7 Online Appendix: Properties of Cost Functionals

In this section, we collect together proofs of properties of cost functionals mentioned in main body of the paper.

7.1 Entropy Reduction Cost Functional

Lemma 18 *The entropy reduction information cost satisfies CPD for all* $\psi \in (\theta_{\min}, \theta_{\max})$.

Proof. For any SCR s, the associated entropy reduction is

$$c(s) = \mathbf{E}[H(s(\theta))] - H[\mathbf{E}(s(\theta))] \ ,$$

where $H : [0,1] \to \mathbb{R}$ is given by

$$H(x) = x \ln x + (1-x)\ln(1-x) \ .$$

Now let $p_1(s) = \mathbf{E}(s(\theta))$ denote the unconditional probability that action 1 is chosen under SCR s. Note that this cost functional is convex and Fréchet differentiable at s with derivative

$$H'(s(\theta)) - H'(p_1(s)) \ .$$

Now since $\psi \in (\theta_{\min}, \theta_{\max})$ and the prior density g is positive over $[\theta_{\min}, \theta_{\max}]$, we have $\mathbf{E}\left(1_{\{\theta \geq \psi\}}\right) \in (0,1)$. Choose $\xi > 0$ such that $\mathbf{E}\left(1_{\{\theta \geq \psi\}}\right) \in (\xi, 1-\xi)$. Then choose $\rho > 0$ small enough such that for all $s \in B_\rho\left(1_{\{\theta \geq \psi\}}\right)$, $p_1(s) \in (\xi, 1-\xi)$. Note that for small $\varepsilon > 0$, $s \in B_\rho\left(1_{\{\theta \geq \psi\}}\right)$ implies $L_\psi^\varepsilon s \in B_\rho\left(1_{\{\theta \geq \psi\}}\right)$. Let $A(s) = \left\{\theta : L_\psi^\varepsilon s(\theta) \neq s(\theta)\right\}$. Now Fréchet differentiability implies that we have

$$c\left(L_\psi^\varepsilon s\right) - c(s) \leq \int_{A(s)} \left[H'\left(L_\psi^\varepsilon s(\theta)\right) - H'\left(p_1\left(L_\psi^\varepsilon s\right)\right)\right]\left(L_\psi^\varepsilon s(\theta) - s(\theta)\right) dG(\theta)$$

and

$$c\left(L_\psi^\varepsilon s\right) - c(s) \geq \int_{A(s)} \left[H'(s(\theta)) - H'(p_1(s))\right]\left(L_\psi^\varepsilon s(\theta) - s(\theta)\right) dG(\theta) \ ,$$

Hence,

$$\left|c\left(L_\psi^\varepsilon s\right) - c(s)\right| \leq \max \left(\begin{array}{c} \left|\int_{A(s)} \left[H'(s(\theta)) - H'(p_1(s))\right]\left(L_\psi^\varepsilon s(\theta) - s(\theta)\right) dG(\theta)\right|, \\ \left|\int_{A(s)} \left[H'\left(L_\psi^\varepsilon s(\theta)\right) - H'\left(p_1\left(L_\psi^\varepsilon s\right)\right)\right]\left(L_\psi^\varepsilon s(\theta) - s(\theta)\right) dG(\theta)\right| \end{array} \right) \ .$$

Since $H'(x)$ is increasing in x, for all $\theta \in A(s)$, both $|H'(s(\theta)) - H'(p_1(s))|$ and $\left|H'\left(L_\psi^\varepsilon s(\theta)\right) - H'\left(p_1\left(L_\psi^\varepsilon s\right)\right)\right|$ are bounded above by

$$K = \max\left(|H'(1-\varepsilon) - H'(\xi)|, |H'(1-\xi) - H'(\varepsilon)|\right) \ .$$

Therefore,

$$\begin{aligned} \left|c\left(L_\psi^\varepsilon s\right) - c(s)\right| &\leq \int_{A(s)} K \cdot \left|L_\psi^\varepsilon s(\theta) - s(\theta)\right| dG(\theta) \\ &= K \cdot \left\|L_\psi^\varepsilon s, s\right\| \ . \end{aligned}$$

This concludes the proof. ■

7.2 The Pairwise-Separable Cost Functional

Lemma 19 *The PS cost functional satisfies A9 (feasible almost perfect discrimination).*

Proof. It suffices to show that $c_{PS}(\widehat{s}_{k,\psi}) < \infty$, i.e., the integral

$$\int_\theta \int_{\theta'} \left|\theta' - \theta\right|^{-\alpha} D\left(\widehat{s}_{k,\psi}(\theta), \widehat{s}_{k,\psi}(\theta')\right) h\left(\theta, \theta'\right) d\theta' d\theta$$

exists.

