14.123: Choice, Decisions, and Utility Theory

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Course Mechanics:

- Four problem sets (40%) and an in-class final exam (60%).
- Each problem set has one problem that is graded.
- You can also submit solutions to other problems for feedback.
- Group work encouraged, write your own solutions.
- We expect you to know proofs <u>iff</u> we sketch them in class or section, point you towards a proof in a text or note, or explicitly assign them. Other results will be stated w/o proof.

Motivation and Overview

- Want at least approximately correct models of individual-level behavior.
- Also want the models to be tractable and relatively parsimonious.
- Standard models of “rational choice” have proved useful in a great many settings but are imperfect.
- So useful to understand when standard models apply, and when and how they should be relaxed.
- “Representation theorems” can help with this: they give conditions on the observables under which various models/representations apply, and may suggest how to change the representations when they don’t.

- We’ll cover both “standard models” and some popular alternatives such as prospect theory and quasi-hyperbolic discounting.

- Look mostly at the theoretical implications of the various models, but also discuss some facts they don’t fit- including facts that don’t fit the behavioral models.

- Focus on classic theoretical and experimental results, but cover some newer ones too.

- Would like everyone in the class to end up with a working knowledge of what the main theories are, what they assume, when they seem to apply, and how to use them.

- Would also like everyone to have some idea of how the related proofs work; people interested in theory should spend more time on the optional readings and proofs.

Tentative Outline

Part 1: Static Choice

- Basics (Choice, Preference, Revealed Preference, Utility Representations)
- Expected utility under risk (objective probabilities)
- Risk aversion and stochastic dominance
- Non-linear risk preferences, e.g. prospect theory and cumulative prospect theory
- Expected utility under uncertainty (subjective probabilities), calibration, and heuristics
- Ambiguity Aversion

Part 2: Dynamic Choice

- Consequentialism and dynamic consistency
- Temptation, self-control, and quasi-hyperbolic discounting
- Discounted expected utility and recursive expected utility

Part 3 Stochastic Choice

- Static Stochastic Choice- individual and aggregate
- Dynamic Stochastic Choice (*not covered in class- papers tba*)

Part 4 Social Choice

- Arrow's Possibility Theorem and Social Choice on Restricted Domains
- Introduction to Mechanism Design and the Gibbard-Satterthwaite Theorem.

Ideally will cover most of the above and have time for discussions- both are important. It's theoretically possible to have too much discussion but empirically rare…

Static Deterministic Choice and Revealed Preference

X : the set of alternatives to choose from

$M(X) := 2^X / \{\varnothing\}$ non-empty subsets of *X* (finite for now)

$M \subseteq M(X)$: the choice sets ("menus") we have data on

Choice correspondence a map $c: M(X) \to M(X)$ s.t. $c(A) \subseteq A$.

(*note this doesn't track the relative choice frequencies within c(A)*)

(*we will assume the analyst knows/observes the value of* $c(A)$ *for every* $A \in M$ *. When* $c(A)$ *isn't a singleton this implies we have "several" observations of choice from A.*)

(*and note that a choice correspondence is supposed to also specify choice on all menus including those we haven't observed.*)

Preference relation $\succsim$ on $X \times X$: $x \succsim y$ means "*x* is weakly preferred to *y*."

Preferences are **complete** if at least one of $x \succsim y$ or $y \succsim x$ for every *x,y.*

Preferences are **transitive** if for all *x,y,z* in *X*, $x \succsim y \succsim z$ implies $x \succsim z$.

Preferences are **anti-symmetric** if $x \succsim y$ and $y \succsim x$ implies $x = y$.

- Complete transitive preferences are often called "rational." (*a better name might be "internally consistent"*)
- Such preferences typically assumed.
- Like them to be "stable enough" to be used for predictions.
- Choices made w/o enough thought may violate transitivity, and w/o a time limit choice may be incomplete.
- In practice "Preferences" may depend on the menu (*examples next lecture*)
- Or on past consumption (*when we do dynamic choice*)
- Both of those possibilities ruled out here.

Every complete transitive preference defines a choice correspondence:

$$c_{\succeq}(A) = \{x \in A : x \succsim y \quad \forall y \in A\}$$

Weak Axiom of Revealed Preference (WARP): If $A, B \in M$, $x, y \in A \cap B$, $x \in C(A)$, and $y \in C(B)$, then $x \in c(B)$.

- WARP implies what is called Sen's α or Arrow's IIA:
  If $x \in A \subseteq B$ and $x \in c(B) \rightarrow x \in c(A)$.

Definition:

- *x* is ***revealed weakly preferred*** to *y,* written $x \succsim^{*} y$, if *x* was chosen in any menu that contains *y.*

- *x* is ***revealed strictly preferred*** to *y,* $x \succ^{*} y$, if *x* but not *y* was chosen in a menu that contains both.

- Choice induced by any complete transitive preference must satisfy WARP.

- Moreover with enough data WARP pins down the generating preferences.

Theorem: (Arrow [1959]) If M contains all subsets of two or three elements, then the choice data c satisfies WARP iff the revealed preference relation is complete and transitive, and under WARP $c_{\succsim^*} = c$.

