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Dealing with Externalities of Payments Systems and Fire Sales: Designs for Optimized Competition among Cryptocurrencies and Market Structure to Remove Pecuniary Externalities (Lecture 7)

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Jain and Townsend (2018) "The Economics of Platforms in a Walrasian Framework"

Kilenthong and Townsend (2019) "A Market Based Solution for Fire Sales and Other Pecuniary Externalities"

The Economics of Platforms in a Walrasian Framework

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Introduction	Model	Examples	Comparative statics	Extensions	Conclusion
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Introduction

Motivation

- ▶ A successful platform needs to intermediate between buyers and sellers.
- ▶ We are interested in platforms where buyers and sellers care about the composition of the platform's users.
  - ▶ causing a potential 'externality' arising from one agent's platform choice on other agents' willingness to join the same platform.
- ▶ A credit card company must attract both consumers and merchants.
- ▶ Dark pools and exchanges must attract buyers and sellers.
- ▶ Internet service providers (ISPs) need to attract content producers and users.

Competition in Cryptocurrencies

- ▶ Though the credit card payments system links directly to policy issues at stake, it also conjures up the image of imperfect competition, as the credit card industry is relatively concentrated with the leading companies being Visa, MasterCard, American Express and Discover.

- ▶ However, crypto currencies are entering and beginning to compete. Coins which feature payments include not only Bitcoin but also Dogecoin, Litecoin, Monero, Ripple, Stellar, and Zcash.

- ▶ This inevitably raises the question of whether there is scope for the existence of multiple coins, that is, can there be multiple payment platforms co-existing.

- ▶ This competitive battle is being waged against the key constraint—the problem of scaling up. Bitcoin's miners are validators using a proof-of-work protocol which consumes significant electricity, yet has limited capacity and slow transaction speed -- an estimated 7 transactions per second for bitcoin.

- ▶ But new entrants with alternative protocols are often faster and cheaper, for example, Stellar with its Federated Byzantine Agreement; or Algorand's using their proof-of-stake protocol.

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Introduction

Questions to answer

▶ In the context of multiple competing platforms is there a Walrasian equilibrium?

▶ Is the Walrasian equilibrium efficient?

▶ Is there a role for regulation due to a possible network externality?

▶ Are these "externalities" something which (only) regulation can deal with?

▶ How do changes in wealth affect prices and subsequently user's welfare?

Solution

- Answer: Under certain assumptions, general equilibrium
  theory can answer these questions—in a manner suggested by
  Meade (1952) and Arrow (1969).
  - We can "internalize" the network externality through ex ante
    contracting.
  - Use "Firms as Clubs" (Prescott and Townsend (2000) and
    General Equilibrium theory.

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Model

Set-up

- ▶ There are two agent types—buyers (A) and sellers (B).
- ▶ Buyers and sellers can trade only on a platform.
- ▶ Buyers and sellers care about the composition of the platform's users.
- ▶ Buyers and sellers each have some capital endowment (κ).
- ▶ There is a single intermediary that connects agents to platforms.

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Model

Agent's utility function

- An agent wants a higher **ratio** of agents of the other type, and **larger** platforms.

- A buyer's (A, i) utility function is:

$$U_{A,i}(N_A, N_B) = U_A(N_A, N_B) = \begin{cases} 0 & \text{if } N_A \text{ or } N_B = 0 \\ \left[\left(\frac{N_B}{N_A}\right)^{\gamma_A} + N_B^{\epsilon_A}\right] & \text{else} \end{cases}$$

- Where $\gamma_A, \gamma_B, \epsilon_A$ and $\epsilon_B > 0$

- Symmetrically, the seller's (B, i) utility function is:

$$U_{B,i}(N_A, N_B) = U_B(N_A, N_B) = \begin{cases} 0 & \text{if } N_A \text{ or } N_B = 0 \\ \left[\left(\frac{N_A}{N_B}\right)^{\gamma_B} + N_A^{\epsilon_B}\right] & \text{else} \end{cases}$$

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Model

Cost of making a platform

▶ A platform is costly to manufacture, increasing in the number of users of each type, and increasing in the set of possible connections.

$$C(N_A, N_B) = \begin{cases} 0 & \text{if } N_A = 0 \text{ or } N_B = 0 \\ c_A N_A + c_B N_B + c N_A N_B + K & \text{else} \end{cases}$$

▶ Where $c_A, c_B, K \geq 0$ and $c > 0$.

▶ Larger platforms are more than proportionally more expensive ($c > 0$)

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Model

Agent's maximization problem

- Agent type (T), subtype (i) buys contracts $b_T(N_A, N_B)$ to join a platform of size and composition (N_A, N_B), subject to:
  - their wealth constraint
  - joining one platform.
  .

- Key tool to convexify commodity space: agents do not buy discrete numbers of contracts instead agent's buy probabilities to join a platform.

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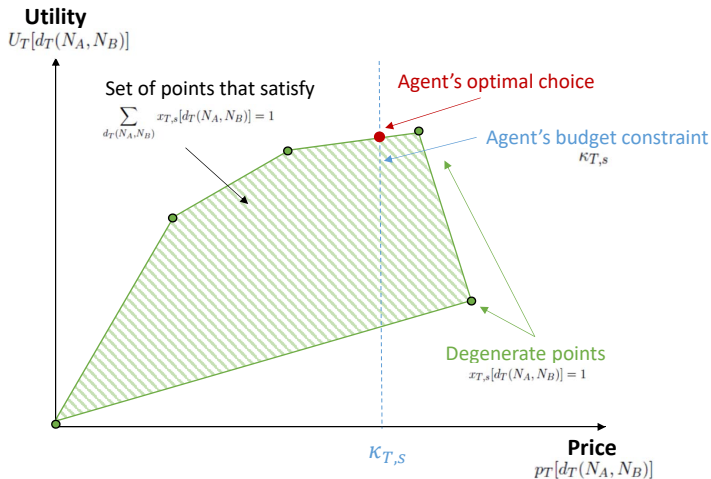**Model**

Agent's maximization problem

▶ Agent (T, i) takes prices $p[b_T(N_A, N_B)]$ as given and solve the following maximization problem:

$$\max_{x_{T,i}} \sum_{N_A, N_B} x_{T,i}[b_T(N_A, N_B)] U_T[b_T(N_A, N_B)] \qquad (1)$$

$$\text{s.t.} \sum_{N_A, N_B} x_{T,i}[b_T(N_A, N_B)] p[b_T(N_A, N_B)] \leq \kappa_{T,i} \qquad (2)$$

$$\sum_{N_A N_B} x_{T,i}[b_T(N_A, N_B)] = 1 \qquad (3)$$

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Model

Agent's maximization problem—graphical illustration

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Model

Intermediary's problem

- The intermediary maximizes the number of platforms $y(N_A, N_B)$ to produce for the given prices $(p[b_T(N_A, N_B)])$ for each position in the platform.

