p^{id}\sigma_{i,t}^{id} + \xi_d w_t \bar{l}_{i,t} - c_{i,t}\right] + \frac{1}{2} V_{\omega\omega}(\sigma_{i,t}^{id})^2 + \lambda_d \left(V^d - V\right), \tag{E.12}$$

where $\sigma_{i,t}^{id} = qk_{i,t}\phi\sigma_{id}$, $\hat{\rho} \equiv \rho - (1-\gamma)\left(\mu_A - \frac{\gamma\sigma_A^2}{2}\right)$, and $\hat{r} \equiv r + \gamma\sigma_A^2 - \mu_A$.

Proof. See Appendix A.6. □

In the first part of Lemma 3, we show that scaled variables are independent of aggregate productivity. This implies that a stationary distribution of scaled wealth exists and there is no need to approximate the aggregate wealth distribution by a finite number of moments as in Krusell and Smith (1998), despite the presence of aggregate risk and incomplete markets. Two assumptions are important to obtain this result: i) a borrowing limit that is proportional to aggregate income; ii) the probability of switching to the disability state is independent of aggregate shocks. Note that the natural borrowing limit in our setting satisfies the first condition. This result echoes the one obtained by Krueger and Lustig (2010), who derive conditions under which uninsurable labor income risk has no effect for the price of aggregate risk. Consistent with their findings, we show that the price of aggregate insurance is $p^{ag} = \gamma\sigma_A$, the same we would obtain in a complete-markets economy.

The second part of the lemma shows that the solution to the entrepreneur's problem shares many of the features we derived without idiosyncratic labor income risk. Labor demand and investment rates are chosen to maximize expected returns, insurance constraints are always binding, and entrepreneurs with low financial wealth have a positive demand for aggregate insurance.

The third part of Lemma 3 shows how the entrepreneur's problem ultimately reduces to a choice of (scaled) consumption and the exposure to idiosyncratic risk, or equivalently, the amount of capital to be employed in the business. In contrast to the problem in Section 3, a closed-form solution to this problem is not available, as the consumption function is not a linear function of total wealth anymore.

E.2.3 The approximate solution

We consider next an approximate solution to problem (E.12). Despite the well-known lack of closed-form solutions, we can provide an analytical characterization using perturbation techniques. In particular, we extend the methods used in Viceira (2001) to a general equilibrium life-cycle model. This approximate solution will allow us to show analytically how borrowing constraints affect entrepreneurs' risk-taking decisions, which is challenging to obtain from problem (E.12) otherwise.

Let $\omega(a)$ denote the average total wealth of entrepreneurs conditional on age a. We consider a log-linear approximation of the consumption and capital functions around the point $(\overline{\omega}, \overline{a})$:

$$\log c(\omega_{i,t}, a_{i,t}) = \log c(\overline{\omega}, \overline{a}) + \psi_{c,\omega}\hat{\omega}_i + \psi_{c,a}\hat{a}_i + \mathcal{O}\left(\hat{\omega}_i^2, \hat{a}_i^2\right) \tag{E.13}$$

$$\log k(\omega_{i,t}, a_{i,t}) = \log k(\overline{\omega}, \overline{a}) + \psi_{k,\omega}\hat{\omega}_i + \psi_{k,a}\hat{a}_i + \mathcal{O}\left(\hat{\omega}_i^2, \hat{a}_i^2\right) \tag{E.14}$$

where $\hat{\omega}_i \equiv \log \omega_i - \log \overline{\omega}$ and $\overline{\omega} \equiv \exp \mathbb{E}[\log \omega_{i,t} | a_{i,t} = \overline{a}]$, given $0 < \overline{a} < T$.

We can write the expressions above in a more compact form:

$$\hat{c}_{i,t} = \psi_{c,\omega}\hat{\omega}_{i,t} + \psi_{c,a}\hat{a}_{i,t}, \qquad \hat{k}_{i,t} = \psi_{k,\omega}\hat{\omega}_{i,t} + \psi_{k,a}\hat{a}_{i,t}, \tag{E.15}$$

up to first order in $(\hat{\omega}_{i,t}, \hat{a}_{i,t})$, where $\hat{z}_{i,t} \equiv \log z(\omega_{i,t}, a_{i,t}) - \log z(\overline{\omega}, \overline{a})$ for $z \in \{c, k\}$.