Let

$$A = \left\{\left(\theta, \theta'\right) \in \mathbb{R}^2 : -k^{-1} \leq \theta - \theta' \leq k^{-1}\right\}$$

and

$$A_1 = \left\{\left(\theta, \theta'\right) \in \mathbb{R}^2 : \theta \geq \psi + k^{-1}/2 \text{ and } \theta' \geq \psi + k^{-1}/2, \text{ or } \theta \leq \psi - k^{-1}/2 \text{ and } \theta' \leq \psi - k^{-1}/2\right\} \ .$$

First note that $\left|\theta' - \theta\right|^{-\alpha}$ is bounded on $\mathbb{R}^2 \backslash A$, thus the integral over $\mathbb{R}^2 \backslash A$ exists. Second, since $D\left(\widehat{s}_{k,\psi}(\theta), \widehat{s}_{k,\psi}(\theta')\right) = 0$ on A_1, we just need to show that the integral over $A \backslash A_1$ exists. Let

$$B_1 = \left\{\left(\theta, \theta'\right) \in A \backslash A_1 : -k^{-1}/2 \leq \theta' \leq k^{-1}/2 \text{ and } 0 \leq \theta - \theta' \leq k^{-1}\right\} \ ,$$

$$B_2 = \left\{\left(\theta, \theta'\right) \in A \backslash A_1 : -k^{-1}/2 \leq \theta' \leq k^{-1}/2 \text{ and } 0 \leq \theta' - \theta \leq k^{-1}\right\} \ ,$$

$$B_3 = \left\{\left(\theta, \theta'\right) \in A \backslash A_1 : -k^{-1}/2 \leq \theta \leq k^{-1}/2 \text{ and } 0 \leq \theta' - \theta \leq k^{-1}\right\} \ ,$$

and

$$B_4 = \left\{\left(\theta, \theta'\right) \in A \backslash A_1 : -k^{-1}/2 \leq \theta \leq k^{-1}/2 \text{ and } 0 \leq \theta - \theta' \leq k^{-1}\right\} \ .$$

Then $A \backslash A_1 = B_1 \cup B_2 \cup B_3 \cup B_4$. We next show that the integral over B_1 exists. Similar calculations can show the existence of the integral over B_2, B_3 and B_4, and are thus omitted.

By definition of a PS cost functional, $D(x_1, x_2)$ is bounded on $[0,1] \times [0,1]$ and $D(x_1, x_2) = O\left(|x_1 - x_2|^\beta\right)$ as $|x_1 - x_2| \to 0$. So there exists a $K > 0$, such that

$$D(x_1, x_2) \leq K \cdot |x_1 - x_2|^\beta \tag{23}$$

on $[0,1] \times [0,1]$. Now

$$\begin{aligned}
& \int_{B_1} \left|\theta - \theta'\right|^{-\alpha} D\left(\widehat{s}_{k,\psi}(\theta), \widehat{s}_{k,\psi}(\theta')\right) h\left(\theta, \theta'\right) d\theta' d\theta \\
\leq & \int_{B_1} \left|\theta - \theta'\right|^{-\alpha} K \cdot \left|\widehat{s}_{k,\psi}(\theta) - \widehat{s}_{k,\psi}(\theta')\right|^\beta h\left(\theta, \theta'\right) d\theta' d\theta \\
= & \int_{B_1} \left(\theta - \theta'\right)^{-\alpha} K \cdot \left(\frac{1}{2} + k(\theta - \psi) - \frac{1}{2} - k\left(\theta' - \psi\right)\right)^\beta h\left(\theta, \theta'\right) d\theta' d\theta \\
\leq & K k^\beta \overline{h} \int_{B_1} \left(\theta - \theta'\right)^{\beta - \alpha} d\theta' d\theta \ ,
\end{aligned}$$

for some $\overline{h} > 0$, where the first inequality is implied by (23), the equality is implied by the definition of $\widehat{s}_{k,\psi}$ and the last inequality is true because $\theta \geq \theta'$ on B_1 and $\frac{h(\theta,\theta')}{g(\theta)g(\theta')}$ is bounded above in the definition of PS cost functionals.