- The intermediary's profits are equal to the number of contracts it sells multiplied by the price of the contract minus the cost of the capital input.

| Introduction | Model | Social Planner's Problem | Welfare Theorems | Examples | Conclusion |
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Model

Intermediary's problem

▶ The total number of agents of each type on a platform must equal the total number of contracts offered for that type.

$$\frac{y_A[b_A(N_A, N_B)]}{N_A} = \frac{y_B[b_B(N_A, N_B)]}{N_B} = y(N_A, N_B) \; \forall b_a \in B_A, \forall b_b \in B_B$$

▶ The intermediary maximizes the number of platforms $y(N_A, N_B)$ to produce for the given prices ($p[b_T(N_A, N_B)]$) for each position in the platform.

▶ The intermediary's profits are equal to the number of contracts it sells multiplied by the price of the contract minus the cost of the capital input.

$$\pi = \max_{y, y_k} \sum_{N_A, N_B} \{p[b_A(N_A, N_B)]N_A + p[b_B(N_A, N_B)]N_B\} \times y(N_A, N_B) - y_k$$

$$\pi = \max_{y, y_k} \sum_{N_A, N_B} \{p[b_A(N_A, N_B)]N_A + p[b_B(N_A, N_B)]N_B\} \times y(N_A, N_B) - y_k$$

$$\sum_{N_A, N_B} y(N_A, N_B)[C(N_A, N_B)] \leq y_\kappa$$

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Model

Intermediary's problem

- The intermediary's FOC w.r.t. to $y(N_A, N_B)$ is:

$$C(N_A, N_B) \geq p[b_A(N_A, N_B)] * N_A + p[b_B(N_A, N_B)] * N_B \quad (4)$$

  - **This condition requires the payments/memberships the platform recieves must cover all of the platform's costs.**

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Model

Competitive equilibrium

- A competitive equilibrium in this economy is
  $(p, x, y) \in L \times X \times Y$ such that

  - For given prices, the allocation solves the consumer and
    platform maximization problems.
  - All markets clear: the demand for each contract equals the
    supply of each contract.
  - Active platforms are populated by numbers of buyers and
    sellers as anticipated (stipulated).

| Introduction | Model | **Social Planner's Problem** | Welfare Theorems | Examples | Conclusion |
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Social Planner's Problem

Social Planner's Problem

$$
\max_{x \geq 0, y \geq 0} \sum_i \lambda_{A,i} \left\{ \sum_{b_A(N_A,N_B)} \alpha_{A,i} x_{A,i}[b_A(N_A,N_B)] U_A(N_A,N_B) \right\}
$$
$$
+ \sum_i \lambda_{B,i} \left\{ \sum_{b_B(N_A,N_B)} \alpha_{B,i} x_{B,i}[b_B(N_A,N_B)] U_B(N_A,N_B) \right\}
$$

$$
\text{s.t.} \quad \sum_{b(N_A,N_B)} x_{T,i}[b_T(N_A,N_B)] = 1 \ \forall T, i
$$
$$
\sum_i \alpha_{T,i} x_{T,i}[b_T(N_A,N_B)] = y(N_A,N_B) \times N_T \ \forall b_T \in B_T, \forall T \in \{A,B\}
$$
$$
\sum_{N_A,N_B} y(N_A,N_B)[C(N_A,N_B)] \leq \sum_{T,i} \alpha_{T,i} \kappa_{T,i}
$$

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Model

Summary of results

- A competitive equilibrium is Pareto optimal.
- Any Pareto optimal allocation can be achieved with transfers between agents:
    - The first and second welfare theorems hold in our modified environment
- The endogenous pricing internalizes the effect of changing the composition of the platform—overcoming any network externality—as in Arrow (1969)

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Examples

Simple example—identical preferences and wealth

- ▶ Consider a platform with 2 subtypes of buyers, and 2 subtypes of sellers
- ▶ There is a measure 1 of each type, a measure 0.5 of each subtype
- ▶ Each agent is equally wealthy

Equilibrium platforms		
Platform Size	Number of Platforms Created	Cost of Production
(N_A, N_B)	$y(N_A, N_B)$	$C(N_A, N_B)$
(2,2)	0.5	8

Equilibrium user utility and platform choice					
Type (T, s)	Wealth $(\kappa_{T,s})$	Platform Joined (N_A, N_B)	Price of Joining $p(d_T[N_A, N_B])$	Pr(joining) $x_{T,s}(d_T[N_A, N_B])$	Utility on Platform $U_T(N_A, N_B)$
A,1	2	(2,2)	2	1	2.41
A,2	2	(2,2)	2	1	2.41
B,1	2	(2,2)	2	1	2.41
B,2	2	(2,2)	2	1	2.41

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Examples

The effect of a single richer subtype

- Let subtype $(B, 2)$ be markedly richer than other types
- $(B, 2)$ will "sponsor" larger platforms—lower prices for Type (A)

Platform Size (N_A, N_B)	Number of Platforms created	Cost of Production
$(3, 2)$	0.25	11
$(1, 2)$	0.25	5

Type (T, i)	Wealth	Platform joined (N_A, N_B)	Price of joining	Pr(joining)	Utility on Platform	Expected Utility
$A, 1$	1.37	$(3, 2)$	1.37	1	2.23	2.23
$A, 2$	1.64	$(3, 2)$	1.37	0.5	2.23	2.53
		$(1, 2)$	1.91	0.5	2.8	
$B, 1$	1.54	$(1, 2)$	1.54	1	1.7	1.7
$B, 2$	3.45	$(3, 2)$	3.45	1	2.96	2.96

