Testing Rank Dependence (Bernheim and Sprenger [2016]).

CPT and all preferences based on rank-dependent probability weights imply a discontinuity in choice as a given reward moves from below to above another.

- Start from $\ell = (p, q, 1-p-q)$ on (x, y, z), $x > y > z$.

$$U(\ell) = \pi(p)u(x) + [\pi(p+q) - \pi(p)]u(y) + [\pi(1) - \pi(p+q)]u(z).$$

- For $m > 0$ s.t. $x > y + m$, define the "lower" *equalizing reduction* $\underline{k}$ so that

$$\pi(p)u(x) + [\pi(p+q) - \pi(p)]u(y) + [\pi(1) - \pi(p+q)]u(z) =$$

$$\pi(p)u(x) + [\pi(p+q) - \pi(p)]u(y+m) + [\pi(1) - \pi(p+q)]u(z - \underline{k}(m)):$$

$\underline{k}$ is the decrease in *z* that just offsets the increase in *y.*

- Note that the term $\pi(p)u(x)$ cancels, so $\underline{k}$ does not depend on *x.*

- Solving $\underline{k}(m) = z - u^{-1}\left(u(z) - \frac{\pi(p+q)-\pi(p)}{1-\pi(p+q)}[u(y+m)-u(y)] \right)$

- Now hold fixed the probability *p* of *x* and *q* of *y*, but suppose $x < y$, and again imagine increasing *y* and decreasing *z.* This defines the "upper" equalizing reduction $\overline{k}(m)$:

$$\pi(q)u(y) + [\pi(p+q) - \pi(q)]u(x) + [\pi(1) - \pi(p+q)]u(z) =$$
$$\pi(q)u(y+m) + [\pi(p+q) - \pi(p)]u(x) + [\pi(1) - \pi(p+q)]u(z - \overline{k}(m))$$
$$\leftrightarrow$$

$$\overline{k}(m) = z - u^{-1}\left(u(z) - \frac{\pi(q)}{1-\pi(p+q)}[u(y+m)-u(y)] \right).$$

- This compensating reduction is also independent of *x,* but its value is different; for the parameters of Tversky Kahneman [1992] it doubles.

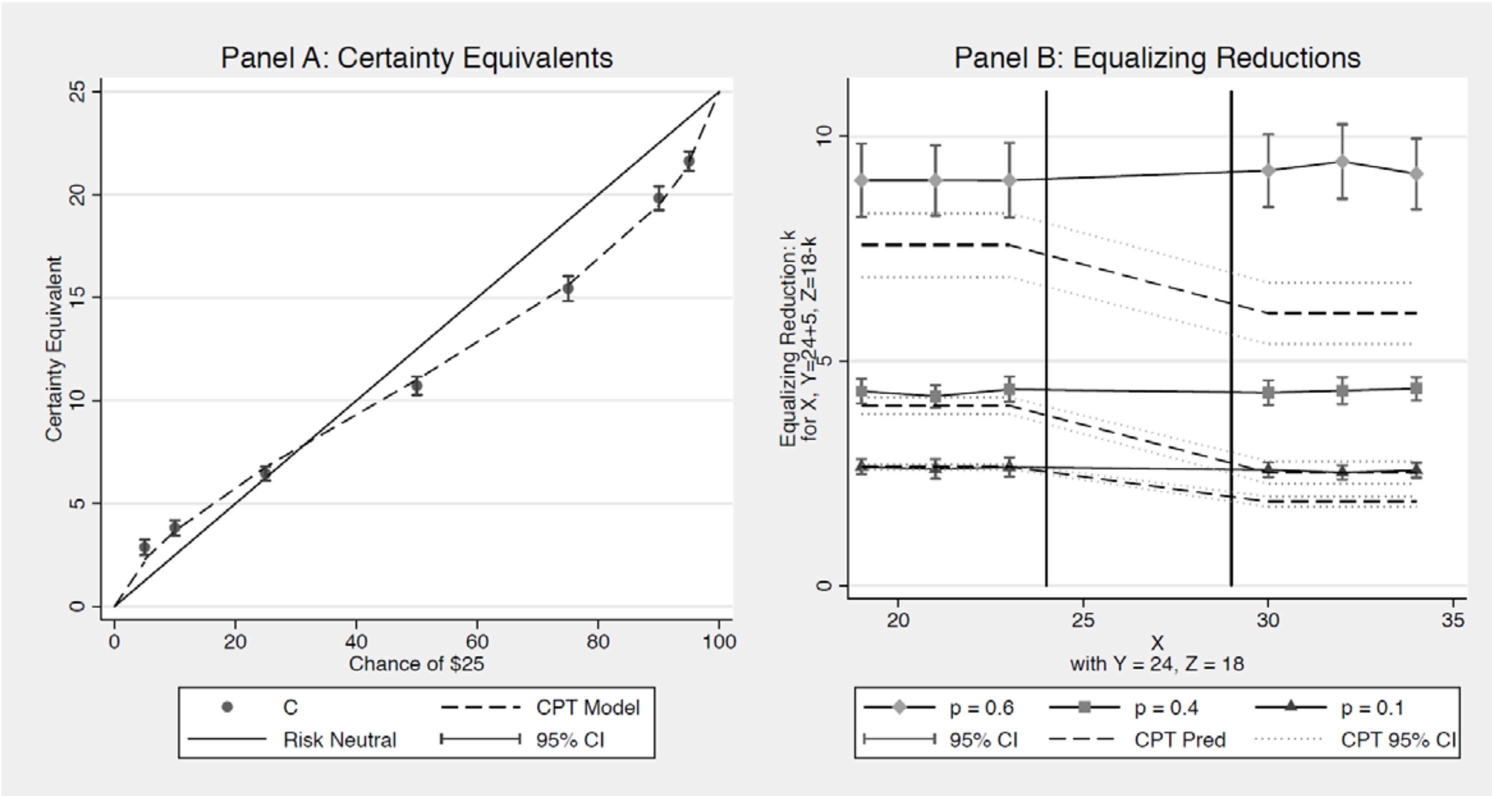
- CPT predicts that the compensating reduction in *z* doesn't depend on x except when *x* switches from less than *y* to greater than *y.* When a switch occurs it could change lot or a little depending on parameters.