Now changing variables from (θ, θ') to (t, t') such that $t = \theta$ and $t' = \theta - \theta'$, we have

$$\begin{aligned}
& \int_{B_1} \left(\theta - \theta'\right)^{\beta - \alpha} d\theta' d\theta \\
= & \int_0^{k^{-1}} \left(t'\right)^{\beta - \alpha} \int_{-k^{-1}/2 + t'}^{k^{-1}/2 + t'} dt \cdot dt' \\
= & \ k^{-1} \int_0^{k^{-1}} \left(t'\right)^{\beta - \alpha} dt' \ .
\end{aligned}$$

This integral exists since $\beta - \alpha + 1 > 0$. Therefore, $c_{PS}\left(\widehat{s}_{k,\psi}\right) < \infty$. ■

Proposition 20 *The PS cost functional satisfies IPD if and only if $\alpha \geq 2$.*

Proof. Let s be a non-decreasing discontinuous SCR and $s\left(\widehat{\theta}_-\right) < s\left(\widehat{\theta}_+\right)$ for some $\widehat{\theta} \in [\theta_{\min}, \theta_{\max}]$.[29] Let

$$s_{\widehat{\theta}}(\theta) = \begin{cases} s\left(\widehat{\theta}_+\right) & \text{if } \theta > \widehat{\theta} \\ s\left(\widehat{\theta}_-\right) & \text{if } \theta \leq \widehat{\theta} \end{cases} \tag{24}$$

[29] We can focus on $\widehat{\theta} \in [\theta_{\min}, \theta_{\max}]$ because the possible $\widehat{\theta}$s of equilibrium SCRs are always in $[\theta_{\min}, \theta_{\max}]$ due to Assumption A3.

and

$$A=\min\left[D\left(s\left(\widehat{\theta}_{-}\right),s\left(\widehat{\theta}_{+}\right)\right),D\left(s\left(\widehat{\theta}_{+}\right),s\left(\widehat{\theta}_{-}\right)\right)\right].$$

Note that $A>0$ since $s\left(\widehat{\theta}_{-}\right)\neq s\left(\widehat{\theta}_{+}\right)$. Then we have

$$\begin{aligned}
c_{PS}(s) &= \int_{\theta}\int_{\theta'}\left|\theta'-\theta\right|^{-\alpha}D\left(s(\theta),s(\theta')\right)h\left(\theta,\theta'\right)d\theta' d\theta \\
&\geq \int_{\theta}\int_{\theta'}\left|\theta'-\theta\right|^{-\alpha}D\left(s_{\widehat{\theta}}(\theta),s_{\widehat{\theta}}(\theta')\right)h\left(\theta,\theta'\right)d\theta' d\theta \\
&= D\left(s\left(\widehat{\theta}_{-}\right),s\left(\widehat{\theta}_{+}\right)\right)\int_{-\infty}^{\widehat{\theta}}\int_{\widehat{\theta}}^{\infty}\left(\theta'-\theta\right)^{-\alpha}h\left(\theta,\theta'\right)d\theta' d\theta \\
&\quad +D\left(s\left(\widehat{\theta}_{+}\right),s\left(\widehat{\theta}_{-}\right)\right)\int_{\widehat{\theta}}^{\infty}\int_{-\infty}^{\widehat{\theta}}\left(\theta-\theta'\right)^{-\alpha}h\left(\theta,\theta'\right)d\theta' d\theta \\
&\geq 2A\cdot\int_{-\infty}^{\widehat{\theta}}\int_{\widehat{\theta}}^{\infty}\left(\theta'-\theta\right)^{-\alpha}h\left(\theta,\theta'\right)d\theta' d\theta\ , \qquad (25)
\end{aligned}$$

where the first inequality follows the monotonicity of s in θ, and the second inequality follows the definition of A. Since g is continuous and strictly positive on $[\theta_{\min},\theta_{\max}]$, it has a strictly positive lower bound on $[\theta_{\min},\theta_{\max}]$. Since $\frac{g(\theta)g(\theta')}{h(\theta,\theta')}$ is bounded above, $h\left(\theta,\theta'\right)$ has a strictly positive lower bound on $[\theta_{\min},\theta_{\max}]\times[\theta_{\min},\theta_{\max}]$. Hence, $\int_{-\infty}^{\widehat{\theta}}\int_{\widehat{\theta}}^{\infty}\left(\theta'-\theta\right)^{-\alpha}h\left(\theta,\theta'\right)d\theta' d\theta$ is integrable if and only if $2-\alpha>0$. Therefore, $\alpha\geq 2$ implies $c_{PS}(s)=\infty$ and thus IPD. For the converse, consider an SCR $s_{\widehat{\theta}}(\cdot)$ defined by (24) such that $D\left(s\left(\widehat{\theta}_{-}\right),s\left(\widehat{\theta}_{+}\right)\right)=D\left(s\left(\widehat{\theta}_{+}\right),s\left(\widehat{\theta}_{-}\right)\right)\equiv A>0$. Immediate from the previous derivation of (25) we obtain that $c_{PS}\left(s_{\widehat{\theta}}\right)=\infty$ if $\alpha\geq 2$ and $c\left(s_{\widehat{\theta}}\right)<\infty$ if $\alpha<2$. Then, IPD implies $c_{PS}\left(s_{\widehat{\theta}}\right)=\infty$ and thus $\alpha\geq 2$. ■