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Comparative statics

How does the equilibrium change as we redistribute wealth? Redistributing wealth *across-* and *within-* agent type.
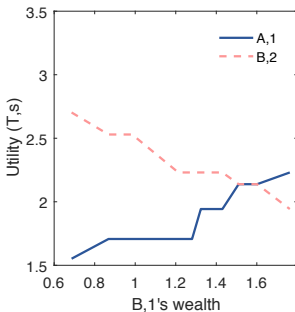
(A) **Across**: Transferring wealth from (A,2) to (B,1)
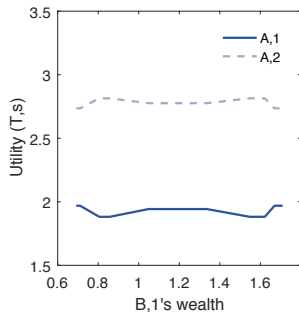
(B) **Within**: Transferring wealth from (B,2) to (B,1)

A Market-Based Solution for Fire Sales and Other Pecuniary Externalities

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Main Ideas

- This paper proposes that a market-based solution with rights and exclusivity can internalize pecuniary externalities.
- This market-based solution to fire sales and other pecuniary externalities can be extended to capture at least the following 6 prototype economies:
  1. Fire Sales Economy (Lorenzoni, 2008).
  2. Liquidity Constrained Economy (Hart and Zingales, 2011).
  3. Collateral Economy (Kilenthong and Townsend, 2014).
  4. Exogenous Incomplete Markets (Greenwald and Stiglitz, 1986).
  5. Moral Hazard with Retrading (Acemoglu and Simsek, 2008; Kilenthong and Townsend, 2011).
  6. Hidden Information with Retrading (Diamond and Dybvig, 1983).

Pecuniary Externalities: The Pollution Problem

- The influence of individual decisions on prices which in turn can cause inefficiencies is akin to pollution, which has a remedy in competitive markets for the rights to pollute.

- Consider an initial economy with two goods, one period, one representative price taking consumer and one representative price taking firm.

- The consumer is endowed with one good which can be consumed or used by firms to produce the second good which the household also values.

- However, that production comes with air or water pollution, which gives the household disutility. The competitive equilibrium in which this pollution is not priced is not at a social optimum; marginal rates of substitution in consumption and production do not line up, as they would in the planner's problem.

Pecuniary Externalities: The Solution to the Pollution Problem

- Solution: Factories have to buy rights to emit pollution, a cost which lowers their profit. Households sell rights to suffer pollution, a revenue added to their budget, and choose how much pollution to allow, effectively issuing permits, and how much to consume of the two goods.

- In the new decentralized market equilibrium, the supply and demand of rights to pollute will be equated by the appropriate price of rights, money changes hands, and the new equilibrium is Pareto optimal, with some but less pollution.

- Of course rights need to be enforced. Firms cannot pollute beyond rights purchased, as in cap and trade.

- The difference between cap and trade and the full market solution is that the quantity of permits are market determined and not fixed by the government.

Pecuniary Externalities: The Over-Saving Problem

- Consider a saving economy with two goods, two periods, and two representative price-taking households, types a and b.
- There is no uncertainty, just intertemporal decisions, over time and within-period decisions, across the two goods, w and z.
- Further suppose that if there were no obstacles to trade and if markets were complete, the environment is such that in a competitive equilibrium type-a would be a lender of both goods and type-b a borrower. However, suppose in contrast that borrowing is not allowed. Only one of the two goods can be stored, good z.
- The relatively rich type a household ends up smoothing consumption over time on its own, not by lending to type b but by saving good z.
- As a result, the price of the storage good z is low in the second period, as type a sells good z in the spot market then. Likewise, the price of good z is high in the first period with saving absorbing some of good z.
- Thus, the relative prices in both periods are moving with saving, but both types take equilibrium prices as given.

Solution to the Over-Saving Problem: Key Intuition

- The stored good is like the input good in the pollution example. The pollution is the impact of the stored good on future relative prices.
- We need a market for rights to trade at those future prices, so that agents internalize the impact of their saving decisions, just as with pollution.
- Agents need to choose in the first period both the price at which they want to trade in the second period and the amount of that trade, there excess demand or supply at that chosen price.
- Agents can choose in the first period one price from among various possible future prices; there is an exchange or trading house earmarked by each possible future price. Rights are traded in these exchanges.
- The saver type a buys rights in the chosen exchange and the number of rights purchased will be time consistent.
- As the borrower type b suffers the damage, type b will choose an exchange indexed by some future price and sell rights in the first period to buy the storage good z in the second period, adding revenue in the first period budget.

Solution to the Over-Saving Problem: Key Intuition (Con't)

- In the decentralized competitive equilibrium, both types choose the same exchange in the first period, indexed by the same price, and the demand for rights of type a and supply of rights by type b are equated there.
- The corresponding future spot market clears at the chosen price. The saver type a decides on storage of good z, a constrained-optimal allocation is achieved, and there is some but less physical storage.
- As with pollution, we required enforcement of rights, including exclusivity.
- An agent chooses only one exchange to buy and sell rights in the first period, for which they pay or receive compensation, and cannot trade in multiple exchanges.
- All spot trade in the second period must take place in the spot markets at the associated designated price that agents have chosen and agents cannot make spot exchanges on the side.

The Relationship to Coase (1960), and Arrow (1969) and Meade (1952)

- Our contribution is related to Coase (1960) in its emphasis on rights. Our pollution example is one of his lead examples.
- However, the Coase theorem is about how any given initial arbitrary distribution of rights would not matter if there were bargaining and no trading frictions, just as the initial allocation of rights to pollute in cap and trade would not matter, as efficiency works through opportunity costs.
- In contrast for us rights are market determined. Thus, closer to what we do is the work of Arrow (1969), following Meade (1952), on the equivalence of solutions to planning problems and competitive equilibria with rights trade in the objects causing non-pecuniary externalities.
- Keys are additional markets and excludability.