- In the Bernheim-Sprenger data, the compensating change doesn't much change as x crosses y at either aggregate or individual level.

- Sounds bad for CPT. But maybe the CPT parameters of their subjects somehow make the predicted jump small- e.g. if most people are close to EU?

- To test this they estimate CPT parameters (of the KT functional form) both for the whole populations and for each subject, using certain equivalents for binary lotteries.

- Aggregate CPT fits aggregate data well, parameter estimates same as in Wu and Gonzales [1996]. Also estimate individual-level CPT parameters.



- For most subjects, the estimates imply a large difference between the two k's, but the observed differences are concentrated near 0, and "the correlation between predicted and actual behavior is indistinguishable from zero."

- One referee suggested that people simply canceled the common probability of *x* across the two lotteries.

- So they added a new experiment on (x, y, z) vs. $(x+m, y-k, z-k)$ so no common outcome probability pairs to cancel.

- CPT predicts that with probabilities (0.4, 0.3, 0.3) and (0.6, 0.2, 0.2) the equalizing reduction should be non-monotonic in *x* as it passes from $x > y$ to $x < y$. Doesn't happen.

- What about giving up stochastic dominance and going back to PT? They elicit certain equivalents for three outcome lotteries that pay $x + \varepsilon$ with probability p/2, $x - \varepsilon$ with probability p/2, and *y* with probability 1 – p, with lotteries chosen so that standard formulations of PT predict a sizable and discontinuous drop in the certainty equivalent at $\varepsilon = 0$. In contrast, CPT implies continuity.

- Contrary to both predictions, we find a discontinuous increase in the certainty equivalent at $\varepsilon = 0$. This behavior implies violations of dominance, but not the type PT predicts.

- Their conclusion: a good theory of choice under uncertainty should account for (1) the inverse S-shaped certainty equivalent profile, (2) the absence of rank-dependence in equalizing reductions, and (3) the sharp drop in certainty equivalents that results from split-ting an event. EU is inconsistent with (1) and (3), while CPT is inconsistent with (2) and (3), and PT is inconsistent with (3).

- "*We hypothesize that the observed behavior results from a combination of standard PT and a form of complexity aversion: people may prefer lotteries with fewer outcomes because they are easier to understand. One can think of the well-known certainty effect as a special case of this more general phenomenon.*"

- My take-aways for now:
  - Substantial evidence for some failures of expected utility theory
  - CPT with an S-shaped probability weighting fits many experiments and is the second-most- used (after EU) theory of risk preferences, including forthcoming paper in the *AER* and *RFS.*
  - But it is far from perfect and should not be adopted uncritically.
  - Can anyone come up with a better model than "PT+complexity aversion"? or formalize complexity aversion so it can be tested?
  - Empirical applications of PT/CPT that don't rely on rank dependence are safer from this critique (but still susceptible to worries about dynamic decision making and firms exploiting non-linear probability weighting, as in the Ebert-Strack and Azevdedo-Gottlieb papers mentioned last time.

