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The Information Problem of Decentralized Monetary Exchange and of Equilibria with High-Velocity Circulating Private Debt: Regulation Using Distributed Ledger Technology (Lecture 9)

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Outline

- Information problem of decentralized monetary exchange
  - Ostroy-Star
- Coordination problem with circulating private debt
  - Townsend-Wallace
- Rehypothecation and broker dealers in NY markets

Problems in Decentralized Exchange: Theory "Money and the Decentralization of Exchange" (Ostroy and Starr 1974)

- Achieving an ideal Walrasian allocation of resources
  - Even with no transaction costs, with meetings and prices known in advance and no strategic behavior
  - Implementing a smart contract
- Two exceptional solutions
  - Abundant liquidity: have everyone so well endowed that all purchases can be financed before sales (it is enough in some environments). Concerns: in some cases the liquidity is someone else's liability.
    - Is this central bank accounts, with consequences?
  - Central warehouse: have a trader so well endowed so as to act as a commodity warehouse or universal brokerage.
    - Is this a trading platform linked to a clearing bank?
- Generic Impossibility Theorem: a *quid pro quo* condition in bilateral exchange must be supplemented with a (centralized or other equivalent) record keeping system so agents can coordinate in trade.
  - Liquidity is not only about the amount of trade, or stuff to clear
  - but also about the timing and nature of transactions, keeping track of this history

Mathematical Formalization of the Trading Process: To Achieve a Multiparty Smart Contract

- J traders and finite number of commodities with given price vector p. A Walrasian environment
- Each trader is endowed with vector b and has excess demand z. If trades are completed successfully, the trader will have b+z commodities.
- Three basic underlying restrictions on z (denote them by U, undelying):
  1. for each trader, $p \cdot z = 0$
  2. $\Sigma_{j \in J} z_j = 0$
  3. z must be feasible given trader's endowment, so $z_c \geq -b_c$ for each commodity c

  A familiar example of U is a competitive equilibrium
- Mechanics of trading. A round of trading has T periods. In each period, traders meet in pairs, and T is the number of periods required so that a trader meets every other trader exactly once. If J is odd, one trader stays out of trading in each period.
- Let a_{it} be the vector that represents the trading outcome for trader i in period t.
  - A trade: positive number, acquisition of each commodity bought, but negative number meaning sale
- Three basic underlying restrictions on *a*
  1. a_{it} should not result in a negative holding of any commodity
  2. commodities must be conserved, if trader i is meeting with trader j, $a_{it} + a_{jt} = 0$
  3. trade should be fair (*quid pro quo*), for every trader i, $p \cdot a_{it} = 0$, payment and clearing
  4. denote them by A, trade actions

Information Restrictions on Trades

- Information restrictions determine which variables can be used as arguments of trading process, as inputs to that function.
- The paper proposes three decentralizing D restrictions D.1, D.2, D.3; with higher number representing greater amount of information.
- It also proposes a centralizing restriction C, which allows two meeting traders to use information on other traders.

- Suppose traders i and j meet in period t:

  D.1. $a_{it} = -a_{jt}$ is a function of traders i and j's initial excess demands z, for traders i and j, endowment b, for traders i and j, and total trading up to period t-1, $\Sigma^{t-1}_{u=1} a_{iu}$ and $\Sigma^{t-1}_{u=1} a_{ju}$

  D.2. $a_{it} = -a_{jt}$ is a function of the arguments in D.1, and in addition the identities of i and j

  D.3. Same as D.2, but instead of the total trading up to t-1, a_{it} is a function of the entire history of trading, $(a_{i1}, a_{i2},\dots, a_{i(t-1)})$ and $(a_{j1}, a_{j2},\dots, a_{j(t-1)})$

  C. Same as D.3, plus the initial excess demand z over all traders, and the trading history $a_{k1}, a_{k2},\dots, a_{k(t-1)}$ for every trader k

Completion of Trades: Fast Chains, Key Broker

- **Theorem 1**: There is a trading rule that (i) uses information C; (ii) satisfies A; and (iii) completes trading within a single round for any (p, Z, B) satisfying U.
  - This tells us that physical decentralization is possible when information is shared. The following theorems limit the types of information used for trading (D.1, D.2, D.3)
- **Theorem 2**: There is NO trading rule that (i) uses information D.3; (ii) satisfies A; and (iii) completes trading within a single round for any (p, Z, B) satisfying U.
  - It implies that some extra assumptions are necessary to allow decentralization, Theorems 3 and 4 are in this line.
- **Theorem 3**: There is a trading rule that (i) uses information D.2; (ii) satisfies A; and (iii) completes trading within a single round for any (p, Z, B) satisfying U such that there exists a trader i such that his endowment is enough to cover all the other traders' buying needs:

$$b_{ic} \geq \sum_{j \neq i} \overset{[z_{jc}]^+}{\cancel{[z_{ic}]^+}} \text{ for every commodity } c.$$

  - It is the case in which trader i serves as a facilitator of trades (giant warehouse/supplier): she should have large enough stock of every commodity. It solves the difficulty of arranging trades when commodities are scarce and trades must be routed along the right chain to ensure that demands be met within a single round

Completion of Trades: Ample Ex Ante Liquidity

- **Theorem 4**: There is a trading rule that (i) uses information D.1; (ii) satisfies A; and (iii) completes trading within a single round for any (p, Z, B) satisfying U such that there exists a commodity m and for every trader i, her endowment of commodity m is enough to fund all her purchases:

$$p_m b_{im} \geq \sum_{c \neq m} p_c [z_{ic}]^+$$

  - It is the case where commodity m serves as some type of money and in which every trader has a large enough stock of commodity m to pay for his purchases. Commodity m helps traders solve problems arising from the restriction of *quid pro quo*.
  - Without *quid pro quo*, easy to find trading process that completes trade within a single round without additional assumptions. However, with the restriction, when two meeting traders' excess demands do not precisely offset each other, they should decide which subset of their demands would be met at the meeting.
  - Money solves this problem because at each meeting, the traders can just fulfill individual demands as much as possible and settle the difference in commodity m

Completion of Trades: The Walras Banker, with Trade Credit, IOUs

- **Theorem 4**: There is a trading rule that (i) uses information D.1; and (ii) completes trading within three rounds for any (p,Z,B) satisfying U in a 'bank credit economy'.
  - In the credit bank economy, every trader can meet their excess demand using money, as in the 'monetary economy' described by theorem 4
  - However, in the bank credit economy, the premise that every trader holds enough money at the beginning is not required Instead, there is a bank that can create money (issue notes/IOUs).
  - In the first round, traders do nothing other than borrowing money from the bank, so all of them have enough money as required by the premise of the monetary economy.
  - In the second round, they make trades as they would do in the monetary economy. At the end of the second round, every trader has met his excess demand and holds the same amount of money that he started the second round with.
  - In the third round, traders pay back what they borrowed from the bank.