The following lemmas show that CPD is satisfied if $\alpha=0$ and it is easier to be satisfied at lower values of α. Since the PS cost functional is continuous in α, there exists some $\widehat{\alpha}\in[0,\min(2,\beta+1)]$ such that CPD is satisfied for $\alpha\in[0,\widehat{\alpha}]$. Due to the technicalities associated with the PS cost functional and the generality of the definitions of CPD and EPD, however, we do not obtain an analytical bound $\widehat{\alpha}$ between CPD and EPD.

Lemma 21 *The PS cost functional satisfies CPD at $\alpha=0$.*

Proof. When $\alpha=0$, the cost functional becomes

$$c_{PS}(s)=\int_{\theta}\int_{\theta'}D\left(s(\theta),s(\theta')\right)h\left(\theta,\theta'\right)d\theta' d\theta\ .$$

Hence, by the triangle inequality,

$$\begin{aligned}\left|c_{PS}\left(L_{\psi}^{\varepsilon}s\right)-c_{PS}\left(s\right)\right| &= \left|\int_{\theta}\int_{\theta'}\left[D\left(\left(L_{\psi}^{\varepsilon}s\right)\left(\theta\right),\left(L_{\psi}^{\varepsilon}s\right)\left(\theta'\right)\right)-D\left(s\left(\theta\right),s\left(\theta'\right)\right)\right]h\left(\theta,\theta'\right)d\theta' d\theta\right| \\ &\leq \int_{\theta}\int_{\theta'}\left|D\left(\left(L_{\psi}^{\varepsilon}s\right)\left(\theta\right),\left(L_{\psi}^{\varepsilon}s\right)\left(\theta'\right)\right)-D\left(s\left(\theta\right),s\left(\theta'\right)\right)\right|h\left(\theta,\theta'\right)d\theta' d\theta \\ &\leq \int_{\theta}\int_{\theta'}\left|D\left(\left(L_{\psi}^{\varepsilon}s\right)\left(\theta\right),\left(L_{\psi}^{\varepsilon}s\right)\left(\theta'\right)\right)-D\left(\left(L_{\psi}^{\varepsilon}s\right)\left(\theta\right),s\left(\theta'\right)\right)\right|h\left(\theta,\theta'\right)d\theta' d\theta \\ &\quad +\int_{\theta}\int_{\theta'}\left|D\left(\left(L_{\psi}^{\varepsilon}s\right)\left(\theta\right),s\left(\theta'\right)\right)-D\left(s\left(\theta\right),s\left(\theta'\right)\right)\right|h\left(\theta,\theta'\right)d\theta' d\theta\ . \qquad (26)\end{aligned}$$

Since $\frac{\partial D(x_1,x_2)}{\partial x_1}$ and $\frac{\partial D(x_1,x_2)}{\partial x_2}$ exist on $[0,1]\times[0,1]$,[30] there exists a $K>0$ such that $|D(x_1',x_2)-D(x_1,x_2)|\leq K\cdot|x_1'-x_1|$ and $|D(x_1,x_2')-D(x_1,x_2)|\leq K\cdot|x_2'-x_2|$ for all $x_1,x_2\in[0,1]$. Hence,

$$\left|D\left(\left(L_{\psi}^{\varepsilon}s\right)\left(\theta\right),\left(L_{\psi}^{\varepsilon}s\right)\left(\theta'\right)\right)-D\left(\left(L_{\psi}^{\varepsilon}s\right)\left(\theta\right),s\left(\theta'\right)\right)\right|\leq K\cdot\left|\left(L_{\psi}^{\varepsilon}s\right)\left(\theta'\right)-s\left(\theta'\right)\right|$$

and

$$\left|D\left(\left(L_{\psi}^{\varepsilon}s\right)\left(\theta\right),s\left(\theta'\right)\right)-D\left(s\left(\theta\right),s\left(\theta'\right)\right)\right|\leq K\cdot\left|\left(L_{\psi}^{\varepsilon}s\right)\left(\theta\right)-s\left(\theta\right)\right|\ .$$

