First price auctions with general information structures: Implications for bidding and revenue

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2. Assumptions about information are hard to test
3. Equilibrium behavior can depend a lot on how we specify information

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  - A tight lower bound on revenue
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- Other results on max revenue, min bidder surplus, min efficiency

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- What predictions can we make about U and R in equilibrium?

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  - Bidders observe everything;
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- True information structure is likely somewhere in between:
  - Bidders have some information about v, but not perfect
  - But exactly how much information do they have?

Lower revenue?

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- $U_1 = \int_{v=0}^{1} v(v - v/2)dv = 1/6$, $U_2 = 0$, $R = 1/3$

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- Problem: hard to say which specification is "correct"
- What welfare predictions do not depend on how we model information?

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- At min R, winning bids have been pushed down "as far as they can go"
- Force pushing back must be incentive to deviate to higher bids
- In EMW, informed bidder strictly prefers equilibrium bid

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$$\begin{aligned} \underline{\beta}(v) &= \frac{1}{\sqrt{v}} \int_{x=0}^{v} x \frac{1}{2\sqrt{x}} dx \\ &= \frac{v}{3} \end{aligned}$$

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- In fact, symmetry/deterministic winning bid are not needed
- Distribution of winning bid has to FOSD $U[0, 1/3]$ in *all* equilibria under *any* information
- $1/6$ is a *global* lower bound on equilibrium revenue

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- Defer proof until general results

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- Bound is characterized by binding *uniform upward incentive constraints*

The plan

- Detailed exposition of minimum bidding
- Maximum revenue/minimum bidder surplus
- Restrictions on information
- Other directions in welfare space (e.g., efficiency)

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- Support of marginals $V = [\underline{v}, \overline{v}] \subseteq \mathbb{R}_+$
- An *information structure* $\mathcal{S}$ consists of
  - A measurable space S_i of signals for each player i, $S = \times_{i=1}^{N} S_i$
  - A conditional probability measure

$$\pi : V^N \to \Delta(S)$$

Equilibrium

- Bidders' strategies map signals to distributions over bids in $[0, \overline{v}]$

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- Bidder i's payoff given strategy profile $\sigma = (\sigma_1, \ldots, \sigma_N)$:

$$U_i(\sigma, \mathcal{S}) = \int_{v \in V} \int_{s \in S} \int_{b \in B^N} (v_i - b_i) \frac{\mathbb{I}_{b_i \geq b_j \ \forall j}}{|\arg\max_j b_j|} \sigma(db|s) \pi(ds|v) P(dv)$$

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- σ is a Bayes Nash *equilibrium* if

$$U_i(\sigma, \mathcal{S}) \geq U_i(\sigma_i', \sigma_{-i}, \mathcal{S}) \ \forall i, \sigma_i'$$

Other welfare outcomes

$$\text{Bidder surplus: } U(\sigma, \mathcal{S}) = \sum_{i=1}^{N} U_i(\sigma, \mathcal{S})$$

$$\text{Revenue: } R(\sigma, \mathcal{S}) = \int_{v \in V^N} \int_{s \in S} \int_{b \in B^N} \max_i b_i \sigma(b|s)\pi(ds|v)P(dv$$

$$\text{Total surplus: } T(\sigma, \mathcal{S}) = R(\sigma, \mathcal{S}) + U(\sigma, \mathcal{S})$$

$$\text{Efficient surplus: } \overline{T} = \int_{v \in V} \max_i v_i P(dv)$$

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- Suggests minimizing equilibrium is efficient, winning bid is constrainted by *loser's (i.e., lowest) value*

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- Similar $\underline{\beta}$, but now depends on *lowest* value
- $Q(dm)$ is distribution of $m = \min\{v_1, v_2\}$ (assume non-atomic)

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- Intuition: coarse information about losers' values lowers revenue
- Consider complete information, all values are common knowledge
- High value bidder wins and pays second highest value

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- If prior is symmetric, believe they are equally likely to be at any point in the distribution *except* the highest
- In equilibrium, winner pays *average of $N-1$ lowest values*:

$$\mu(v_1, \ldots, v_N) = \frac{1}{N-1}\left(\sum_{i=1}^{N} v_i - \max_i v_i\right)$$

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- Let $\underline{H}(b) = Q(\underline{\beta}^{-1}(b))$

Main result

Theorem (Minimum winning bids)

1. *In any equilibrium under any information structure in which the marginal distribution of values is* P*, the distribution of winning bids must first-order stochastically dominate* $\underline{H}$*.*

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1. *In any equilibrium under any information structure in which the marginal distribution of values is* P*, the distribution of winning bids must first-order stochastically dominate* $\underline{H}$*.*
2. *Moreover, there exists an information structure and an efficient equilibrium in which the distribution of winning bids is exactly* $\underline{H}$*.*

Implications

Corollary (Minimum revenue)

Minimum revenue over all information structures and equilibria is $\underline{R}$.