Subjective Probability and Subjective Expected Utility

- Choice under risk: random consequences, known probabilities
- Choice under uncertainty: some or all of the probabilities unknown.
- ***Subjective Expected Utility:*** EU but where the probabilities p represent subjective belief.
- Two main formalisms of this, Savage and Anscombe-Aumannn.
- In Savage, <u>all</u> probabilities are subjective, there are infinitely many subjective states; choice of σ-algebra on the states matters.
- Strzalecki goes over Savage's representation, we'll skip it.

Anscombe-Aumann *Ann. Math Stat* [1963]

- Some probabilities objectively known, only finitely many subjective states.

- This lets us calibrate the subjective probabilities to the objective ones, and simplifies the math.

- *Z*: finite set of outcomes that agent cares about.

- *S*: finite state space: contingencies that determine outcomes.

- $\mathcal{F} := \left(\Delta(Z)\right)^S$

- An *act* $f \in \mathcal{F}$ gives the objective lottery $f(s)$ for each *s*. E.g. "If the Patriots win next year's Super Bowl, roll a die and get $100 if it comes up 1, 2, or 3."

- The idea is that the decision maker has a subjective probability distribution on the states in *S.* An act says what probability distribution on outcomes she gets for each state, so an act combined with a probability distribution on states gives a distribution on outcomes.

- Writing $\mathcal{F} := \left(\Delta(Z)\right)^S$ implicitly assumes that every *z* is possible in every state. If there are distinct sets Z_s for each *s,* then we can't separately identify the agent's utility function and their beliefs.

- We assume the analyst (and the agent!) know *Z* and *S.*

- For $\alpha \in [0,1]$ and $f, g \in \mathcal{F}$ the *mixture* $\alpha f + (1-\alpha)g$ is the act that for each *s* gives the lottery $\alpha f(s) + (1-\alpha)g(s) \in \Delta(Z)$.

- We will identify lotteries over $\mathcal{F}$ with their mixture (thus identifying compound lotteries), so we will implicitly also get a representation of preference on probability distributions on Z^S .

- Note that when #S=2 we are saying that ½(a,a)+ ½ (b,b) is the same as ½ (a,b)+ ½ (b,a): both reduce to the same point in $\mathcal{F}$.

- And remember that $\dim(\mathcal{F}) = \#S \bullet (\#Z - 1) \leq (\#Z)^{\#S} - 1 = \dim(\Delta(Z^S))$.

  For example with 3 states and 3 outcomes, $\dim(\mathcal{F}) = 6 < 8 = \dim(\Delta(Z^S))$
  So we are "compressing" 2 dimensions that some people might conceivably care about.

- Our goals:

- characterize when a complete transitive preference on $\mathcal{F}$ has a subjective expected utility representation of the form

$$V(f) = \sum_s u(f(s))p(s)$$

- understand the extent to which u and p are pinned down.

- *Note:* If we want to pin down the agent's subjective probability distribution *p,* then it's important that the utility function doesn't depend on the state: if $V(f) = \sum_s u_s(f(s))p(s)$ and q has full support, then

$V(f) = \sum_s \left(u_s(f(s))\frac{p(s)}{q(s)} \right) q(s)$ represents the same preferences- for example can take q to be uniform.

- **Axiom (Mixture Continuity)** For all $f, g, h \in \mathcal{F}$ the sets

  $$\{\alpha \in [0,1] : \alpha f + (1-\alpha) g \succsim h\}$$

  and

  $$\{\alpha \in [0,1] : \alpha f + (1-\alpha) g \precsim h\}$$

  are closed in [0,1].

Notation: For $z \in Z$ or $p \in \Delta(Z)$ let $\vec{z}$ and $\vec{p}$ denote the "constant acts" that give *z* or *p* in every state.

- Any preference on the space of state-contingent acts $\mathcal{F} := \left(\Delta(Z)\right)^S$ induces a preference on constant acts:

  for $p, q \in \Delta(Z)$ say that $p \succsim q$ iff $\vec{p} \succsim \vec{q}$.