An Example Economy: Timing and Commodities

- Two periods: $t = 1, 2$.
- Two physical goods: good w and good z.
  - Good w in not storable, used as a numeraire good.
  - Good z is storable $\implies$ endogenous saving, k^h.
- There are $H = 2$ types each with continuum agents of mass $\alpha^h = \frac{1}{2}$.
- Endowment of agent $h = a, b$: $\left(e_{w1}^h, e_{z1}^h, e_{w2}^h, e_{z2}^h \right)$. The endowment profiles are such that an agent type a is well endowed with 3 units of both goods in period $t = 1$ relative to one unit of both at $t = 2$, a savings type, and vice versa for type b, a want-to-be-borrowing type.
- Storage technology: One unit of good z will become $R = 1$ units of good z at date $t = 2$.
- Each agent can trade in spot market at date $t = 2$: τ_{w2}^h denote spot trade for good w and τ_{z2}^h denote spot trade for good z.
- As a result, we can rewrite the utility function as

$$u^h \left(c_{w1}^h, c_{z1}^h \right) + u^h \left(e_{w2}^h + \tau_{w2}^h, e_{z2}^h + Rk^h + \tau_{z2}^h \right). \tag{1}$$

Review of the Basics: First Best Planner Problem

Definition (Basic Planner Problem)

$$\max_{\left(c_{w1}^h, c_{z1}^h, k^h, \tau_{w2}^h, \tau_{z2}^h\right)_h} \sum_h \lambda^h \alpha^h \left[u^h \left(c_{w1}^h, c_{z1}^h\right) + u^h \left(e_{w2}^h + \tau_{w2}^h, e_{z2}^h + Rk^h + \tau_{z2}^h\right) \right] \quad (2)$$

subject to

$$k^h \geq 0, \forall h, \quad (3)$$

$$\sum_h \alpha^h c_{w1}^h = \sum_h \alpha^h e_{w1}^h, \quad (4)$$

$$\sum_h \alpha^h \left[c_{z1}^h + k^h \right] = \sum_h \alpha^h e_{z1}^h, \quad (5)$$

$$\sum_h \alpha^h \tau_{\ell 2}^h = 0, \forall \ell = w, z. \quad (6)$$

The Intertemporal Euler Equations for the Planner Problem

- The intertemporal Euler equations hold with possible inequality if the non-negativity constraint on saving $k^h \geq 0$ is binding or adjusted by a Lagrange multiplier:

$$\frac{u^a_{z1}}{u^a_{w1}} = \frac{u^b_{z1}}{u^b_{w1}}, \frac{u^a_{z2}}{u^a_{w2}} = \frac{u^b_{z2}}{u^b_{w2}}, \tag{7}$$

$$u^h_{z1} = R u^h_{z2} + \frac{\mu^h}{\lambda^h \alpha^h}, \forall h = a, b, \tag{8}$$

where $u^h_{\ell t} \equiv \frac{\partial u^h \left(c^h_{wt}, c^h_{zt} \right)}{\partial c_{\ell t}}$ for $\ell = w, z; t = 1, 2$, and μ^h is a Lagrange multiplier associated with $k^h \geq 0$.

- There is an entire class of first best allocations as solutions to the planner problem indexed by λ-weights, which pin down levels.

$$\lambda^h \alpha^h u^h_{\ell t} = \mu_{\ell t}, \forall h = a, b; \ell = w, z; t = 1, 2, \tag{9}$$

where $\mu_{\ell t}$ are Lagrange multipliers on the resource constraints for goods $\ell = w, z$ at $t = 1, 2$.

Competitive Equilibrium Complete Markets

Definition (Complete Markets)

A competitive equilibrium with complete markets is a specification of prices p_t of good z in period $t = 1, 2$, and the price of financial security Q at $t = 1$; consumptions $\left(c_{w1}^h, c_{z1}^h\right)$ at $t = 1$, saving and financial securities $\left(k^h, \theta^h\right)$ decisions made at $t = 1$, and trades $\left(\tau_{w2}^h, \tau_{z2}^h\right)$ at $t = 2$ for each type $h = a, b$ such that (i) at $t = 2$, taking $\left(p_2, k^h, \theta^h\right)$ as given parameters at $t = 2$, agent type $h = a, b$ solves

$$V^h\left(k^h, \theta^h, p_2\right) = \max_{\tau_{w2}^h, \tau_{z2}^h} u^h\left(e_{w2}^h + \tau_{w2}^h, e_{z2}^h + Rk^h + \theta^h + \tau_{z2}^h\right) \tag{10}$$

subject to the budget constraint in period $t = 2$,

$$\tau_{w2}^h + p_2 \tau_{z2}^h = 0, \tag{11}$$

Competitive Equilibrium Complete Markets

Definition (Complete Markets, Con't)

(ii) at $t = 1$, taking (p_1, Q) and $V^h\left(k^h, \theta^h, p_2\right)$ from $t = 2$ as given, agent type $h = a, b$ solves

$$\max_{c_{w1}^h, c_{z1}^h, k^h, \theta^h} u^h\left(c_{w1}^h, c_{z1}^h\right) + V^h\left(k^h, \theta^h, p_2\right) \tag{12}$$

subject to

$$k^h \geq 0, \tag{13}$$

$$c_{w1}^h + p_1\left(c_{z1}^h + k^h\right) + Q\theta^h = e_{w1}^h + p_1 e_{z1}^h, \tag{14}$$

Competitive Equilibrium Complete Markets

Definition (Complete Markets, Con't)

(iii) market clearing conditions hold:

$$\sum_h \alpha^h c_{w1}^h = \sum_h \alpha^h e_{w1}^h, \tag{15}$$

$$\sum_h \alpha^h \left[c_{z1}^h + k^h \right] = \sum_h \alpha^h e_{z1}^h, \tag{16}$$

$$\sum_h \alpha^h \tau_{\ell 2}^h = 0, \forall \ell = w, z, \tag{17}$$

$$\sum_h \alpha^h \theta^h = 0. \tag{18}$$

The Intertemporal Euler Equations for Competitive Equilibrium with Complete Markets