Note that this log-linear approximation is only feasible when total wealth is always positive, that is, $\omega_{i,t} > 0$ for all i and t. This condition holds in the solution to the entrepreneurs' problem given the combination of the natural borrowing limit with an unbounded marginal utility as $c_{i,t}$ approaches zero (see, e.g., the discussion in Chamberlain and Wilson 2000). If an entrepreneur were to borrow the maximum amount, such that $\omega_{i,t} = 0$, then consumption would be zero in the advent of a disability shock, a state with infinite marginal utility. An entrepreneur is better off by reducing borrowing and avoiding this possibility.

The next proposition provides a characterization of the first-order approximation of the entrepreneurs' problem.

Proposition 5. *Suppose the economy is in a stationary equilibrium and consider a first-order approximation of the policy functions and wealth dynamics around $(\overline{\omega}, \overline{a})$. Then,*

i. Consumption is an age-dependent concave function of total wealth given by

$$c_{i,t} = f_c(a_{i,t})\omega_{i,t}^{\psi_{c,\omega}}, \tag{E.16}$$

where $0 < \psi_{c,\omega} < 1$ and $f_c'(a_{i,t}) > 0$ for ξ_d sufficiently small.

ii. The demand for capital is given by

$$\frac{qk_{i,t}}{n_{i,t}} = \frac{1 + (1 - \xi_d)\frac{h_{i,t}}{n_{i,t}}}{\gamma\psi_{c,\omega}} \frac{p^{id}}{\phi\sigma_{id}}. \tag{E.17}$$

iii. Log total wealth evolves according to

$$d\log\omega_{i,t} = [\tilde{\psi}_{\omega,0} + \psi_{\omega,a}a_{i,t} + \psi_{\omega,\omega}\log\omega_{i,t}]\,dt + \frac{p^{id}}{\gamma\psi_{c,\omega}}dZ_{i,t}, \tag{E.18}$$

where $\psi_{\omega,\omega} < 0$ and $\psi_{\omega,a} < 0$ for ξ_d sufficiently small.

Proof. See Appendix A.7. □

The first part of Proposition 5 describes the consumption function. Consumption is a strictly concave function of total wealth, as it is typical of problems with uninsurable labor income risk, where $\psi_{c,\omega} \in (0, 1)$ represents the elasticity of consumption with respect to total wealth. This is in contrast to the consumption function derived in Proposition 1, where consumption is a linear function of $\omega_{i,t}$, that is, $\psi_{c,\omega} = 1$. For ξ_d sufficiently small, consumption is an increasing function of age given $\omega_{i,t}$, reflecting the impact of entrepreneurs' finite horizon.[43]

The second part of Proposition 5 gives the demand for capital and it corresponds to the main result of this section. The fact that entrepreneurs cannot borrow against future labor income has two (opposite) effects on the demand for capital. First, entrepreneurs have fewer resources available to them when $\xi_d > 0$, which tends to reduce the business scale. This effect is particularly more pronounced for entrepreneurs who are close to the borrowing limit, that is, $-n_{i,t}$ is close to $(1 - \xi_d)h_{i,t}$. Second, the limited pledgeability of human wealth *reduces* the effective risk aversion of entrepreneurs, as $\gamma\psi_{c,\omega} < \gamma$. The fraction of human wealth that cannot be used to fund a business investment acts as a buffer against future shocks, which makes the entrepreneur less concerned about taking investment risks.

It can be shown that uninsurable labor income risk reduces the scale of the business relative to an economy with $\xi_d = 0$ if and only if

$$n_{i,t} < \left[\frac{\xi_d}{1 - \psi_{c,\omega}} - 1\right] h_{i,t}. \tag{E.19}$$

Tighter borrowing constraints, captured by $\xi_d > 0$, reduce investment in the business for poor entrepreneurs, but they increase investment in the business for rich entrepreneurs. Given that these two forces move in opposite directions, the aggregate effect of borrowing constraints tends to be muted. Importantly, for all values of ξ_d, a declining human-financial wealth ratio over the life cycle causes entrepreneurs' exposure to the business to decline with age, as in the baseline model.

Entrepreneurs have an age-dependent target for wealth: $\frac{\tilde{\psi}_{\omega,0}+\psi_{\omega,a}a}{|\psi_{\omega,\omega}|}$. Entrepreneurs build up wealth when $\omega_{i,t}$ is below target and they decumulate wealth when $\omega_{i,t}$ is above target. When $\psi_{\omega,a} < 0$, the target on total wealth drifts down with age, again an implication of the entrepreneurs' finite horizon.