- **Axiom (Monotonicity)** If $\overrightarrow{f(s)} \succsim \overrightarrow{g(s)}\ \forall s$ then $f \succsim g$.

  Implicit assumption here that preference is state-independent. Suppose $S = \{s', s''\}$*, and* z' *is good in state* s' *and bad in in state* s'' *. Then it could be that* $\overrightarrow{z''} \succ \overrightarrow{z'}$ but $(z'', y) \prec (z', y)$.

- **Axiom (Non-triviality)** There are $f, g \in \mathcal{F}$ s.t. $f \succ g$.

- **Axiom (Independence)** For all $f, g, h \in \mathcal{F}$ and $\alpha \in (0,1)$,

$$f \succsim g \text{ iff } \alpha f + (1-\alpha)h \succsim \alpha g + (1-\alpha)h \ .$$

Theorem (Anscombe-Aumann style): A complete transitive preference $\succsim$ on $\mathcal{F}$ satisfies mixture continuity, monotonicity, non-triviality, and independence iff there is a linear $u:\Delta(Z)\rightarrow\mathbb{R}$ and a $p\in\Delta(S)$ s.t. $V(f)=\sum_s u(f(s))p(s)$ represents $\succsim$. Moreover, p is unique and u is unique up to affine transformations.

Intuition: Calibrate subjective probabilities to objective ones.

Suppose $x,y\in Z$, $x\succ y$.

Then for each $E\subseteq 2^S$ find the α s.t. $\alpha\vec{x}+(1-\alpha)\vec{y}\sim\overrightarrow{x_E y}$, where $\overrightarrow{x_E y}$ means the act that pays *x* for $s\in E$ and *y* for $s\in S-E$.

This is our candidate for $p(E)$.

Need to show $p(E)$ doesn't depend on the choice of $x,y\in Z$, and that *p* is a probability distribution.

Idea of proof:

1. The induced preference on $\Delta(Z)$ satisfies the vN-M axioms so it is represented by a non-trivial utility function *u* that is linear in the objective probabilities.

   Normalize the range of *u* to be $U = [-1, 1]$.

2. A *utility act* is a map $\tau : S \to U$. Define $\succsim^*$ on U^S by $\tau \succsim^* \tau'$ iff $f \succsim g$ for some $f, g \in \mathcal{F}$ s.t. $\tau = u \circ f$ and $\tau' = u \circ g$.

<u>Claim:</u> From monotonicity, $\succsim^*$ doesn't depend on the choice of *f,g* so it's a complete transitive preference.

Proof of claim:

Suppose there are $f, f', g, g' \in \mathcal{F}$ s.t. $\tau = u \circ f = u \circ f'$ and $\tau' = u \circ g = u \circ g'$

Then for each *s* $f(s) \sim f'(s)$. So $\overrightarrow{f(s)} \sim \overrightarrow{f'(s)}$ for all s so from monotonicity $f \sim f'$.

Similarly $g \sim g'$.

So $f \succsim g$ iff $f' \succsim g'$ iff $\tau \succsim^* \tau'$.

3. Let $I(\tau) \in [-1,1]$ (for "indifference") be the number s.t. the constant utility act $\overrightarrow{I(\tau)}$ is indifferent to τ. $I(\tau)$ *will end up being the expected utility of* τ *in our representation, but first we have to show it is well defined.*

C*laim:* $I(\tau)$ exists and is unique.

Proof sketch:

- Define the upper and lower contour sets
$$U(\tau) = \left\{ \eta \in [-1,1] : \vec{\eta} \succsim \tau \right\} \text{ and } L(\tau) = \left\{ \eta \in [-1,1] : \vec{\eta} \precsim \tau \right\} .$$

- These sets are non-empty, closed (from mixture continuity) and their union is [-1,1], so they have a non-empty intersection.

- From the definition of utility acts this intersection is unique (*otherwise we'd have* $u > u'$ and $(u,u,....u) \sim (u',u',...,u')$).

- Define the intersection to be $I(\tau)$.

- Note that by construction $\tau \succsim^* \sigma$ iff $I(\tau) \geq I(\sigma)$, so *I* represents $\succsim^*$.

Now show that the indifference function has the specified linear form.

4. Use independence to show that $I(\lambda\tau) = \lambda I(\tau)$.
   a. First do this for $\lambda \in (0,1)$:

      Fix any $\tau \in U^S$, and any $f \in \mathcal{F}$ s.t. $\tau = u \circ f$.

      Pick $p, q \in \Delta(Z)$ s.t. $u(p) = I(\tau)$ and $u(q) = 0$.