Key Example of the Difficulty: Proof of Theorem 2, Kyungmin Kim

Theorem 2: There is no trading rule that (i) uses information D.3; (ii) satisfies A; and (iii) completes trading within a single round for any (p, Z, B) satisfying U.

Proof: As the original paper attempted to, I find two economies such that in a certain meeting, (i) the two traders cannot decide which economy they are in; and (ii) they have to make different trades depending on which economy they are in.

There are five traders, $1, 2, ..., 5$, there are five periods, $t = 1, 2, ..., 5$, and there are four types of goods, $1, 2, 3$ and 4. The sequence of meetings is as follows:

$$t = 1 : \overline{25}, \overline{34}, \overline{1}.$$
$$t = 2 : \overline{15}, \overline{23}, \overline{4}.$$
$$t = 3 : \overline{12}, \overline{45}, \overline{3}.$$
$$t = 4 : \overline{14}, \overline{35}, \overline{2}.$$
$$t = 5 : \overline{13}, \overline{24}, \overline{5}.$$

Two traders under the same line are meeting with each other.

First, each type of good has the same price, so $p = (1, 1, 1, 1)$.

Let N be the matrix of excess demands at the beginning, where

$$N = \begin{bmatrix} 4 & -2 & -2 & 0 \\ -4 & 1 & 1 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Each row represents a trader and each column represents a type of good. For example, $N_{12} = -2$ means that trader 1's excess demand for good 2 is -2.

Let B be the matrix of endowment, with B_{ij} representing the number of good j that trader i has:

$$B = \begin{bmatrix} 0 & 2 & 2 & 0 \\ 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Below, I go through individual periods. Trader 5 has zero endowment, so he can never trade.

In $t = 1$, trader 2 meets with 5, and 3 meets with 4. Since trader 5 does not trade, only traders 3 and 4 can trade. Both trader 3 and trader 4 have only 1 unit of good 4. Therefore, they do not make any trade, and no trade occurs in $t = 1$.

In $t = 2$, trader 1 meets with 5, and 2 meets with 3. Therefore, only traders 2 and 3 can make a nontrivial trade. After trading with 2, trader 3 does not trade with anyone until he trades with trader 1 in $t = 5$.

Both traders 1 and 3 want to hold zero amount of good 4. Therefore, trader 3 must give 1 unit of good 4 to trader 2 in $t = 2$; otherwise, some good 4 must remain with traders 1 and 3 and their excess demands will not be met. In return, trader 2 gives 1 unit of good 1 to trader 3, since it is the only type of good that he has. After $t = 2$, the matrix of excess demands N_2 and good holdings B_2 are:

$$N_2 = N = \begin{bmatrix} 4 & -2 & -2 & 0 \\ -3 & 1 & 1 & 1 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}; B_2 = B = \begin{bmatrix} 0 & 2 & 2 & 0 \\ 3 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Trader 1 demands none of good 2, currently has 2 units. Has use for 1 only. So, give 1 unit of good to Trader 2 and 2 units of good 3.

In $t = 3$, trader 1 meets with 2 and 4 meets with 5. Therefore, only traders 1 and 2 can make a nontrivial trade. Trader 2 needs to give 3 units of good 1 to trader 1 because only trader 1 demands good 1; after $t = 3$, trader 2 meets with trader 4 in $t = 5$, so $t = 3$ is the last chance for trader 2 to pass his stock of good 1 to trader 1.

In return, trader 1 needs to give 3 units of goods 2 or 3 to trader 2. However, in $t = 5$, trader 1 needs to give 1 unit of good 2 to trader 3;

To complete the proof, I construct a similar but different economy. The price vector is the same: $p = (1, 1, 1, 1)$. The matrix of initial excess demands, denoted by M, is

$$M = \begin{bmatrix} 4 & -2 & -2 & 0 \\ -4 & 1 & 1 & 2 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Traders 1 and 2 excess demands have not changed. History of Traders 3 and 4 at t = 1 is different but Traders 1 and 2 cannot know this.

This is the entire history of trades of 2 and 3. Only under C do they know more.

M can be obtained from N by exchanging row 3 with row 4. The endowment is also given in the same way: The endowment of trader i in good j is $[-M_{ij}]^+$.

Going through the same steps as I did for the previous economy, I find that trader 1 needs to give 2 units of good 2 and 1 unit of good 3 to trader 2 in $t = 3$. In the previous economy, trader 1 needed to give 1 unit of good 2 and 2 units of good 3 to trader 2.

Finally, I show that traders 1 and 2 cannot know whether they are in economy N or M in $t = 3$. In both economies, the history of trader 1's excess demands is simply a repeat of $(4, -2, -2, 0)$ because he never trades before $t = 3$. In both economies, the history of trader 2's excess demands is: Before meeting with trader 3 in $t = 2$, the excess demand is $(-4, 1, 1, 2)$, and after the meeting in $t = 2$, it is $(-3, 1, 1, 1)$.

This completes the proof.

“CIRCULATING PRIVATE DEBT: AN EXAMPLE WITH A COORDINATION PROBLEM” TOWNSEND-WALLACE (1987)

- A seemingly central observation for monetary economics is that some objects-often referred to as monies-appear in exchange much more frequently than other objects. In this paper, we present and study a model that generates a version of this observation for private securities; in the model, some securities get traded frequently, or circulate, whereas others do not. In this and other respects, the securities in our model resemble historically observed bills of exchange.

- The quantities of securities that are required to support an equilibrium and that are issued by individuals at the same time in spatially and informationally separated markets must satisfy restrictions not implied by individual maximization and by market clearing in each separate market.
- The utility-maximizing choices of quantities of securities, the strategies of individuals, are not in general unique but must somehow be coordinated across informationally separated markets if they are to be consistent with the existence of an equilibrium.
- This (coordination) problem arises only in some versions of our model. There is a close connection between its appearance and that of circulating securities
- problems-perhaps in the form of chaotic conditions-sometimes arise in credit markets with unregulated issue of private securities that play an important role in exchange.

- More general: Let *J* denote the number of locations and *T* the number of dates, the commodity space has dimension *JT.* We assume that each person gets utility from commodities and has positive endowments of commodities in a proper subset whose elements correspond to the location-date combinations that the person visits
- One commodity for each location-date combination
- one good that is indexed by location and date
- goods indexed by one location-date combination cannot be transformed into goods indexed by another location-date combination
- there is no transportation, production, or storage technology for goods.