Notes:

- If *M* contains all binary menus the revealed preference relation is complete, but needn't be transitive; the proof shows that seeing all 3-element subsets is enough.

- In many cases we won't see choice from all binary menus, and then the revealed preference relation needn't be complete.

- A proof is in MWG.

Proof is simpler when *X* is finite, *M* is all of $M(X)$ (*i.e. all menus observed*) and choice is single-valued. In this case Arrow's condition simplifies to "Sen's α":

If $c(B) \in A \subseteq B$ then $c(A) = c(B)$.

Theorem: If the choice function *c* is single valued, it satisfies Sen's α iff the revealed preference relation $\succsim^*$ is complete, transitive, and anti-symmetric, and $c = c_{\succsim^*}$

proof of the only if: Suppose choice is single valued and satisfies Sen's α and $M = M(X)$. Define the "binary revealed preference" relation $\succsim^{**}$ by $x \succsim^{**} y$ iff $x = c(x, y)$ (restricting the revealed preference comparisons to 2-item menus.)

- $\succsim^{**}$ is complete, because we see a choice from every pair.
- $\succsim^{**}$ is asymmetric because it is single-valued.
- For transitivity, suppose $x \succsim^{**} y \succsim^{**} z$ where x,y, z are distinct. Then $x = c(x, y)$ and $y = c(y, z)$; want to show that $x = c(x, z)$ so that $x \succsim^{**} z$.

To show $x = c(x, z)$, first use Sen's α to infer that $x = c(x, y, z)$:
-can't have $z = c(x, y, z)$ as then we'd need $z = c(y, z)$
-can't have $y = c(x, y, z)$ as then $y = c(x, y)$.
So $x = c(x, y, z)$, and then Sen's α implies that $x = c(x, z)$.

- Now argue that $c = c_{\succsim^{**}}$: if $x = c(A)$ and $y \in A$ then from Sen's α $x = c(x, y)$, so $x \succsim^{**} y$.

- And this implies the overall revealed preference $\succsim^{*}$ relation is transitive, complete, and antisymmetric.

WARP is vacuous if we only see choices on pairs. What can we say then, or more generally with too little data to appeal to Arrow's theorem?

Definition: The *transitive closure* $\succsim^*$ of a binary relation $\succsim$ is given by $x \succsim^* y$ if there is a finite sequence $z_1, ..., z_n$ with $x \succsim z_1 \succsim .... \succsim z_n \succsim y$.

Let $\succsim$ be the "directly preferred" relation defined by $x \succsim y$ if there is $A \in M$ with $x, y \in A$ and $x \in c(A)$. Let $\succsim^*$ be the *transitive closure* of $\succsim$.

Say that *c* is *congruent* if for all $A \in M$ and all $x, y \in A$, if $x \succsim^* y$ and $y \in c(A)$ then $x \in c(A)$. *(like WARP but on the constructed preference $\succsim^*$, so only indirectly on observed choice)*

Theorem (Richter *Ema* [1966]) Choice data *c* is congruent iff $c = c_{\succsim}$ for some complete transitive preference $\succsim$.

Note that this result doesn't assume we have any data, so it doesn't pin down the rationalizing preferences.

Utility Representations

Utility function $u : X \to \mathbb{R}$.

Preference $\succeq$ is **represented by *u*** if for all $x, y \in X$, $x \succeq y \Leftrightarrow u(x) \geq u(y)$.

Theorem: Suppose *X* is finite or countably infinite. Then there is a utility function that represents $\succsim$ iff $\succsim$ is complete and transitive.

Proof sketch (for the "if", other direction is immediate)

Let $W(x) := \{z \in X : x \succsim z\}$ be the points that are no better than *x.*

Let *p* be any strictly positive probability distribution on *X,* and set
$u(x) := \sum_{z \in W(x)} p(x)$.

Want to show that $x \succeq y \Leftrightarrow u(x) \geq u(y)$.

By transitivity, $x \succeq y$ implies $W(x) \supseteq W(y)$

(*if* $z \in W(y)$ *then* $y \succsim z$ *so by transitivity* $x \succsim z$ *so* $z \in W(x)$)

So $x \succeq y \to u(x) \geq u(y)$

Now argue that $\neg(x \succeq y)$ implies $\neg(u(x) \geq u(y))$.
Then $y \notin W(x)$ so by completeness $y \succsim x$, so $W(y) \supseteq W(x)$.
And since $y \succsim y$ (*completeness*), so $W(x)$ is a strict subset of $W(y)$ and so $\neg(u(x) \geq u(y))$

Thus u represents $\succeq$.

Often convenient to represent objects by real numbers or real vectors (for example commodity bundles) so we can for example always have exact solutions to first-order conditions.

Need additional conditions to have utility representations when X is uncountably infinite (lexicographic preference is the standard counterexample: e.g. if $X = \mathbb{R}^2$ set $x \succsim y$ if $x_1 > y_1$ or $x_1 = y_1$ and $x_2 > y_2$).

Existence of utility functions has been shown at varying degrees of generality.