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Minimum revenue over all information structures and equilibria is $\underline{R}$.

Corollary (Maximum bidder surplus)

Maximum total bidder surplus over all information structures and equilibria is $\overline{T} - \underline{R}$.

Proof methodology

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(start with #2)

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  $\implies$ distribution of highest signal is $Q(s)$
- Signals are correlated with values s.t.
  - Highest signal is true average lowest value, i.e.,

$$\mu(v_1, \ldots, v_n) = \max\{s_1, \ldots, s_n\}$$

Minimizing equilibrium and information

- Bidders receive independent signals $s_i \sim Q^{1/N}(s_i)$ $\implies$ distribution of highest signal is $Q(s)$
- Signals are correlated with values s.t.
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- All bidders use the monotonic pure-strategy $\underline{\beta}(s_i)$

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$$\sigma(s_i) = \frac{1}{Q^{\frac{N-1}{N}}(s_i)} \int_{x=\underline{v}}^{s_i} x\, Q^{\frac{N-1}{N}}(dx) = \underline{\beta}(s_i)$$

Downward deviations

- Expectation of the bidder with the highest signal is $\tilde{v}(s_i) \geq s_i$
- Downward deviator obtains surplus

$$(\tilde{v}(s_i) - \underline{\beta}(m))Q^{\frac{N-1}{N}}(m)$$

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$$\begin{aligned}
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- Well-known that IPV surplus is single peaked: if $m < s_i$,

$$\Longrightarrow (s_i - \underline{\beta}(m))Q^{\frac{N-1}{N}}(dm) - \underline{\beta}'(m)Q^{\frac{N-1}{N}}(dm) \geq 0$$

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- By deviating up to win on this event, gain m in surplus

Upward deviations

- Upward deviator's surplus

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- In effect, correlation between others bids' and losing values induces adverse selection s.t. losing bidders are indifferent to deviating up

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- Claim is that construction attains a lower bound
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- Note

$$H(b) = \int_{v \in V^N} \sum_{i=1}^{N} H_i(b|v) P(dv)$$

Relaxed program

- Also impose *uniform upward incentive constraints* (IC):

$$\underbrace{\int_{v \in V^N} \int_{x=\underline{v}}^{b} (b-x) H_i(dx|v) P(dv)}_{\text{loss when would have won}}$$

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over $\{H_i(b|v)\}$ subject to (Feas) and (IC)

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- Note: Objective and constraints are *linear* in H_i

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- For example, with $N = 2$, can create symmetric solution:

$$\widetilde{H}_1(b|v_1, v_2) = \frac{1}{2}\left(H_1(b|v_1, v_2) + H_2(b|v_2, v_1)\right)$$
$$\widetilde{H}_2(b|v_1, v_2) = \frac{1}{2}\left(H_2(b|v_1, v_2) + H_1(b|v_2, v_1)\right)$$

Average losing values III

- Consider a bidder who uniformly deviates up, so they *always* win when the equilibrium winning bid is b
- Say there is a value profile v at which b is sometimes the winning bid
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- But someone has to win in equilibrium...
- Incremental gain from winning when you would lose in equilibrium is the *average losing value* given $[v]$:

$$\mu(v) = \frac{1}{N-1}\left(\sum_{i=1}^{N} v_i - \text{expected winner's value}\right)$$

Efficiency

- Can rewrite gain from upward deviating as

$$\int_{v \in V^N} \int_{x=\underline{v}} (\mu(v) - b) \frac{N-1}{N} \sum_i H_i(dx|v) P(dv)$$

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- Can always induce efficient allocation without changing $H(b)$: If $v_i = \max v$, set

$$\widetilde{H}_i(b|v) = \frac{1}{|\arg\max v|} \sum_{j=1}^{N} H_j(b|v)$$

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subject to

$$0 \leq H(b|m) \leq 1 \tag{Feas}$$

and

$$\begin{aligned} \frac{1}{N} \int_{m=\underline{v}}^{\overline{v}} \int_{x=\underline{v}}^{b} (b-x) H(dx|m) Q(dm) \\ \geq \frac{N-1}{N} \int_{m=\underline{v}}^{\overline{v}} (m-b) H(b|m) Q(dm) \end{aligned} \tag{IC}$$