- spatial separation limit trades in securities in what seems to be a natural way
- at a particular time, a person can only trade securities with someone he or she meets.
- Second, although securities can be transported, they can move only with a person.
- Finally, we do not allow people to renege on their debts or to counterfeit others' debts. Securities or debts in our model take the form of promises to pay stated amounts of goods that are date and location specific. We assume that if the promise is presented at the relevant date and location, then it is honored.

Table V-1. Who Meets Whom When

	Location	
Date	1	2
1	(1,2)	(3,4)
2	(1,3)	(2,4)
3	(1,2)	(3,4)
4	(1,3)	(2,4)

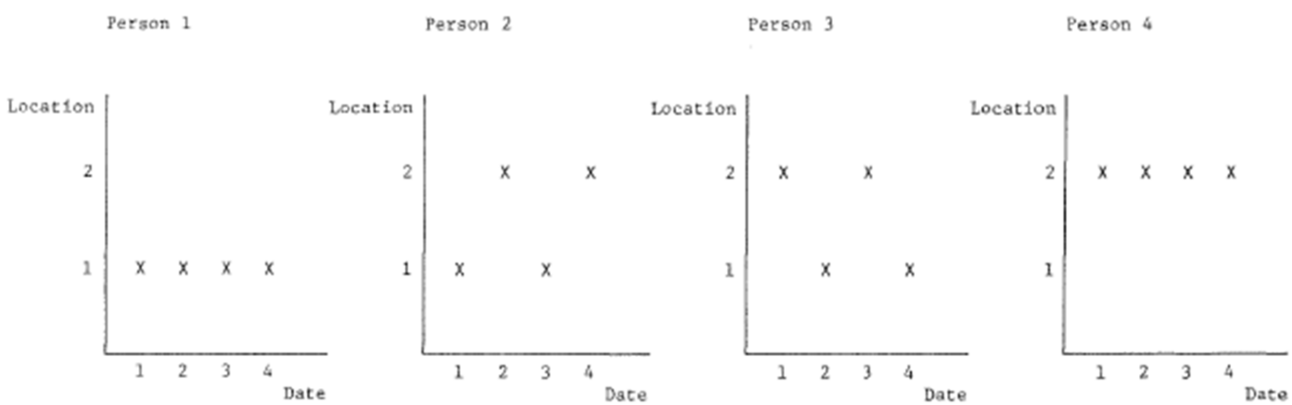


Figure V-1. Relevant commodity subspaces in the economy shown in table V-1.

- If $T = 2$-that is, if the economy lasts only two periods-then no trade is possible in the table V-I economy under our security trading rules.
- For example, person 1 cannot sell a promise to person 2 because person 2 can neither redeem it at date 2 nor pass it on to person 4, who has no use for it at date 2, the assumed last date
- absence of double coincidence: $T = 2$ no pair of persons has endowments and cares about a common two-dimensional subspace of the commodity space
- in the sense that there can exist redistributions of the endowments that give rise to allocations that are Pareto-superior to the endowment allocation. Put differently, if all four people were together at some time zero and traded in complete (location and date contingent) markets, something we rule out, then the endowment would not necessarily be a competitive equilibrium

- We assume an economy with G persons, each of whom lives T periods. At each time t, each person g can be paired with some other person or with no one. These pairings occur at (isolated) locations. Thus, we assume that person g is assigned to some location i at each time t and that in that location there either is or is not a single trading partner.

If person g is in location i at time t, then he or she is endowed with some positive number of units, w^g_{it}, of the consumption good at location i, date t. For other location-date combinations, the endowment is zero. Let w^g denote the entire JT dimensional endowment vector for person g. Also let c^g_{it} denote the nonnegative number of units of location i-date t consumption of person g and let c^g denote the entire JT dimensional consumption vector for person g. Preferences of each person g are described by a utility function $U^g(c^g)$ that is continuous, concave, and strictly increasing in the T-dimensional subspace that is relevant for g.

We restrict attention to securities that can be redeemed. Thus, if d^f_{st}, which is nonnegative, denotes securities issued by person f at time s to pay d^f_{st} units of the consumption good where f will be at time t, we consider only triplets (f,s,t) with the property that there is a path or chain of pairings leading from where f is at s to where f is at t.

We let $p^f_{st}(i,u)$ be the price per unit of d^f_{st} at location i, date u, in units of good (i,u). However, we define such a price only for pairs (i,u) that potentially admit of a nontrivial trade in d^f_{st}. (This allows us to avoid having to determine a price for d^f_{st} in a market where demand and supply are identically zero and also allows us to restrict attention to positive prices.) Thus, suppose h and g meet at (i,u). We say that h is a *potential demander* of d^f_{st} at (i,u) if there is a route from h at u to f at t. We say that h is a *potential supplier* of d^f_{st} at (i,u) if there is a route from f at s to h at u. We say there is a *market* in d^f_{st} at (i,u) if and only if h is a potential demander and g is a potential supplier at (i,u), or vice versa.

We let $d_{st}^{fg}(i,u)$ be the excess demand by g at (i,u) for d_{st}^{f}. In terms of this notation, our debt trading rules are

$$\begin{aligned} &\sum_{u=s}^{t} d_{st}^{ff}(\cdot,u) \geq 0 \text{ for each } f \\ &\sum_{u=s}^{t'} d_{st}^{fg}(\cdot,u) \geq 0 \text{ for each } t' \geq s \text{ and } g \neq f \end{aligned} \tag{1}$$

The first inequality says that f must end up demanding as much as f issues, which expresses our no-reneging rule. The second says that $g \neq f$ cannot supply d_{st}^{f} without having previously acquired it. Finally, as a convention,

Then, as budget constraints for any person g, we may write

$$w_{iu}^{g} \geq c_{iu}^{g} + \sum d_{st}^{fg}(i,u)\, p_{st}^{f}(i,u), \tag{2}$$

there being T such constraints, one for each (i,u) that g visits. The summation in (2) is over all securities—all (f,s,t)—for which a market exists

DEFINITION. A debt equilibrium is a specification of consumption and debt demands—c^{g} and d^{g} for each $g = 1, 2, \ldots, G$—and positive security prices, $p_{st}^{f}(i,u)$, such that

(i) c^{g} and d^{g} maximize $U^{g}(c^{g})$ subject to (1) and (2)

(ii) $\Sigma_{g}\,(c_{iu}^{g} - w_{iu}^{g}) = 0$ for each (i,u) and $\Sigma_{g}\, d_{st}^{fg}(i,u) = 0$ for each (i,u) and all potentially redeemable d_{st}^{f}.