Plugging the above two inequalities into (26), we obtain

$$\begin{aligned}&\left|c_{PS}\left(L_{\psi}^{\varepsilon}s\right)-c_{PS}\left(s\right)\right| \\ \leq\ &\int_{\theta}\int_{\theta'}K\cdot\left|\left(L_{\psi}^{\varepsilon}s\right)\left(\theta'\right)-s\left(\theta'\right)\right|h\left(\theta,\theta'\right)d\theta' d\theta+\int_{\theta}\int_{\theta'}K\cdot\left|\left(L_{\psi}^{\varepsilon}s\right)\left(\theta\right)-s\left(\theta\right)\right|h\left(\theta,\theta'\right)d\theta' d\theta \\ \leq\ &\int_{\theta}\int_{\theta'}K\cdot\left|\left(L_{\psi}^{\varepsilon}s\right)\left(\theta'\right)-s\left(\theta'\right)\right|K'g\left(\theta'\right)g\left(\theta\right)d\theta' d\theta+\int_{\theta}\int_{\theta'}K\cdot\left|\left(L_{\psi}^{\varepsilon}s\right)\left(\theta\right)-s\left(\theta\right)\right|K'g\left(\theta'\right)g\left(\theta\right)d\theta' d\theta \\ =\ &KK'\cdot\int_{\theta}\left\|L_{\psi}^{\varepsilon}s,s\right\|g\left(\theta\right)d\theta+KK'\cdot\int_{\theta'}\left\|L_{\psi}^{\varepsilon}s,s\right\|g\left(\theta'\right)d\theta' \\ =\ &2KK'\cdot\left\|L_{\psi}^{\varepsilon}s,s\right\|\ ,\end{aligned}$$

where the second inequality follows because $\frac{h\left(\theta,\theta'\right)}{g(\theta)g(\theta')}$ is bounded above by some $K'>0$. Therefore, c_{PS} satisfies CPD when $\alpha=0$. ■

Lemma 22 *If the PS cost functional satisfies CPD at some $\alpha\geq 0$, then it satisfies CPD at all $\alpha'\in[0,\alpha]$.*

Proof. To avoid confusion, let $c_{PS}^{\alpha}(\cdot)$ denote the PS cost functional with parameter α. Since $c_{PS}^{\alpha}(\cdot)$ satisfies CPD, for any $\psi\in\mathbb{R}$ and $\varepsilon\in(0,1/2)$, there exists a $\rho>0$ and $K>0$ such that

$$\left|c_{PS}^{\alpha}\left(L_{\psi}^{\varepsilon}s\right)-c_{PS}^{\alpha}\left(s\right)\right|\leq K\cdot\left\|L_{\psi}^{\varepsilon}s,s\right\|$$

[30] The proof goes through under a weaker condition that $\frac{\partial}{\partial x_i}D(x_1,x_2)$ exists for all $x_i\in(0,1)$ and $x_j\in[0,1]$, $i,j\in\{1,2\}$, $i\neq j$.

for all monotonic $s \in B_\rho\left(1_{\{\theta \geq \psi\}}\right)$. Without loss of generality, we can choose a sufficiently small $\rho > 0$. Then by the construction of operator L_ψ^ε, there exists an interval $[\theta_1, \theta_2]$ such that for any monotonic $s \in B_\rho\left(1_{\{\theta \geq \psi\}}\right)$, $L_\psi^\varepsilon s$ and s differ only in $[\theta_1, \theta_2]$. Fix a $z > 0$. Then

$$
\begin{aligned}
& \left|c_{PS}^{\alpha'}\left(L_\psi^\varepsilon s\right) - c_{PS}^{\alpha'}\left(s\right)\right| \\
= & \left|\int_\theta \int_{\theta'} \left|\theta' - \theta\right|^{-\alpha'} \left[D\left(\left(L_\psi^\varepsilon s\right)(\theta), \left(L_\psi^\varepsilon s\right)(\theta')\right) - D\left(s(\theta), s(\theta')\right)\right] h(\theta, \theta')\, d\theta' d\theta\right| \\
\leq & \left|\int_{\mathbb{R}^2 \setminus [\theta_1 - z, \theta_2 + z] \times [\theta_1 - z, \theta_2 + z]} \left|\theta' - \theta\right|^{-\alpha'} \left[D\left(\left(L_\psi^\varepsilon s\right)(\theta), \left(L_\psi^\varepsilon s\right)(\theta')\right) - D\left(s(\theta), s(\theta')\right)\right] h(\theta, \theta')\, d\theta' d\theta\right| \\
& + \left|\int_{[\theta_1 - z, \theta_2 + z] \times [\theta_1 - z, \theta_2 + z]} \left|\theta' - \theta\right|^{-\alpha'} \left[D\left(\left(L_\psi^\varepsilon s\right)(\theta), \left(L_\psi^\varepsilon s\right)(\theta')\right) - D\left(s(\theta), s(\theta')\right)\right] h(\theta, \theta')\, d\theta' d\theta\right| \\
= & \left|\int_{(-\infty, \theta_1 - z) \cup (\theta_2 + z, \infty)} \int_{[\theta_1, \theta_2]} \left|\theta' - \theta\right|^{-\alpha'} \left[D\left(\left(L_\psi^\varepsilon s\right)(\theta), \left(L_\psi^\varepsilon s\right)(\theta')\right) - D\left(s(\theta), s(\theta')\right)\right] h(\theta, \theta')\, d\theta' d\theta\right| \\
& + \left|\int_{[\theta_1, \theta_2]} \int_{(-\infty, \theta_1 - z) \cup (\theta_2 + z, \infty)} \left|\theta' - \theta\right|^{-\alpha'} \left[D\left(\left(L_\psi^\varepsilon s\right)(\theta), \left(L_\psi^\varepsilon s\right)(\theta')\right) - D\left(s(\theta), s(\theta')\right)\right] h(\theta, \theta')\, d\theta' d\theta\right| \\
& + \left|\int_{[\theta_1 - z, \theta_2 + z] \times [\theta_1 - z, \theta_2 + z]} \left|\theta' - \theta\right|^{-\alpha'} \left[D\left(\left(L_\psi^\varepsilon s\right)(\theta), \left(L_\psi^\varepsilon s\right)(\theta')\right) - D\left(s(\theta), s(\theta')\right)\right] h(\theta, \theta')\, d\theta' d\theta\right|
\end{aligned} \tag{27}
$$