- The intertemporal Euler equations for competitive equilibrium with complete markets are

$$p_1 = \frac{\mu_{z1}}{\mu_{w1}}, p_2 = \frac{\mu_{z2}}{\mu_{w2}}, Q = \frac{\mu_{z2}}{\mu_{w1}}. \tag{19}$$

Table: Equilibrium allocations with externalities.

	k^h	c_{w1}^h	c_{z1}^h	c_{w2}^h	c_{z2}^h	$U^h(\mathbf{c}^h)$
$h = a$	0	2	2	2	2	-2.00
$h = b$	0	2	2	2	2	-2.00

Competitive Equilibrium with Incomplete Markets, Saving Only

Definition (Incomplete Markets)

A competitive equilibrium with incomplete markets, specifically saving k^h only and no securities, is a specification of prices p_t of good z in period $t = 1, 2$; consumptions $\left(c_{w1}^h, c_{z1}^h\right)$ at $t = 1$, saving k^h decision made at $t = 1$, and trades $\left(\tau_{w2}^h, \tau_{z2}^h\right)$ at $t = 2$ for each type $h = a, b$ such that

(i) at $t = 2$, taking $\left(p_2, k^h\right)$ as given parameters, agent type $h = a, b$ solves for trades $\left(\tau_{w2}^h, \tau_{z2}^h\right)$

$$V^h\left(k^h, p_2\right) = \max_{\tau_{w2}^h, \tau_{z2}^h} u^h\left(e_{w2}^h + \tau_{w2}^h, e_{z2}^h + Rk^h + \tau_{z2}^h\right) \tag{20}$$

subject to the budget constraint in period $t = 2$,

$$\tau_{w2}^h + p_2 \tau_{z2}^h = 0, \tag{21}$$

Competitive Equilibrium with Incomplete Markets, Saving Only

Definition (Incomplete Markets, Con't)

(ii) at $t=1$, taking p_1 and $V^h\left(k^h, p_2\right)$ from $t=2$ as given, agent type $h=a,b$ solves

$$\max_{c_{w1}^h, c_{z1}^h, k^h} u^h\left(c_{w1}^h, c_{z1}^h\right) + V^h\left(k^h, p_2\right) \tag{22}$$

subject to

$$k^h \geq 0, \tag{23}$$

$$c_{w1}^h + p_1\left(c_{z1}^h + k^h\right) = e_{w1}^h + p_1 e_{z1}^h, \tag{24}$$

Competitive Equilibrium with Incomplete Markets, Saving Only

Definition (Incomplete Markets, Con't)

(iii) market clearing conditions hold:

$$\sum_h \alpha^h c_{w1}^h = \sum_h \alpha^h e_{w1}^h, \tag{25}$$

$$\sum_h \alpha^h \left[c_{z1}^h + k^h \right] = \sum_h \alpha^h e_{z1}^h, \tag{26}$$

$$\sum_h \alpha^h \tau_{\ell 2}^h = 0, \forall \ell = w, z. \tag{27}$$

The Intertemporal Euler Equations for Competitive Equilibrium with Incomplete Markets

- The intertemporal Euler equations for competitive equilibrium with incomplete markets are

$$p_1 = \frac{u_{z1}^h}{u_{w1}^h} = \frac{u_{z2}^h}{u_{w1}^h}R + \frac{\eta^h}{\lambda^h \alpha^h u_{w1}^h}, \forall h = a, b. \qquad (28)$$

- Agent type b borrowing nothing and agent type a will be holding physical saving on its own to smooth consumption over time.

- The price of good z in period $t = 1$ is $p_1^{ex} = \left(\frac{4}{4-k^{ex}}\right)^2 = 2.2948$, and at date 2 is $p_2^{ex} = 0.5570$. Note that the price of good z is high at $t = 1$ relative to the first best.

Table: Equilibrium allocations with externalities.

	k^h	c_{w1}^h	c_{z1}^h	c_{w2}^h	c_{z2}^h	$U^h(\mathbf{c}^h)$
$h = a$	1.36	2.69	1.78	1.33	1.78	-2.2527
$h = b$	0	1.31	0.87	2.67	3.58	-2.5724

Constrained Planner Problem

Definition

$$\max_{\left(c_{w1}^h, c_{z1}^h, k^h\right)_h} \sum_h \lambda^h \alpha^h \left[u^h \left(c_{w1}^h, c_{z1}^h \right) + V^h \left(k^h, p_2 \left(k^a, k^b \right) \right) \right] \qquad (29)$$

subject to

$$k^h \geq 0, \forall h, \qquad (30)$$

$$\sum_h \alpha^h c_{w1}^h = \sum_h \alpha^h e_{w1}^h, \qquad (31)$$

$$\sum_h \alpha^h \left[c_{z1}^h + k^h \right] = \sum_h \alpha^h e_{z1}^h. \qquad (32)$$

Note that the value function V^h is already defined in the agent maximization problem (20) with the pricing function $p_2 \left(k^a, k^b \right)$ inserted.

Pecuniary Externality

- The necessary conditions for constrained optimality are given by

$$\frac{u_{z1}^h}{u_{w1}^h} = \frac{u_{z2}^h}{u_{w1}^h}R + \frac{\mu^h}{\lambda^h\alpha^h u_{w1}^h} + \frac{1}{\lambda^h u_{w1}^h}\sum_{\tilde{h}}\lambda^{\tilde{h}}\alpha^{\tilde{h}}\frac{\partial V^{\tilde{h}}}{\partial p_2}\frac{\partial p_2}{\partial k^h}, \forall h = a, b. \quad (33)$$

Definition

Distinction between Constrained-Optimal and Incomplete Markets Allocations, when borrowing is not allowed: The solutions to the Planner Problem (29) are termed constrained-efficient allocations. When (33) is different from (28), the competitive equilibrium with no borrowing is not constrained-efficient.

Definition

Pecuniary Externality: A pecuniary externality arises when the last term in (33) is non zero.