Definition: A preference relation $\succsim$ a topological space is *continuous* if its upper and lower contour sets $\{y \in X : y \succsim x\}$ and $\{y \in X : x \succsim y\}$ are closed. (For $X \subseteq \mathbb{R}^n$ this boils down to the condition that for any sequence $\{x^n, y^n\}$ with $x^n \to x, y^n \to y,$ and $x^n \succsim y^n$ we have $x \succsim y$.)

Theorem (Debreu [1954]): If X is a convex subset of $\mathbb{R}^n$, then $\succsim$ is complete, transitive and continuous iff it has a representation by a continuous utility function.

(more generally Debreu's theorem applies to any topological space with a countable base, but $\mathbb{R}^n$ covers most applications.)

(Continuity can't be falsified in any finite data set, and is very convenient. Also if we don't assume it, there's no guarantee we can approximate the true preferences using finite data, see e.g. Chambers, Echenique and Lambert [2017] "Preference Identification.")

(personal philosophy: use finite or infinite models depending on what's more convenient, no strong view on which is "better" or "more realistic.")

Proof sketch:

1. Show that if $x \succ y$ there is $z \in X$ s.t. $x \succ z \succ y$. (*uses X convex and preferences continuous. One way to show this is constructive: if it isn't true, build sequences* $\{x_t\}$ *and* $\{y_t\}$ *along the line from x to y by at each step evaluating the midpoint* m_t *(which is either better than x or worse than y); if it's better than* x_t *then set* $x_{t+1} = m_t$*,* $y_{t+1} = y_y$ *if not move y instead.*)

2. Show that if *Y* is dense in *X* then for every $x, y \in X$ there is $z \in Y$ s.t. $x \succ z \succ y$. (*uses continuity of preferences*).

3. Let $\overline{E}$ be the maximal (most preferred) elements of *X*- the *x* such that no $z \in X$ is strictly better (*could be empty),* let $\underline{E}$ be the minimal elements (*also possibly* empty) and set $E = \overline{E} \cup \underline{E}$.

4. Let $Y = \{y_1, y_2, ...\}$ be a countable dense subset of *X-E.* (*since* $X \subseteq \mathbb{R}^n$ *such a Y exists and by step 1 it is infinite.)* Define a utility function on *Y* as follows: $u(y_1) = 1/2$. If $y_2 \succ y_1$ then $u(y_2) = 3/4$, if $y_1 \succ y_2$ $u(y_2) = 1/4$. And interpolate utility for later y's "in between" previous ones, e.g. if $y_2 \succ y_3 \succ y_1$ then $u(y_3) = (1/2 + 3/4)/2 = 5/8$.

5. Extend *u* to all of *X* by setting $u(x) = 1$ for $x \in \overline{E}$, $u(x) = 0$ for $x \in \underline{E}$, and $u(x) = \sup\{u(y) \mid x \succ y, y \in Y\}$ for $x \notin Y \cup E$. Show that *u* is continuous and generates $\succsim$.

Optional HW: fill in the missing steps

Discussion:

- If u represents $\succsim$, it is also represented by v iff there is a strictly increasing function $f:\mathbb{R}\to\mathbb{R}$ s.t. $v = f \circ u$.

- "As if" representations, no necessary link to what makes the decision maker happy or is good for them.

- (*personal view*) : It may not always be clear that people's choices are in their best interest (*and more on this later on),* but political economy considerations still argue for using choice and revealed preference for evaluating social welfare.

Classic Revealed Preference Theory: $X = \mathbb{R}^n_+$, free disposal (so $x \geq y \to x \succsim y$), menus = budget sets.

Here the data is a set *D* of pairs (p, x). (price vector and consumption)

The data satisfies the *Generalized Axiom of Revealed Preference* ("GARP") if for every finite sequence $(p_1, x_1), \ldots, (p_n, x_n)$ of points in *D,*
$p_1(x_2 - x_1) \leq 0, \ldots, p_{n-1}(x_n - x_{n-1}) \leq 0 \to p_n(x_1 - x_n) \geq 0$.

Theorem: (Afriat *IER* [1967], Fostel et al *ET* [2004]): If *D* is finite then the data satisfies GARP iff it can be generated by a continuous, concave, strictly increasing utility function. (*strictly increasing: weakly increasing and* $x_i > y_i \; \forall i \to x \succ y$.)
Proof uses linear algebra and duality- not constructive.

Reny [2015] says that D can be "rationalized by utility function *u*" if *u* generates the data and in addition $p(y - x) < 0 \to u(y) < u(x)$: strictly affordable bundles are strictly worse than the chosen one.

Theorem (Reny *Ema* [2015]) If D is finite or infinite the data satisfies GARP iff it can be rationalized by a strictly increasing quasiconcave utility function.

Proof defines a utility function as the limit of utility functions that each "work" for a given pair (endowment, price vector) and shows that this limit has the desired properties.

R*eadings:* Huber et al *J Cons Research* [1982], Huber et al *J Marketing Research* [2014], Strzalecki 5.1 and 5.2 and/or MWG 6A, 6B.