      Then $f \sim \vec{p}$ so by Independence $\lambda f + (1-\lambda)\vec{q} \sim \lambda \vec{p} + (1-\lambda)\vec{q}$.

      So $\lambda\tau \sim^* \lambda I(\tau)$, so $I(\lambda\tau) = I(\lambda I(\tau)) = \lambda I(\tau)$

      (because $I(u,u,..,u) = u$.)

b. For $\lambda > 1$ and $\lambda\tau \in U^S$ start with $\lambda\tau$ and multiply by $1/\lambda$.

c. Then extend *I* to all of $\mathbb{R}^S$ by setting $I(\tau) = \|\tau\|_1 I\left(\tau / \|\tau\|_1\right)$.

We took $U = [-1,1]$ *instead of* $U = [0,1]$ *so we could do this.*

5. Now show *I* is additive, that is that $I(\lambda\tau + (1-\lambda)\sigma) = \lambda I(\tau) + (1-\lambda)I(\sigma)$.

6. Since *I* is a linear operator on $\mathbb{R}^S$ there is a unique vector *p* s.t. $I(\tau) = \sum_s p(s)\tau(s)$.

(To see this let $e_1 = (1,0,0,...,0), e_2 = (0,1,0,...,0),...e_{\#S} = (0,....,0,1)$,
Then $\tau = \sum_s \tau(s)e_s$ so $I(\tau) = I\left(\sum_s \tau(s)e_s\right) = \sum_s \tau(s)I(e_s)$, so
$p(s) = I(e_s)$.)

7. It remains to show that this p is a probability distribution, i.e. each entry non-negative and sums to 1: HW.

8. Now set $V(f) = \sum_s u\left(f(s)\right) p(s)$ and we're done.

Eliciting Subjective Beliefs (for agents who fit the subjective EU model)

- Calibrate to objective lotteries given a list: ask the decision maker if she would rather win $10 if (Event E) or with probability *p*= [0,.1, .2, …,1]? Then pick a probability *q* at random (say uniform on [0,1]) and give her the choice she selected.

- Becker-DeGroot Marshak (BDM) (*also used to assess certain equivalents in risk tasks)* : ask the subject for the *p* that makes her indifferent, then pick true *p* at random.

  In both cases it's obvious *to economists* that truthful reporting is optimal; some debate among experimenters about how obvious it is to subjects. (see e.g. Harrison and Rustrom [2008]).

- This requires asking about $\#S-1$ pairwise comparison of states.

- Or use a "proper scoring rule": ask the agent to report the vector *p* and pay him (in cash) $z(p,s)$.

  z is a *proper scoring rule* if for any belief *p,*

  $$p \in \arg\max_{q \in \Delta(S)} \sum_s p(s) z(q,s) \ .$$

- Standard examples are the logarithmic rule $z(p,s) = -\ln(p(s))$ (*which is nice because it only depends on the report's value at the realized state but very sensitive to low probabilities)* and the quadratic scoring rule $z(p,s) = 1 + 2p(s) - \sum_{s' \in S} p(s')^2$.

- If subjects are risk neutral these are both proper scoring rules.

- If not- can pay in lottery tickets instead of money, e.g. one ticket= .01 chance of winning $20 so has utility .1*u(20).

Note that the subjective probability is purely subjective, and needn't correspond to any objective randomness, because you can have subjective beliefs about knowable facts that you don't happen to know.

Define a *subjective 90% confidence interval* for a number to be any interval s.t. you're indifferent between winning a prize if the number is in the interval or winning with objective probability .9.

Note this means you're also indifferent between winning if the number is outside the interval or winning with objective probability .1.

w/o looking anything up or talking to classmates, write down 90% subjective confidence intervals for the following:

- Minimum distance (in kilometers) from Earth to Mars.
- GDP of Croatia in 2010.
- Number of goals scored in English premier league last season.
- Population of Hawaii at 2010 census
- Number of Ph.D.s granted in the U.S. in 2013 (all fields)
- Current exchange rate US$ to Indonesian rupiah: 1 US$=?
- Age of Drew's cat (in years)
- The year that Queen Elizabeth 1st was born
- The number of Bank of America ATM's in Cambridge MA
- NBC's estimate of number of people at the Boston Common Women's March on Jan. 21 2017.

Reading for next time: Strzalecki 7.5, 10.3, 10.4, 10.5 optional, Halevy *Ema* [2007]