4. Debt Equilibria and Complete-Markets Equilibria

In this section we establish equivalence for the table V-1, $T = 4$ economy between the equilibrium allocations and prices of complete date-location contingent markets and the allocations and prices of debt equilibria. This proves useful in describing the transaction pattern implications of the theory and the coordination problem. We begin by showing that any complete-markets equilibrium (CME) consumption allocation can be supported by a debt equilibrium (DE).

To show that any CME can be supported by a DE in the table V-1, $T = 4$ economy, we start with a given CME. This we describe by individual consumption excess demands, $e^g_{it} \equiv c^g_{it} - w^g_{it}$, and by associated prices, s_{it} (in terms of an abstract unit of account). These constitute a CME if they satisfy:

$$\sum_i \sum_t e^g_{it} s_{it} = 0 \text{ for each } g \tag{3}$$

$$\sum_g e^g_{it} = 0 \text{ for each } (i,t) \tag{4}$$

A corresponding DE consists of positive debt prices and nonnegative, market-clearing debt quantities such that

(a) the debt quantities and the given CME e^g_{it}'s satisfy each person's debt budget constraints

(b) the debt quantities and the given CME e^g_{it}'s are utility maximizing for each person given those debt prices

Our first step is to produce candidate debt prices for the table V-1, T = 4 economy. This candidate is produced by matching the terms of trade

$p^1_{14}(1,4) = 1$.] Thus, we let $p^1_{14}(1,1) = s_{14}/s_{11}$. In general, then, each debt price is taken to be a ratio of CME prices with the numerator corresponding to the redemption location-date and the denominator to the location-date of the current trade.

For noncirculating debts, then, our candidate is

$$\begin{aligned}
&(p^1_{13}(1,1), p^3_{13}(2,1), p^1_{24}(1,2), p^2_{24}(2,2)) \\
&\quad = (p^2_{13}(1,1), p^4_{13}(2,1), p^3_{24}(1,2), p^4_{24}(2,2)) \\
&\quad = (s_{13}/s_{11}, s_{23}/s_{21}, s_{14}/s_{12}, s_{24}/s_{22}), \qquad (5)
\end{aligned}$$

whereas, for circulating debts, it is

$$\begin{bmatrix} p^1_{14}(1,1), p^1_{14}(2,2), p^1_{14}(2,3) \\ p^2_{14}(1,1), p^2_{14}(1,2), p^2_{14}(2,3) \\ p^3_{14}(2,1), p^3_{14}(2,2), p^3_{14}(1,3) \\ p^4_{14}(2,1), p^4_{14}(1,2), p^4_{14}(1,3) \end{bmatrix} = \begin{bmatrix} s_{14}/s_{11}, s_{14}/s_{22}, s_{14}/s_{23} \\ s_{24}/s_{11}, s_{24}/s_{12}, s_{24}/s_{23} \\ s_{14}/s_{21}, s_{14}/s_{22}, s_{14}/s_{13} \\ s_{24}/s_{21}, s_{24}/s_{12}, s_{24}/s_{13} \end{bmatrix}. \qquad (6)$$

We can immediately indicate that this implies that satisfaction of (a) implies satisfaction of (b). To see this, multiply the debt constraint for e_{it}^{g} [equation (2)] by s_{it} and sum over i and t. Using (5) and (6), the result is (3), in which debt quantities do not appear. Thus, at prices given by (5) and (6), the debt constraints for any person are at least as constraining as (3). Therefore, if we can produce market-clearing debt quantities, d_{st}^{f}'s, which make the CME e_{it}^{g}'s feasible choices subject to the budget constraints (2), then they are certainly utility-maximizing choices. That is, (a) implies (b).

3 for excess demand

8 for location date market clearing

16 for location dates not visited = 4 x 4

Two people in same island

To motivate how we produce debt quantities, recall that a CME consists of arbitrary s_{it}'s and e_{it}^{g}'s that satisfy (3), (4), and zero restrictions for those e_{it}^{g}'s that correspond to (i,t)'s that g does not visit. For the table V-1, T = 4 economy, there are 3 + 8 + 16 *independent* constraints on the 32 e_{it}^{g}'s. This leaves us free to choose 5 e_{it}^{g}'s arbitrarily, but not any 5. For example, e_{11}^{1} and e_{11}^{2} cannot both be chosen arbitrarily because (4) and the zero restrictions imply that these sum to zero. Similarly, e_{11}^{1}, e_{12}^{1}, e_{13}^{1}, e_{14}^{1} cannot each be chosen arbitrarily since (3) must be satisfied. We arrive at candidates for equilibrium debt quantities by finding some that satisfy constraints (1) and (2) and the relevant debt market clearing conditions for a set of e_{it}^{g}'s that can be chosen arbitrarily.

4 people
x 4 dates
x 2 locations

Budget for
person 1

For the table V-1, $T = 4$ economy, the following equations are the debt budget constraints, at prices satisfying (5) and (6), for five e_{it}^{g}'s that *can* be chosen arbitrarily:

$$(e_{21}^{4}, e_{22}^{4}, e_{11}^{1}, e_{12}^{1}, e_{13}^{1})' = Ad \tag{7}$$

$$A = \begin{bmatrix} s_{23}/s_{21} & 0 & 0 & 0 & 0 & 0 & -s_{14}/s_{21} & s_{24}/s_{21} \\ 0 & s_{24}/s_{22} & 0 & 0 & -s_{14}/s_{22} & 0 & s_{14}/s_{22} & 0 \\ 0 & 0 & s_{13}/s_{11} & 0 & s_{14}/s_{11} & -s_{24}/s_{11} & 0 & 0 \\ 0 & 0 & 0 & s_{14}/s_{12} & 0 & s_{24}/s_{12} & 0 & -s_{24}/s_{12} \\ 0 & 0 & -1 & 0 & 0 & 0 & -s_{14}/s_{13} & s_{24}/s_{13} \end{bmatrix}$$

and $d = (d_{13}^{4} - d_{13}^{3}, d_{24}^{4} - d_{24}^{2}, d_{13}^{1} - d_{13}^{2}, d_{24}^{1} - d_{24}^{3}, d_{14}^{1}, d_{14}^{2}, d_{14}^{3}, d_{14}^{4})'$. Note that zeros in the A matrix do not denote zero debt prices, but rather that the particular debt cannot be traded at the relevant location-date combination.