Note that the ratio of consumption to financial wealth can be written as

$$\frac{c_{i,t}}{n_{i,t}} = f_c(a_{i,t})\omega_{i,t}^{-(1-\psi_{c,\omega})}\left(1 + \frac{h_{i,t}}{n_{i,t}}\right). \tag{E.20}$$

As typically $f_c(\cdot)$ is increasing with age and $\omega_{i,t}$ is decreasing with age on average, we obtain that the first two terms in the expression above increase with age. Figure 3 shows that the human-financial wealth ratio is declining with age. In line with our discussion in Section 3, the consumption-financial wealth ratio then depends on two forces that move in opposite directions with age.

Therefore, we conclude that introducing uninsurable labor income risk and borrowing

[43]The condition on ξ_d is necessary, as with uninsurable labor income risk the slope of the labor income profile also plays a role. The effect of a finite horizon on consumption is attenuated when labor income declines with age, and this effect is amplified when labor income increases with age.

constraints does not change substantially our results, while it adds a significant layer of complexity to the analysis.

E.3 Endogenous occupational choice

In this subsection, we introduce an occupational choice into the households' problem. Moreover, we assume that wage earners have a finite horizon and imperfect altruism in the same way as entrepreneurs. For simplicity, we abstract from limited pledgeability and ex-ante heterogeneity on entrepreneurs and once again focus on a stationary equilibrium.

E.3.1 The occupational choice

At the beginning of life, a household can choose to become an entrepreneur or a wage earner. To become an entrepreneur, household i must pay a fixed cost $\varphi_i \tilde{y}_{i,t}$, where φ_i is a cost parameter draw from a distribution $F_\varphi(\cdot)$ with support $[\underline{\varphi}, \overline{\varphi}]$. Let $\tilde{V}_t(\tilde{n}_i, a)$ denote the value function of a household that chose to become an entrepreneur and $\tilde{V}_t^w(\tilde{n}_i, a)$ the value function of a household who chose to become a wage earner. In contrast to the model from Section 3, a wage earner lives for T periods and derives the same utility of bequests as entrepreneurs.

A household that inherits financial wealth $\tilde{n}_i$ will choose to become an entrepreneur if

$$\tilde{V}_t(\tilde{n}_i - \varphi_i \tilde{y}_t, 0) > \tilde{V}_t^w(\tilde{n}_i, 0). \tag{E.21}$$

The value function of an entrepreneur can be written, after normalization, as $V(n,a) = \zeta(a)^{-\frac{1}{\gamma}} \frac{(n+h(a))^{1-\gamma}}{1-\gamma}$. Similarly, the value function of a wage earner can be written as $V^w(n,a) = \zeta_w(a)^{-\frac{1}{\gamma}} \frac{(n+h_w(a))^{1-\gamma}}{1-\gamma}$.

The condition for becoming an entrepreneur can then be written as

$$\zeta(0)^{\frac{1}{\gamma(\gamma-1)}} (n_i + h(0) - \varphi_i y) > \zeta_w(0)^{\frac{1}{\gamma(\gamma-1)}} (n_i + h_w(0)). \tag{E.22}$$

Rearranging the expression above, we obtain that a household becomes an entrepreneur if $\varphi_i < \varphi^*(n_i)$, where the threshold $\varphi^*(n_i)$ is given by

$$\varphi^*(n_i) \equiv \frac{1}{y} \left[\left(\left(\frac{\zeta(0)}{\zeta_w(0)} \right)^{\frac{1}{\gamma(\gamma-1)}} - 1 \right) n_i + \left(\frac{\zeta(0)}{\zeta_w(0)} \right)^{\frac{1}{\gamma(\gamma-1)}} h(0) - h_w(0) \right]. \tag{E.23}$$

It can be shown that $\zeta(0) > \zeta_\omega(0)$ for $\gamma > 1$, so households who received larger bequests are more likely to become entrepreneurs. The difference between $\zeta(0)$ and $\zeta_\omega(0)$ is increasing in p^{id}, the shadow price of idiosyncratic risk.