where the second equality follows the fact $L_\psi^\varepsilon s$ and s differ only in $[\theta_1, \theta_2]$. Since $\frac{\partial D(x_1, x_2)}{\partial x_1}$ and $\frac{\partial D(x_1, x_2)}{\partial x_2}$ exist on $[0,1] \times [0,1]$,[31] there exists a $K_1 > 0$ such that $|D(x_1', x_2) - D(x_1, x_2)| \leq K_1 \cdot |x_1' - x_1|$ and $|D(x_1, x_2') - D(x_1, x_2)| \leq K_1 \cdot |x_2' - x_2|$ for all $x_1, x_2 \in [0,1]$. Then, the first term in the right hand side of (27) is

$$
\begin{aligned}
& \left|\int_{(-\infty, \theta_1 - z) \cup (\theta_2 + z, \infty)} \int_{[\theta_1, \theta_2]} \left|\theta' - \theta\right|^{-\alpha'} \left[D\left(\left(L_\psi^\varepsilon s\right)(\theta), \left(L_\psi^\varepsilon s\right)(\theta')\right) - D\left(s(\theta), s(\theta')\right)\right] h(\theta, \theta')\, d\theta' d\theta\right| \\
\leq & \int_{(-\infty, \theta_1 - z) \cup (\theta_2 + z, \infty)} \int_{[\theta_1, \theta_2]} \left|\theta' - \theta\right|^{-\alpha'} \left|D\left(s(\theta), \left(L_\psi^\varepsilon s\right)(\theta')\right) - D\left(s(\theta), s(\theta')\right)\right| h(\theta, \theta')\, d\theta' d\theta \\
\leq & K' \int_{(-\infty, \theta_1 - z) \cup (\theta_2 + z, \infty)} \int_{[\theta_1, \theta_2]} z^{-\alpha'} K_1 \cdot \left|\left(L_\psi^\varepsilon s\right)(\theta') - s(\theta')\right| g(\theta')\, d\theta' g(\theta)\, d\theta \\
\leq & z^{-\alpha'} K' K_1 \cdot \int_{(-\infty, \theta_1 - z) \cup (\theta_2 + z, \infty)} \left\|L_\psi^\varepsilon s, s\right\| g(\theta)\, d\theta \\
\leq & z^{-\alpha'} K' K_1 \cdot \left\|L_\psi^\varepsilon s, s\right\| ,
\end{aligned}
$$

where the first inequality holds because $\left(L_\psi^\varepsilon s\right)(\theta) = s(\theta)$ for $\theta \in (-\infty, \theta_1 - z) \cup (\theta_2 + z, \infty)$, and the second inequality follows that $\left|\theta' - \theta\right|^{-\alpha'} \leq z^{-\alpha'}$ for $\theta \in (-\infty, \theta_1 - z) \cup (\theta_2 + z, \infty)$

[31] The proof goes through under a weaker condition that $\frac{\partial}{\partial x_i} D(x_1, x_2)$ exists for all $x_i \in (0,1)$ and $x_j \in [0,1]$, $i, j \in \{1, 2\}$, $i \neq j$.