To see that there are *nonnegative* debt quantities that satisfy (7) for arbitrary s_{it}'s and an arbitrary left-hand side (LHS) of (7), consider an equivalent set of equations obtained by replacing the last equation of (7) by itself plus a multiple (s_{11}/s_{13}) of the third equation, namely

$$[e^4_{21}, e^4_{22}, e^1_{11}, e^1_{12}, e^1_{13} + (s_{11}/s_{13})e^1_{11}]' = [A_1, A_2, A_3, A_4, A_5 + (s_{11}/s_{13})A_3]'d, \quad (8)$$

where A_i denotes the ith row of the matrix A. Note that in each of the first four equations of (8), there appears (with a nonzero coefficient) a difference between noncirculating debts that do not appear in any other equation. Thus, for any quantities of the other debts, each of the first four equations can be satisfied by choosing *nonnegative* quantities of the noncirculating debts that appear in that equation only. This allows us to choose nonnegative quantities of the circulating debts in any way that satisfies the last equation of (8), namely

$$e^1_{13} + (s_{11}/s_{13})e^1_{11} = (s_{14}/s_{13})(d^1_{14} - d^3_{14}) - (s_{24}/s_{13})(d^2_{14} - d^4_{14}). \quad (9)$$

Equation (9) is easily satisfied; if the LHS is positive (negative), it can be satisfied by setting at zero all but $d^1_{14}(d^3_{14})$.

By a payments matrix we mean an N by N matrix, where N is the number of objects observed in a debt equilibrium, in which the (i,j)-th element is one if object i is observed to trade for object j and is zero otherwise. Thus, for a table V-1, $T = 4$ economy, N equals the number of distinct consumption goods—eight—plus the number of distinct private securities issued in an equilibrium. And, if the transaction pattern is such that each consumption good gets traded for one circulating security and one noncirculating security, then there are two nonzero elements in each row corresponding to a consumption good or to a noncirculating debt, and there are four in each row corresponding to a circulating debt. Note, by the way, that nontrivial spatial setups seem not to produce equilibria in which *one* object trades for *every* other object.

By the transaction velocity of an object, we mean the ratio of the average amount traded per date to the average stock, a pure number per unit time. For example, for a table V-1, $T = 4$ economy, the following transaction velocity pattern among objects shows up in a debt equilibrium. For a consumption good at location i, date t, the *average* stock outstanding may be taken to be the total endowment divided by 4 (at dates other than t, the stock of this good is zero), whereas the average amount traded per date is the amount traded at t divided by 4. Thus, the transaction velocity is in the interval (0,1). Computed in a similar way, the transaction velocity of noncirculating debt in such an economy is 2/3 (such debt is outstanding for three dates and the entire stock is traded at two of those dates), whereas that of circulating debt is unity (the maximum possible velocity given our choice of time unit).

Our coordination problem bears some resemblance to a result obtained by Ostroy (1973) and Ostroy-Starr (1974) in their study of the decentralization of exchange. In their model, knowledge of equilibrium prices of commodities is not enough to guide people to the trades that produce the equilibrium allocation in one round of bilateral trading if the trading rules are informationally decentralized. In our model, knowledge of current and future equilibrium prices of securities is not enough to guide people to the quantities of securities required to support an equilibrium if security transactions in other markets are not observed. Of course, both private debt in our model and money in their model alleviate a quid pro quo requirement and facilitate the attainment of equilibrium. There is a sense, though, in which the monetary exchange process is informationally centralized in their model. It requires that budget balance information be transmitted to a monetary authority or requires that there be implicit agreement about which commodity is to be used to cover budget deficits and surpluses. One interpretation of our coordination problem is that a debt equilibrium also requires centralization.

Spector and Townsend (2019) "Notes on Townsend-Wallace"

- **Preferences:** $U^i = \sum_{t=1}^{4} \ln\left(c_t^i\right)$.
- **Endowments:** aggregate endowment is constant at $\Omega = \bar{\omega} + \underline{\omega}$, and agents alternate between the high and low states (with agents 1 and 4 starting with $\bar{\omega}$ and agents 2 and 3 starting with $\underline{\omega}$ in $t = 1$).
- **CME**: everyone consumes $\frac{\bar{\omega}+\underline{\omega}}{2}$ in every period, and all prices are 1.
- Suppose debts issued at $t = 1$ do not support the CME, and we want to find the equilibrium for $t = 2$ onwards (assuming everyone knows the debts that were issued at $t = 1$). The equilibrium is given by 10 prices and 2 quantities: the prices of the four circulating debts at $t = 2$ and $t = 3$, and the prices and quantities of the two the non-circulating debts issued at $t = 2$.
- We can reduce the dimensionality by noting that the price of the non-circulating debts is pinned-down by the aggregate endowment (by homotheticity), and there are arbitrage conditions between prices (e.g., agent 1 can save/borrow between periods $t = 2$ and $t = 4$ either by buying non-circulating debt or by buying and reselling circulating debts).