Monotonicity

- Only part of (IC) that depends on correlation between b and m is

$$\hat{m}(b) = \int_{m=\underline{v}}^{\overline{v}} m\, H(b|m) Q(dm),$$

i.e., average losing value when winning bid is less than b

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  i.e., the α lowest m are associated with the α lowest b
- Implies a *deterministic winning bid* $\beta(m)$ s.t. for all m,

$$H(\beta(m)) = Q(m)$$

Relaxed program III

- Relaxed program is reduced to what we assumed in example:

$$\min \int_{m=\underline{v}}^{\overline{v}} f(\beta(m))Q(dm)$$

subject to $\beta(m) \geq \underline{v}$ and

$$\frac{1}{N}\int_{x=\underline{v}}^{m}(\beta(m)-\beta(x))Q(dx) \geq \frac{N-1}{N}\int_{x=\underline{v}}^{\overline{v}}(x-\beta(m))Q(dx) \tag{IC}$$

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- Minimize $\beta(m)$ pointwise by $\beta(\underline{v}) = \underline{v}$ and (IC) binding everywhere
- Solution is precisely $\underline{\beta}$!

Wrapping up

- $\underline{H}$ solves the relaxed program for an arbitrary $f(\max b)$
- Must therefore be FOSD by any equilibrium $H(b)$
- Construction attains $\underline{H}$, so proof of theorem is complete

Maximum revenue

- With pure common value, no-information and complete information induce full surplus extraction
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Theorem (Maximum revenue and minimum bidder surplus)

For every $\epsilon > 0$, there exists an information structure and equilibrium such that revenue is at least $\overline{T} - \epsilon$ and bidder surplus is at most ϵ.

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- We refer to above model as *unknown values*: bidder need not know anything about value
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- Weak dominance: players do not bid more than own value

A lower bound bidder surplus

- If bid b, always win when others' values are less than b
- Lower bound on bidder surplus $\underline{U}_i(v_i)$ from best responding to "worst case" in which others bid their values:

$$\underline{U}_i(v_i) = \max_b \left\{ (v_i - b) \int_{\{v_{-i} | \max_{j \neq i} v_j \leq b\}} P(dv_{-i}|v_i) \right\}$$

- Integrate over values to obtain an ex-ante bound $\underline{U}_i$

Maximum revenue/minimum bidder surplus

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Proof sketch

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- Uses ideas from "The Limits of Price Discrimination", BBM 2015

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- With unknown values, likelihood may depend on b and distribution of others' values
- Example: higher winning bids occur when values are higher on average
- If equilibrium is efficient and v_i is low, I am unlikely to win in equilibrium at high bids
- Increase in probability of winning from upward deviation varies with v_i

Binary known values

- Case we can solve completely: $v_i \in \{v_L, v_H\}$
- Setting first considered by Maskin and Riley (1985)
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- General known values minimum revenue is an open question

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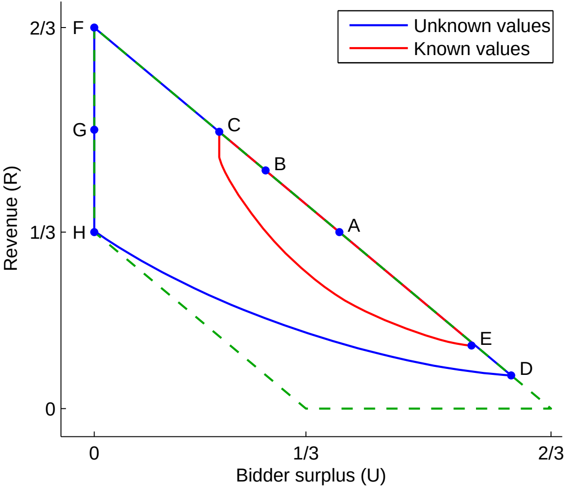
- Solved numerically for two bidder i.i.d. $U[0, 1]$ model

Welfare set



- Note: Lower bound on efficiency

What can we do with this?

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  - Impact of reserve prices/entry fees
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- Context:
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Thank you!