Transition to the Decentralization: Market Maker Problem

- We can transform the planner problem to a fully equivalent one in which the planner is a market maker choosing price p_2 and rights to trade at that price, and then finding savings $\boldsymbol{k}$ consistent with the pricing function $p_2\left(k^a, k^b\right)$ to support that price p_2 with all possible prices considered.

  1. First step, one can work with the inverse equilibrium price map, that is, designating p_2 first and then filling in the requisite k^h, $h = a; b$.
  2. Second step, we can then replace the inverse equilibrium price map by simply rewriting clearing condition (6) for good $\ell = w$ as

  $$\sum_h \alpha^h \Delta^h\left(k^h\left(p_2\right), p_2\right) = 0, \forall p_2, \tag{34}$$

  where $\Delta^h\left(k^h\left(p_2\right), p_2\right) \equiv \tau_{w2}^{h*}\left(k^h\left(p_2\right), p_2\right)$.
  3. Third step, write out the entire vector of variables for any agent h given the planner's choice of p_2: $\mathbf{x}^h\left(p_2\right) = \left[\mathbf{c}_1^h\left(p_2\right),\ k^h\left(p_2\right),\right.$
  $\left.\Delta^h\left(k^h\left(p_2\right), p_2\right)\right]$.
  4. Fourth step, we let the market maker choose the fraction $\delta^h\left(p_2\right)$ of type h assigned to p_2-exchange varying over all possible prices p_2.

The Market Maker Problem

$$\max_{[\delta^h(p_2), \mathbf{x}^h(p_2)]_{h,p_2}} \sum_h \sum_{p_2} \lambda^h \alpha^h \delta^h(p_2) \left[u^h \left(c_{w1}^h(p_2), c_{z1}^h(p_2) \right) + V^h \left(k^h(p_2), p_2 \right) \right] \tag{35}$$

subject to

$$\delta^h(p_2) k^h(p_2) \geq 0, \forall h; p_2, \tag{36}$$

$$\sum_{p_2} \sum_h \alpha^h \delta^h(p_2) c_{w1}^h(p_2) = \sum_h \alpha^h e_{w1}^h, \tag{37}$$

$$\sum_{p_2} \sum_h \alpha^h \delta^h(p_2) \left[c_{z1}^h(p_2) + k^h(p_2) \right] = \sum_h \alpha^h e_{z1}^h, \tag{38}$$

$$\sum_h \alpha^h \delta^h(p_2) \Delta^h \left(k^h(p_2), p_2 \right) = 0, \forall p_2. \tag{39}$$

The Necessary Conditions for the Market Maker Problem

- The necessary conditions can be derived in two parts.
  1. First, for each h with fraction $\delta^h(p_2) > 0$, for a given choice of p_2, the planner is choosing $c_{w1}^h(p_2)$, $c_{z1}^h(p_2)$ and $k^h(p_2)$ to maximize (35) subject to (36)-(39):

  $$p_1 = \frac{u_{z1}^h}{u_{w1}^h} = \frac{u_{z2}^h}{u_{w1}^h}R + \frac{\mu^h(p_2)}{\lambda^h \alpha^h u_{w1}^h} - \frac{\mu_\Delta(p_2)}{\mu_{w1}}\Delta_k^h\left(k^h(p_2), p_2\right), \forall h = a, b, \quad (40)$$

  where the derivative of $\Delta_k^h\left(k^h, p_2\right) \equiv \frac{\partial\Delta^h\left(k^h, p_2\right)}{\partial k}$.

  2. Second, regarding the global problem, the overall choice of p_2, for each h with fraction $\delta^h(p_2) > 0$, satisfies the following condition:

  $$\lambda^h V_p^h\left(k^h(p_2), p_2\right) = \mu_\Delta(p_2)\Delta_p^h\left(k^h(p_2), p_2\right) \quad (41)$$

  The planner trades off the marginal benefit (cost) on V^h from choosing p_2 with the marginal cost (benefit) on excess demand from choosing p_2.

Decentralization with Individually Chosen Rights to Trade Δ^h at prices P_Δ

Definition (Competitive Equilibrium with Rights to Trade)

A competitive equilibrium with rights to trade is a specification of allocation $\left[\mathbf{x}^h(p_2), \delta^h(p_2), \boldsymbol{\tau}^h(p_2)\right]$, price of good z at $t = 1$, p_1, spot prices p_2 for active and potential spot markets at $t = 2$, and the prices of the rights to trade $[\mathbf{P}_\Delta(p_2)]_{p_2}$ such that

(i) at $t = 2$, taking $k^h(p_2)$ as predetermined and p_2 as given, agent type $h = a, b$ solves for trades $\boldsymbol{\tau}^h(p_2)$:

$$V^h\left(k^h, p_2\right) = \max_{\tau_{w2}^h, \tau_{z2}^h} u^h\left(e_{w2}^h + \tau_{w2}^h, e_{z2}^h + Rk^h + \tau_{z2}^h\right) \tag{42}$$

subject to the budget constraint in period $t = 2$,

$$\tau_{w2}^h + p_2 \tau_{z2}^h = 0, \tag{43}$$

Decentralization with Individually Chosen Rights to Trade Δ^h at prices P_Δ

Definition (Competitive Equilibrium with Rights to Trade, Con't)

(ii) at $t = 1$, agent type h takes prices p_1 and $P_\Delta(p_2)$ as given and solves for the choice of p_2 and associated $\mathbf{x}^h(p_2)$ to

$$\max_{\left[\mathbf{x}^h(p_2), \delta^h(p_2)\right]_{p_2}} \sum_{p_2} \delta^h(p_2) \left[u^h\left(c^h_{w1}(p_2), c^h_{z1}(p_2)\right) + V^h\left(k^h(p_2), p_2\right) \right] \quad (44)$$

subject to

$$\delta^h(p_2) k^h(p_2) \geq 0, \forall p_2,$$

$$\sum_{p_2} \delta^h(p_2) \left[c^h_{w1}(p_2) + p_1 c^h_{z1}(p_2) + p_1 k^h(p_2) \right.$$

$$\left. + P_\Delta(p_2) \Delta^h\left(k^h(p_2), p_2\right) \right] \leq e^h_{w1} + p_1 e^h_{z1},$$