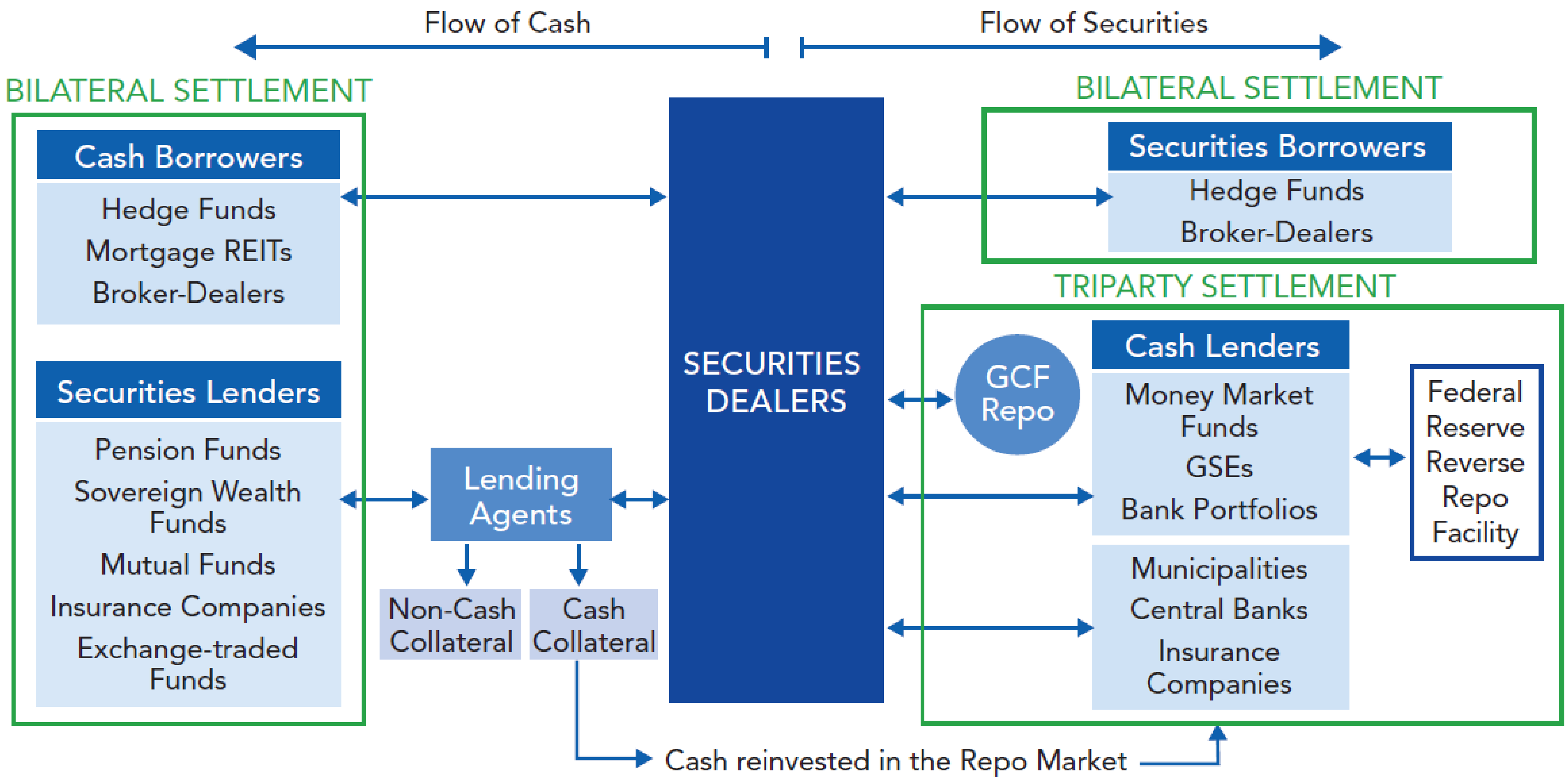
- Suppose $\bar{\omega} = 1$, $\underline{\omega} = 0.5$, and consider two scenarios:
  - Scenario 1: CME with all circulating debt issued in location 1 (and non-circulating debts are calculated to support the CME).
  - Scenario 2: location 1 issues the same debts as in Scenario 1, but now location 2 also issues the same debts (i.e., everyone thinks, mistakenly, that all long-term debt is being issued in their location).

		Location 1 Sc.1	Location 1 Sc.2		Location 2 Sc.1	Location 2 Sc.2
$t=1$	d_{14}^1	0.50	0.50	d_{14}^3	0.00	0.00
	d_{14}^2	0.00	0.00	d_{14}^4	0.00	0.50
	d_{13}^1	-0.25	-0.25	d_{13}^4	0.25	-0.25
$t=2$	c_1	0.75	0.66	c_2	0.75	0.84
	c_3	0.75	0.84	c_4	0.75	0.66
	p_{14}^4	–	0.36	p_{14}^1	1.00	0.36
	p_{24}^1	1.00	1.00	p_{24}^4	1.00	1.00
	d_{24}^1	-0.25	-0.16	d_{24}^4	0.25	-0.16
$t=3$	c_1	0.75	1.03	c_3	0.75	0.47
	c_2	0.75	0.47	c_4	0.75	1.03
	p_{14}^4	–	0.56	p_{14}^1	1.00	0.56
$t=4$	c_1	0.75	0.66	c_2	0.75	0.84
	c_3	0.75	0.84	c_4	0.75	0.66

	Sc.1	Sc.2
U_1	-0.86	-0.80
U_2	-0.86	-1.10
U_3	-0.86	-1.10
U_4	-0.86	-0.80

Repo Market Structure

Figure 1: Key Secured Financing Market Participants



Notes: REITs = real estate investment trusts. GCF = general collateral financing trades. GSEs = government-sponsored enterprises.
Source: OFR analysis

Baklanova, Caglio, Cipriani and Copeland (2016) "The U.S. Bilateral Repo Market: Lessons from a New Survey"

Rehypothecation

FEDS Notes

December 21, 2018

The Ins and Outs of Collateral Re-use

Sebastian Infante, Charles Press, and Jacob Strauss (University of Minnesota)[1]

Executive Summary

In this article, we empirically document how primary dealers use and re-use collateral in the United States. Using confidential supervisory data, we precisely map the flow of collateral to and from individual dealers and identify whether the collateral used in those transactions is encumbered or rehypothecated. From these data, we can characterize how different cash and secured financing transactions affect dealers' balance sheets and present some stylized facts of their operations. We present three measures of collateral use and re-use at the dealer level to proxy for the amount of collateral circulation in the U.S. financial system. We find evidence from one of our measures that the degree of collateral circulation is significantly higher for U.S. Treasury securities and highlight the special role repurchase agreements (repos) play in their intermediation. Characterizing dealers' use and re-use of collateral contributes to ongoing research aimed at understanding how collateral circulation improves market functioning by increasing the availability of collateral but may also lead to financial fragility by increasing the amount of interconnectedness in the financial system.

Repo market coordination: the case for a smart contract

Daniel Aronoff and Robert Townsend

MIT

February 28, 2020

- Repo's are short term collateralized loans with a peculiar legal structure.
- The repo market (aka the money market) has greater than $1.5 trillion daily turnover. It is the largest volume market in the world by orders of magnitude. The US treasury collateral market is the largest segment.
- The repo market is decentralized and connected. Broker-dealers make repo trades in isolated markets, then enter into repo trades with each other. This creates a possibility of multiple equilibria in trading volumes and mis-coordination.
- Treasuries reside on a ledger at the US Treasury. Repo contracts are standardized.

Research questions

✓Can we solve coordination problems by placing the treasuries on a DL to which we append repo contracts?

✓Can we solve coordination problems if, in addition, we append a smart contract onto the DL?

✓Is there a feasible implementation to do this?

A repo trade has 2 legs. Let's consider a repo trade between a Money Market Fund "*MM*" and a Broker-Dealer "*BD*" say, Goldman Sachs.

First leg: *MM* purchases treasuries "*T*" from *BD* for a unit price of p_1. *T* moves from *BD* to *MM* and a commercial bank deposit "*R*" moves from *MM* to *BD*.
Second leg: *BD* purchases *T* from *MM* for a unit price of p_2.The objects move back to their initial positions.

The "repo rate" for this trade is $r_{MM} = \frac{p_2 - p_1}{p_1}$
It is the implicit interest rate paid by *BD* for borrowing *R* from *MM*.