As the cost parameter is drawn independently of the bequest a household receives, then the mass of entrepreneurs in a stationary equilibrium is given by

$$\chi_e = F_\varphi \left(\varphi^*(n(0))\right), \tag{E.24}$$

where $n(0)$ is the average financial wealth of newborn entrepreneurs.

In a stationary equilibrium, the mass of entrepreneurs is constant. As $\zeta(0)$, $\zeta_w(0)$, and $n(0)$ depend on the interest rate and the aggregate and idiosyncratic risk premia, then the share of entrepreneurs in the economy depends on the equilibrium expected returns.

E.3.2 Wage earners' problem and equilibrium determination

The optimal consumption-wealth ratio and demand for insurance for wage earners are now given by

$$\frac{c_{j,t}}{\omega_{j,t}} = \zeta_w(a) = \frac{\bar{r}_w}{1-\psi e^{-\bar{r}_w(T-a_j)}}, \qquad \theta^{ag}_{j,t} = h_{j,t}\sigma_A - \frac{p^{ag}}{\gamma}\omega_{j,t}, \tag{E.25}$$

where

$$\bar{r}_w = \frac{1}{\gamma}\rho_w + \left(1-\frac{1}{\gamma}\right)\left(r + \frac{(p^{ag})^2}{2\gamma}\right). \tag{E.26}$$

The price of aggregate insurance, wages, and the relative price of capital are the same as in the baseline model:

$$p^{ag} = \gamma\sigma_A, \qquad w = (1-\alpha)K^\alpha, \qquad q = \Phi_0 + \Phi_1(g+\delta). \tag{E.27}$$

Finite lives for wage earners change the determination of the interest rate. The interest rate is now jointly determined with the capital-labor ratio and the price of idiosyncratic risk by conditions (32), (33), and the market clearing condition for consumption

$$\int_0^T \frac{\bar{r}\omega(a)}{1-\psi e^{-\bar{r}(T-a)}} f(a)da + \int_0^T \frac{\bar{r}_w\omega_w(a)}{1-\psi e^{-\bar{r}_w(T-a)}} f(a)da = \alpha K^\alpha - \iota K, \tag{E.28}$$

where

$$\omega(a) = \omega(0)e^{\left(r+\gamma\sigma_A^2+\frac{(p^{id})^2}{\gamma}-\mu_A-\bar{r}\right)a}\frac{1-\psi e^{-\bar{r}(T-a)}}{1-\psi e^{-\bar{r}T}}, \qquad \omega_w(a) = \omega_w(0)e^{\left(r+\gamma\sigma_A^2-\mu_A-\bar{r}_w\right)a}\frac{1-\psi e^{-\bar{r}_w(T-a)}}{1-\psi e^{-\bar{r}_w T}}.$$

Assuming finite lives for wage earners would change the calibration of ρ_w but otherwise would not affect our main results.

E.4 Financial autarky

We consider next the case of financial autarky, where entrepreneurs have no access to either aggregate or idiosyncratic insurance. To shut down idiosyncratic insurance, we must set $\phi = 1$. To capture the absence of aggregate insurance, we will focus on the special case where there is no demand for aggregate insurance, so the solution would coincide with the case where entrepreneurs have no access to insurance.

Suppose that $h_{i,t} = 0$, so entrepreneurs have no labor income, and assume that $n_{j,t} = 0$ for $j \in \mathcal{W}_t$, so wage earners have no financial wealth. The first assumption implies that $\frac{qk_{i,t}}{n_{i,t}}$ is equalized across entrepreneurs, and the second assumption implies that $qk_{i,t} = n_{i,t}$. As the demand for aggregate insurance is given by $\theta^{ag}_{i,t} = (qk_{i,t} - n_{i,t})\sigma_A$, we obtain that entrepreneurs do not demand aggregate insurance. The solution will then coincide with the case where aggregate insurance is not available.

Under these assumptions, the price of idiosyncratic risk, given in Equation (33), specializes to the following expression:

$$p^{id} = \gamma\sigma_{id}. \tag{E.29}$$

The risk premium is then given by

$$p^{ag}\sigma_A + p^{id}\sigma_{id} = \gamma\left[\sigma_A^2 + \sigma_{id}^2\right], \tag{E.30}$$

and the aggregate Sharpe ratio relative to the idiosyncratic Sharpe ratio is given by $p^{ag}/p^{id} = \sigma_A/\sigma_{id}$.