Decentralization with Individually Chosen Rights to Trade Δ^h at prices P_Δ

> **Definition (Competitive Equilibrium with Rights to Trade, Con't)**
>
> (iii) market-clearing conditions hold:
>
> $$\sum_{p_2} \sum_h \alpha^h \delta^h(p_2) c_{w1}^h(p_2) = \sum_h \alpha^h e_{w1}^h, \tag{45}$$
>
> $$\sum_{p_2} \sum_h \alpha^h \delta^h(p_2) \left[c_{z1}^h(p_2) + k^h(p_2) \right] = \sum_h \alpha^h e_{z1}^h, \tag{46}$$
>
> $$\sum_h \alpha^h \delta^h(p_2) \Delta^h \left(k^h(p_2), p_2 \right) = 0, \forall p_2, \tag{47}$$
>
> $$\sum_h \delta^h(p_2) \alpha^h \tau_{\ell 2}^h(p_2) = 0, \forall \ell = w, z; p_2. \tag{48}$$

The Necessary Conditions for the Decentralization

- Using the similar steps, we can write the necessary conditions for type h maximization as follows.

$$p_1 = \frac{u_{z1}^h}{u_{w1}^h} = \frac{u_{z2}^h}{u_{w1}^h} R + \frac{\eta^h(p_2)}{u_{w1}^h} - P_\Delta(p_2) \Delta_k^h\left(k^h(p_2), p_2\right), \forall h = a, b. \tag{49}$$

- The two equations, (40) and (49), are identical when we match the Lagrange multipliers and prices from the planner problem and the new decentralized equilibrium using the following conditions:

$$P_\Delta(p_2) = \frac{\mu_\Delta(p_2)}{\mu_{w1}} \text{ and } \eta^h = \frac{\mu^h}{\lambda^h \alpha^h}. \tag{50}$$

- Similar to the market maker problem,

$$\left(\frac{1}{\eta_{bc,1}^h}\right) V_p^h\left(k^h(p_2), p_2\right) = P_\Delta(p_2) \Delta_p^h\left(k^h(p_2), p_2\right) \tag{51}$$

Numerical Solutions for Competitive Equilibrium with Rights to Trade

- The competitive equilibrium with rights to trade has *one and only one active exchange*, $p_2^{op} = 0.5974$, even though all exchanges are available in principle a priori for trade.

Table: Constrained Optimal Allocation with Pareto weights $\lambda^1 = 0.778$ and $\lambda^2 = 0.222$.

	k^h	c_{w1}^h	c_{z1}^h	c_{w2}^h	c_{z2}^h	$\Delta^h\left(p_2^{op}\right)$	$U^h\left(\mathbf{c}^h\right)$
$h = a$	1.18	2.61	1.84	1.30	1.68	0.30	-2.2934
$h = b$	0	1.39	0.98	2.70	3.50	-0.30	-2.3904

Table: Equilibrium prices of rights to trade in spot markets $P_\Delta\left(p_2\right)$ at price p_2.

	$p_2 = 0.5770$	$p_2^{op} = 0.5974$	$p_2 = 0.6181$
$P_\Delta\left(p_2\right)$	1.1383	1.2116	1.2840

General Economy with Collateral Constraints

- Consider an economy with S states at $t = 2$, i.e., s, each of which occurs with probability π_s, $\sum_s \pi_s = 1$.
- Each agent type h is endowed with $\left(e_{w1}^h, e_{z1}^h\right)$ at date $t = 1$ and $\left(e_{w2s}^h, e_{z2s}^h\right)$ in state s at date $t = 2$.
- Utility functions u^h are strictly concave with regularity conditions.
- There are J securities. Let $\mathbf{D} = [D_{js}]$ be the payoff matrix of those assets at $t = 2$ where D_{js} is the payoff of asset j in units of good w in state s.
- Let θ_j^h denote the amount of the jth security acquired by an agent of type h at $t = 1$ with $\boldsymbol{\theta}^h \equiv \left[\theta_j^h\right]_j$.
- The collateral constraint states that there must be sufficient collateral in value to honor all promises:

$$p_{2s} R_s k^h + \sum_j D_{js} \theta_j^h \geq 0, \forall s, \tag{52}$$

where R_s is the state contingent return on the collateral.

The Planner Problem for General Economy

Definition

The Pareto problem with Pareto weights $\left[\lambda^h\right]_h$ is defined as follows.

$$\max_{\left[\mathbf{x}^h, \delta^h(\mathbf{p})\right]_h} \sum_{h,\mathbf{p}} \lambda^h \alpha^h \delta^h(\mathbf{p}) \left[u^h\left(c_{w1}^h(\mathbf{p}), c_{z1}^h(\mathbf{p})\right) + \sum_s \pi_s V_s^h\left(k^h(\mathbf{p}), \theta^h(\mathbf{p}), \mathbf{p}\right)\right]$$

subject to

$$\delta^h(\mathbf{p}) k^h(\mathbf{p}) \geq 0, \forall h; \mathbf{p}, \tag{53}$$

$$\delta^h(\mathbf{p}) \left[p_{2s} R_s k^h(\mathbf{p}) + \sum_j D_{js} \theta_j^h(\mathbf{p})\right] \geq 0, \forall h; \mathbf{p}; s, \tag{54}$$

$$\sum_{\mathbf{p}} \sum_h \delta^h(\mathbf{p}) \alpha^h c_{w1}^h(\mathbf{p}) = \sum_h \alpha^h e_{w1}^h; \tag{55}$$

$$\sum_{p_2} \sum_h \delta^h(\mathbf{p}) \alpha^h \left[c_{z1}^h(\mathbf{p}) + k^h(\mathbf{p})\right] = \sum_h \alpha^h e_{z1}^h; \tag{56}$$

$$\sum_h \delta^h(\mathbf{p}) \alpha^h \theta_j^h(\mathbf{p}) = 0, \forall j; \mathbf{p}; \tag{57}$$

$$\sum_h \delta^h(\mathbf{p}) \alpha^h \Delta_s^h(\mathbf{p}) = 0, \forall s; \mathbf{p} \tag{58}$$