A Repo Contract Between a BD and an MM

		BD	MM
FIRST LEG	R	$\leftarrow$	P_1T
	T	T $\rightarrow$	
SECOND LEG	R	P_2T $\rightarrow$	
	T	$\leftarrow$	T

Some crucial repo facts:

- The repo contract, entered into on the first leg, includes the (re)purchase obligation for the second leg.
- The repo contracts are standardized.
- When collateral T is transferred, so is ownership of T.
- The holder of T collateral has the legal right to sell it or use it as collateral in another transaction - *Rehypothecation*.
- In the event the borrower on the first leg fails to repurchase on the second leg, the lender's right to retain or sell T cannot be 'stayed' by a bankruptcy court. It is an absolute right.

- **MM** manages R on behalf of investors. Dual mandate: Safety ("don't lose") and liquidity (access to cash on short notice). Solution: exchange R for T with an agreement to return the objects after a short duration (day/week/month). *This is the essence of a repo contract*.

- A risk manager "**RM**" owns T and desires to borrow R in order to boost yield. e.g. pension fund required to hold portfolio of T and under pressure to earn higher return to meet payout obligations.

- **BD** intermediates trade in R and T between its clients, *MM* and *RM*.

Key takeaway

✓ *MM* and *RM* have gains from trading objects R and T, but they need to go through BD to consummate trades (an empirical fact).

Example: A Matched-Book repo trade

MM initially holds *R*. It requires the ability to quickly transfer bank deposits, yet wants to hold *T* to avoid the unsecured bank deposit risk.

At the same time, *RM* desires to borrow *R* and is willing to offer *T* as collateral.

Solution
BD purchases *T* from *RM* with a promise to re-purchase at a later date (maybe the next day). This provides *RM* with the *R* it desires and it provides *BD* with *T* it needs to convey to *MM*.

Simultaneously, *MM* purchases *T* from *BD* with a promise to re-purchase at a later date. This provides *MM* with the *T* it desires, along with access to a deposit on the repurchase date, and it provides *BD* with the *R* it needs to convey to *RM*.

Note that *BD* merely moves objects between its clients; it does not retain any objects. This is what is meant by *Matched Book*.

Example: A Matched-Book repo trade

The movement of objects in the matched-book repo trade is depicted below.

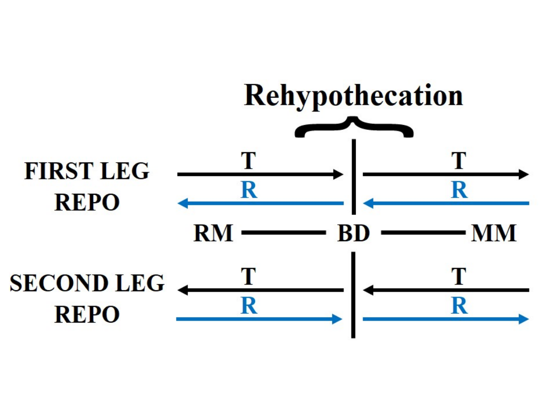


Figure 2

The inter-dealer repo market

In fact, there are multiple BD's, each serving a non-identical client base. Suppose (i) RM_4 desires more R than the MM clients of BD_2 are willing to provide and (ii) MM_1 desires more T than the RM clients of BD_1 are willing to provide.

Solution The BD's can satisfy their client excess demands by transacting an *inter-dealer* repo, conveying the objects between RM_4 andMM_1.

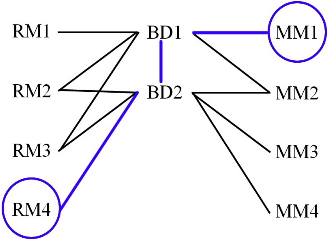


Figure 3

We model the repo market as a repeating sequence of the following timing game:

t_1: First leg of client repos BD's intermediate repo trades with their clients. Matched-book at the max volume. BD chooses how much excess demand to absorb and carry into inter-dealer market.

t_2: First leg of inter-dealer repos BD's enter into repo trades on matching excess inventory. e.g. if BD_1 has R inventory and BFD_2 has T inventory, they will make repo trade. If inventory values don't match up, the BD with the most inventory will hold inventories it cannot trade.

t_3,t_4:Second leg of client and inter-dealer repos.

Key takeaway

✓In t_1 BD's make strategic choice of the amount of client excess demand to accommodate. This can lead to (i) multiple equilibria and (ii) mis-coordination in t_2.

Repo market timing

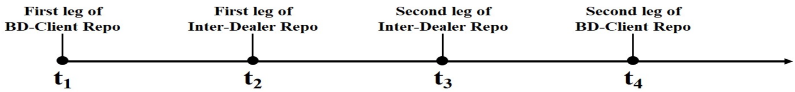


Figure 5

Participation in the repo market induces an ordering of repo rates.

- **MM**To induce BD to intermediate matched book trades between RM and MM at t_1, the BD must earn a positive spread. Thus $r_{RM} > r_{MM}$.
- **RM** To induce RM to borrow in the repo market, the repo rate r_{RM} must be lower than the interest rate charged by a bank, r_B. Thus, $r_B > r_{RM}$.
- **BD** To induce the BD's to carry inventories into t_2 and trade with each other, the following must obtain.
  ▷ For BD_1 to accommodate an excess demand for T from its MM clients by borrowing R, it must anticipate that it can lend the R to BD_2 for a profit. Thus,$r_{BD} > r_{MM}$.
  ▷ For BD_2 to accommodate an excess demand for R from its RM clients by borrowing R from a bank at rate r_B (which, by the above exceeds the r_{RM} it earns from its RM client), it must anticipate that it can borrow R from BD_1 at a lower rate and pay off its bank loan while earning a positive net interest margin. Thus $r_{RM} > r_{BD}$.

Here is a snapshot of what is going on.

t_1 (Before inter-dealer repos) BD_1 accommodates an excess demand for T by borrowing at interest rate r_{MM}, which is a loss to BD_1.
BD_2 accommodates an excess demand for R by its clients by borrowing from a bank at interest rate r_B. BD_2 is losing money, since $r_B > r_{BD}$.

t_2 (After inter-dealer repos) The brokers enter into a repo trade at interest rate r_{BD}. This enables BD_1 to turn a profit, since $r_{BD} > r_{MM}$. This enables BD_2 to turn a profit, since $r_B > r_{RM} > r_{BD}$.