and $\theta' \in [\theta_1, \theta_2]$, and that $\frac{h(\theta,\theta')}{g(\theta)g(\theta')}$ is bounded above by some $K' > 0$. By a symmetric argument, the second term in the right hand side of (27) is also bounded by $z^{-\alpha'} K' K_1 \cdot \left\| L_\psi^\varepsilon s, s \right\|$. Since $\alpha - \alpha' \geq 0$, $\left|\theta' - \theta\right|^{\alpha - \alpha'}$ is bounded for $(\theta, \theta') \in [\theta_1 - z, \theta_2 + z] \times [\theta_1 - z, \theta_2 + z]$, then there is a $K_2 > 0$ such that the third term in the right hand side of (27) is

$$
\begin{aligned}
& \left| \int_{[\theta_1 - z, \theta_2 + z] \times [\theta_1 - z, \theta_2 + z]} \left|\theta' - \theta\right|^{\alpha - \alpha'} \left|\theta' - \theta\right|^{-\alpha} \left[D\left(\left(L_\psi^\varepsilon s\right)(\theta), \left(L_\psi^\varepsilon s\right)(\theta') \right) - D\left(s(\theta), s(\theta') \right) \right] h(\theta, \theta')\, d\theta' d\theta \right| \\
\leq\ & K' K_2 \cdot \left| \int_{[\theta_1 - z, \theta_2 + z] \times [\theta_1 - z, \theta_2 + z]} \left|\theta' - \theta\right|^{-\alpha} \left[D\left(\left(L_\psi^\varepsilon s\right)(\theta), \left(L_\psi^\varepsilon s\right)(\theta') \right) - D\left(s(\theta), s(\theta') \right) \right] g(\theta')\, g(\theta)\, d\theta' d\theta \right| \\
\leq\ & K' K_2 \cdot \left| c_{PS}^{\alpha}\left(L_\psi^\varepsilon s\right) - c_{PS}^{\alpha}(s) \right| \\
\leq\ & K' K_2 K \cdot \left\| L_\psi^\varepsilon s, s \right\| \ .
\end{aligned}
$$

Hence, (27) becomes

$$
\begin{aligned}
& \left| c_{PS}^{\alpha'}\left(L_\psi^\varepsilon s\right) - c_{PS}^{\alpha'}(s) \right| \\
\leq\ & 2 z^{-\alpha'} K' K_1 \cdot \left\| L_\psi^\varepsilon s, s \right\| + K' K_2 K \cdot \left\| L_\psi^\varepsilon s, s \right\| \\
=\ & \left(2 z^{-\alpha'} K_1 + K_2 K \right) K' \cdot \left\| L_\psi^\varepsilon s, s \right\| \ .
\end{aligned}
$$

Therefore, $c_{PS}^{\alpha'}$ satisfies CPD. ■

7.3 The Fisher Cost Functional

Lemma 23 *The Fisher cost functional satisfies sub-modularity.*

Proof. Let s_1 and s_2 be two SCRs. It is straightforward to see that $c_{Fisher}(s_2 \vee s_1) + c_{Fisher}(s_2 \wedge s_1) = c_{Fisher}(s_1) + c_{Fisher}(s_2)$. Let $A = \{\theta \in \mathbb{R} : s_2(\theta) \geq s_1(\theta)\}$ and $B = \{\theta \in \mathbb{R} : s_2(\theta) < s_1(\theta)\}$. Then,

$$
\begin{aligned}
& c_{Fisher}(s_2 \vee s_1) + c_{Fisher}(s_2 \wedge s_1) \\
= & \int_A \frac{\left([g(\theta)\, s_2(\theta)]'\right)^2}{g(\theta)\, s_2(\theta)} + \frac{\left([g(\theta)(1 - s_2(\theta))]'\right)^2}{g(\theta)(1 - s_2(\theta))} d\theta + \int_B \frac{\left([g(\theta)\, s_1(\theta)]'\right)^2}{g(\theta)\, s_1(\theta)} + \frac{\left([g(\theta)(1 - s_1(\theta))]'\right)^2}{g(\theta)(1 - s_1(\theta))} d\theta \\
& + \int_A \frac{\left([g(\theta)\, s_1(\theta)]'\right)^2}{g(\theta)\, s_1(\theta)} + \frac{\left([g(\theta)(1 - s_1(\theta))]'\right)^2}{g(\theta)(1 - s_1(\theta))} d\theta + \int_B \frac{\left([g(\theta)\, s_2(\theta)]'\right)^2}{g(\theta)\, s_2(\theta)} + \frac{\left([g(\theta)(1 - s_2(\theta))]'\right)^2}{g(\theta)(1 - s_2(\theta))} d\theta \\
= & \int_A \frac{\left([g(\theta)\, s_1(\theta)]'\right)^2}{g(\theta)\, s_1(\theta)} + \frac{\left([g(\theta)(1 - s_1(\theta))]'\right)^2}{g(\theta)(1 - s_1(\theta))} d\theta + \int_B \frac{\left([g(\theta)\, s_1(\theta)]'\right)^2}{g(\theta)\, s_1(\theta)} + \frac{\left([g(\theta)(1 - s_1(\theta))]'\right)^2}{g(\theta)(1 - s_1(\theta))} \\
& + \int_A \frac{\left([g(\theta)\, s_2(\theta)]'\right)^2}{g(\theta)\, s_2(\theta)} + \frac{\left([g(\theta)(1 - s_2(\theta))]'\right)^2}{g(\theta)(1 - s_2(\theta))} d\theta + \int_B \frac{\left([g(\theta)\, s_2(\theta)]'\right)^2}{g(\theta)\, s_2(\theta)} + \frac{\left([g(\theta)(1 - s_2(\theta))]'\right)^2}{g(\theta)(1 - s_2(\theta))} d\theta d\theta \\
= & c_{Fisher}(s_1) + c_{Fisher}(s_2) \ .
\end{aligned}
$$