Decentralization for General Economy with Δ^h at prices P_Δ

Definition (Competitive Equilibrium for General Economy)

A competitive equilibrium with rights to trade is a specification of allocation $\left[\mathbf{x}^h, \boldsymbol{\tau}^h, \delta^h\right]_h$, price of good z at $t = 1$, p_1, spot prices $\mathbf{p} = [p_{2s}]_s$ for active and potential spot markets at $t = 2$, and the prices of securities and the rights to trade $[\mathbf{Q}(\mathbf{p}), \mathbf{P}_\Delta(\mathbf{p})]_\mathbf{p}$ such that

(i) in state s at date $t = 2$, taking $\left(k^h(\mathbf{p}), \theta^h(\mathbf{p}), \mathbf{p}\right)$ as given, agent type $h = a, b$ solves

$$\max_{\tau^h_{w2s}(\mathbf{p}), \tau^h_{z2s}(\mathbf{p})} u^h\big(e^h_{w2s} + \sum_j D_{js}\theta^h_j(\mathbf{p}) + \tau^h_{w2s}(\mathbf{p}), e^h_{z2s} + R_s k^h(\mathbf{p}) + \tau^h_{z2s}(\mathbf{p})\big)$$

subject to the budget constraint in period $t = 2$,

$$\tau^h_{w2}(\mathbf{p}) + p_{2s}\tau^h_{z2}(\mathbf{p}) = 0, \tag{59}$$

where the optimum is defined as the value function $V^h_s\left(k^h(\mathbf{p}), \theta^h(\mathbf{p}); \mathbf{p}\right)$.

Decentralization for General Economy with Δ^h at prices P_Δ

Definition (Competitive Equilibrium for General Economy, Con' t)

(ii) at date $t = 1$, for any agent type h as a price taker, $\left[\mathbf{x}^h(\mathbf{p}), \delta^h(\mathbf{p})\right]_{\mathbf{p}}$ solves

$$\max_{\mathbf{x}^h, \delta^h} \sum_{\mathbf{p}} \delta^h(\mathbf{p}) \left[u^h(c_{w1}^h(\mathbf{p}), c_{z1}^h(\mathbf{p})) + \sum_s \pi_s V_s^h\left(k^h(\mathbf{p}), \boldsymbol{\theta}^h(\mathbf{p}), \mathbf{p}\right) \right] \tag{60}$$

subject to

$$\delta^h(\mathbf{p}) \, k^h(\mathbf{p}) \geq 0, \forall h; \mathbf{p},$$

$$\delta^h(\mathbf{p}) \left[p_{2s} R_s k^h(\mathbf{p}) + \sum_j D_{js} \theta_j^h(\mathbf{p}) \right] \geq 0, \forall h; \mathbf{p}; s,$$

$$\sum_{\mathbf{p}} \delta^h(\mathbf{p}) \left[c_{w1}^h(\mathbf{p}) + p_1 \left[c_{z1}^h(\mathbf{p}) + k^h(\mathbf{p}) \right] + \mathbf{Q}(\mathbf{p}) \cdot \boldsymbol{\theta}^h(\mathbf{p}) \right.$$
$$\left. + \mathbf{P}_\Delta(\mathbf{p}) \cdot \boldsymbol{\Delta}^h(\mathbf{p}) \right] \leq e_{w1}^h + p_1 e_{z1}^h,$$

Decentralization for General Economy with Δ^h at prices P_Δ

> **Definition (Competitive Equilibrium for General Economy, Con't)**
>
> (iii) market-clearing conditions hold:
>
> $$\sum_{\mathbf{p}} \sum_{h} \delta^h(\mathbf{p}) \alpha^h c_{w1}^h(\mathbf{p}) = \sum_{h} \alpha^h e_{w1}^h; \qquad (61)$$
>
> $$\sum_{p_2} \sum_{h} \delta^h(\mathbf{p}) \alpha^h \left[c_{z1}^h(\mathbf{p}) + k^h(\mathbf{p}) \right] = \sum_{h} \alpha^h e_{z1}^h; \qquad (62)$$
>
> $$\sum_{h} \delta^h(\mathbf{p}) \alpha^h \theta_j^h(\mathbf{p}) = 0, \forall j; \mathbf{p}; \qquad (63)$$
>
> $$\sum_{h} \delta^h(\mathbf{p}) \alpha^h \Delta_s^h(\mathbf{p}) = 0, \forall s; \mathbf{p}; \qquad (64)$$
>
> $$\sum_{h} \delta^h(\mathbf{p}) \alpha^h \tau_{\ell 2}^h(\mathbf{p}) = 0, \forall \ell = w, z; \mathbf{p}. \qquad (65)$$

The Welfare and The Existence Theorems

Theorem

*With non-satiation of preferences, a competitive equilibrium with rights to trade in **p**-exchanges is constrained Pareto optimal.*

Theorem

Any constrained Pareto optimal allocation with strictly positive Pareto weights $\lambda^h > 0, \forall h$ can be supported as a competitive equilibrium with rights to trade with transfers.

Theorem

With local non-satiation of preferences and positive endowments, a competitive equilibrium with rights to trade exists.

Conclusion

- We show how markets in rights and exclusivity internalize pecuniary externalities.

- Ex ante competition can achieve a constrained-efficient allocation: create contemporary markets for rights to trade in future spot markets at all possible designated prices that could prevail.

- With the appropriate ex ante design of exchanges, we can let markets for the rights to trade solve the problem.

- This market-based solution to fire sales and other pecuniary externalities can be extended to capture at least the following 6 prototype economies:
  1. Fire Sales Economy (Lorenzoni, 2008).
  2. Liquidity Constrained Economy (Hart and Zingales, 2011).
  3. Collateral Economy (Kilenthong and Townsend, 2014).
  4. Exogenous Incomplete Markets (Greenwald and Stiglitz, 1986).
  5. Moral Hazard with Retrading (Acemoglu and Simsek, 2008; Kilenthong and Townsend, 2011).
  6. Hidden Information with Retrading (Diamond and Dybvig, 1983).