	Inventory Cost Before Inter-Dealer Repo		Inventory Cost After Inter-Dealer Repo	
	Interest Rate In	Interest Rate Out	Interest Rate In	Interest Rate Out
$\mathbf{BD_1}$	N/A	$\mathbf{r_{MM}}$	$\mathbf{r_{BD}}$	$\mathbf{r_{MM}}$
$\mathbf{BD_2}$	$\mathbf{r_{RM}}$	$\mathbf{r_B}$	$\mathbf{r_{RM}}$	$\mathbf{r_{BD}}$

Figure 6

The ordering of repo interest rates

Pulling together the ordered pairs yields

$$r_B > r_{RM} > r_{BD} > r_{MM}$$

The data corroborates the predictions of our model.

	OFR Bilateral repo pilot study	FRBNY centrally cleared repo series		
	r_{RM}	r_{MM}	$r_{MM} + r_{BD}$	
Date	Bilateral repo Mean 'securites-in'**- Median	Tri-party GC-O/N	Tri-party + GCF-O/N^VV	Tri-Party + GFC + FICC bilateral^VVV
1.12.2015		0.06	0.06	0.08
2.10.2015		0.07	0.07	0.09
3.10.2015	0.11	0.06	0.07	0.08
Ave from 11.20.2014 - 3.29.2018---		43	43.2	46.9

Model Elements

Agents Two broker-dealers, BD_1 and BD_2 and RM's and MM's.
Price formation Repo rates are exogenous. They comply with the ordering derived from the model. r_{BD} is the Nash Bargaining Solution (see below).
Timing Repeating sequence as per above.
Preferences Client demands R or T. BD's gain utility from repo spreads. Matched-book intermediation between RM and MM is risk-free. Absorbing excess client demand has risk that other BD might not have matching inventory to trade.
Strategic interaction Clients are non-strategic. BD's choose the amount of excess demand to accommodate. Assume BD_1 faces excess demand for T; BD_2 faces excess demand for R.

Key takeaway
✓A BD must forecast inventories its counterpart will bring to the inter-dealer market. BD's are subject to two competing objectives, (i) avoid loss from holding inventories in excess of inter-dealer demand and (ii) make profit from volume of inter-dealer inventory repos.

Model Elements

Client excess demand

Client aggregate excess demand (in terms of R) is drawn from a probability distribution. The joint distribution is known to both BD's.

$x_1^T, x_2^R \sim f(x_1^T, x_2^R)$
where x_1^T is excess demand for T faced by BD_1 and x_2^R is excess demand for R faced by BD_2.

$\hat{x}_i$ is the amount of client excess demand BD_i absorbs and carries into the inter-dealer market as inventory $i \in \{1, 2\}$.

Model Elements

BD utility functions

BD_i's utility from accommodating client excess demand, $U(\pi_i)$, is concave and differentiable, where

$$\pi_i = \frac{1}{2}(r_{RM} - r_{MM})) \times Q - (\hat{x}_i - Q)L_i, \ Q = \min\{\hat{x}_1, \hat{x}_2\}.$$

The terms of π_i have the following interpretation.

$\frac{1}{2}(r_{RM} - r_{MM}) = r_{BD}$ is the Nash Bargaining solution to the inter-dealer rate,

The first term, $\frac{1}{2}(r_{RM} - r_{MM}) \times Q$, is the profit BD_i makes by entering into a repo trade in the inter-dealer market at t_2.

The second term, $(\hat{x}_i - Q)L_i$, is the loss BD_i suffers by carrying inventories in excess of its counterpart into the inter-dealer market, where $L_1 = r_{MM}$, the rate BD_1 pays to satisfy the excess demand of MM for T and $L_2 = r_B - r_{RM}$, the net rate BD_2 pays to satisfy the excess demand of RM for R .

Model Elements

Welfare

We characterize welfare associated with excess client demands, broker inventories and repo trading in the inter-dealer market.

BD's face the risk of holding excess inventories. $U(\pi_i)$ is maximized when the following 2 conditions obtain: (i) $Q = \min\{x_1^T, x_2^R\}$ and (ii) $\hat{x}_i = Q, i \in \{1, 2\}$. This is the Pareto Optimum (PO) associated with the highest utility levels for the *BD*'s, since it is the maximum amount of inter-dealer repo trading consistent with no excess inventories.[1]

Key takeaway

✓We want to explore whether a public repo DL will push outcomes in the direction of the maximal PO.

[1] The computation of welfare is more complicated when clients are included. There may be a social optimum which involves a *BD* absorbing excess demand (accumulating inventories) it cannot trade in the inter-dealer market.

Mis-coordination and risk aversion

BD's are risk averse; they prefer certainty over a risky bet.

Conjecture

BD_1 chooses inventory $\hat{x}_1$, after realization of its client excess demand, x_1^T. It knows the joint distribution f from which BD_2's client inventory is drawn, but it does not observe the draw. BD_1 also knows that BD_2 faces an identical decision problem in choosing its inventory $\hat{x}_2$.

- Each *BD* might choose a low volume of inventories to limit potential loss from holding excess inventories, since uncertainty over counterpart inventories will likely cause $\hat{x}_1 \neq \hat{x}_2$, which imposes an inventory carry cost on the *BD* with the most inventory.

- The fact that each *BD* knows the other *BD* faces the same decision problem, exerts additional incentive to lower its inventory volume.

✓A DL would enable each *BD* to see the other *BD*'s inventory at t_1. Will the DL drive inventory volume to the max PO?

Multiple Equilibrium

Consider what happens under **perfect foresight** as when the repo trades are on a public DL.

- Each *BD* knows ex ante the inventories its counterpart will carry into the inter-dealer market.
- In t_1, BD_1 faces excess demand of x_1^T = - $p_1 T^{'}$. and BD_2 faces excess demand of x_2^R $= R'$ and

It would be PO for each *BD* to carry $Q = \min(R^{'}, p_1 T^{'})$ of inventories. It is also an equilibrium. Any deviation will reduce the profit of the deviating *BD*. She will incur a loss from any addition to her inventory and a decline in profit on any reduction in inventory.

But...the same reasoning applies to *any* deviation from *any* level of matching inventories below Q. Thus, any level of matching inventories below Q is an equilibrium when there is perfect foresight!

✓It doesn't look like the DL helps...but there is something else that can be done with a DL.

Smart Contracts

Incentive Compatibility RM and MM are not strategic; they announce their true demand for objects. Under our assumption that price is exogenously determined, the clients have an incentive to tell the truth. They gain nothing, and risk loss, by announcing falsely.

Aggregation of demand If the clients posted their demand for repo trades to the DL, the excess demand in each client market could be calculated and thereby the max PO level of inventories for each BD could be determined.

Smart contract A smart contract appended to a public repo DL could achieve the max PO under the following rule:

"Each BD shall fulfill repo transactions equal to $Q = \min\{x_1^T, x_2^R\}$ above the matched-book volume."[2]

[2] Also need to create allocation rule for oversubscribed demand.