■

7.4 The Additive Noise Cost Functional

Here we show that the additive noise cost functional c_{AN} is not submodular, by constructing a counterexample. Suppose ε is uniform on $\left[-\frac{1}{2}, \frac{1}{2}\right]$. Let $b_\psi = 1_{\{x \geq \psi\}}$ be the step function behavioral strategy where a player invests if and only if his signal is above ψ. Then the induced stochastic choice rule $\widetilde{s}_{k,b_\psi}$ is equal to the slope k threshold approximation of $1_{\{\theta \geq \psi\}}$, i.e.,

$$\widetilde{s}_{k,b_\psi}(\theta) = \int_{-1/2}^{1/2} b_\psi\left(\theta + \frac{1}{k}\varepsilon\right) d\varepsilon = \int_{-1/2}^{1/2} 1_{\varepsilon \leq k(\theta - \psi)} = \widehat{s}_{k,\psi}(\theta)$$

Since k is the maximum slope of $\widehat{s}_{k,\psi}$, we have

$$\frac{d\widetilde{s}_{k,b}(\theta)}{d\theta} \leq k \ , \tag{28}$$

where the inequality is an equality if and only if the behavioral strategy is the switching strategy b_ψ for some switching cutoff ψ. Now consider $\widetilde{s}_{k_1,b_\psi}$ and $\widetilde{s}_{k_2,b_\psi}$, where $k_2 > k_1 > 0$. Note that $\widetilde{s}_{k_1,b_\psi}$ and $\widetilde{s}_{k_2,b_\psi}$ intersect at $(\psi, 1/2)$, so that

$$\left(\widetilde{s}_{k_1,b_\psi} \vee \widetilde{s}_{k_2,b_\psi}\right)(\theta) = \begin{cases} \widetilde{s}_{k_1,b_\psi}(\theta) & \text{if } \theta < \psi \\ \widetilde{s}_{k_2,b_\psi}(\theta) & \text{if } \theta \geq \psi \end{cases}$$

and

$$\left(\widetilde{s}_{k_1,b_\psi} \wedge \widetilde{s}_{k_2,b_\psi}\right)(\theta) = \begin{cases} \widetilde{s}_{k_2,b_\psi}(\theta) & \text{if } \theta < \psi \\ \widetilde{s}_{k_1,b_\psi}(\theta) & \text{if } \theta \geq \psi \end{cases} .$$

So k_2 is the maximal slope of both $\widetilde{s}_{k_1,b_\psi} \vee \widetilde{s}_{k_2,b_\psi}$ and $\widetilde{s}_{k_1,b_\psi} \wedge \widetilde{s}_{k_2,b_\psi}$. Inequality (28) thus implies $c_{AN}\left(\widetilde{s}_{k_1,b_\psi} \vee \widetilde{s}_{k_2,b_\psi}\right) = c(k_2)$ and $c_{AN}\left(\widetilde{s}_{k_1,b_\psi} \wedge \widetilde{s}_{k_2,b_\psi}\right) = c(k_2)$. Therefore,

$$\begin{aligned} c_{AN}\left(\widetilde{s}_{k_1,b_\psi}\right) + c_{AN}\left(\widetilde{s}_{k_2,b_\psi}\right) &= \widehat{c}(k_1) + \widehat{c}(k_2) \\ &< 2\widehat{c}(k_2) \\ &= c_{AN}\left(\widetilde{s}_{k_1,b_\psi} \vee \widetilde{s}_{k_2,b_\psi}\right) + c_{AN}\left(\widetilde{s}_{k_1,b_\psi} \wedge \widetilde{s}_{k_2,b_\psi}\right) , \end{aligned}$$

a violation of submodularity.