Continuous Consumption Rules with Non-Exponential Discounting

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Abstract

In a three period cake eating problem with non-exponential discounting, there is a continuous equilibrium if the utility function satisfies convex absolute risk tolerance and the discount weights are "hyperbolic". We discuss if and when this sufficient condition can be adapted to cope with alternative discount weights, positive interest rates, uncertain income and longer horizons.

1. Introduction

A decision maker will consume in T periods (where T may equal ∞). Her utility in period t from consumption stream

$$\mathbf{x} = (x_1, x_2, ...) \in \mathbb{R}_+^T$$

is given by

$$U_t(\mathbf{x}) = \sum_{\tau=t}^{T} \delta_{\tau-t} u(x_\tau)$$

where $\boldsymbol{\delta} = (\delta_0, ..., \delta_{T-1}) \in (0,1]^T$ satisfies

$$\delta_0 = 1 \text{ and } \delta_{t+1} < \delta_t$$

for all t and the within period utility function $u(\cdot)$ is "well-behaved," with

$$\begin{aligned} u'(\cdot) &> 0 \\ u''(\cdot) &< 0 \\ \text{and } u'(x) &\to \infty \text{ as } x \to 0. \end{aligned}$$

Consider a simple *cake-eating problem*, where the decision maker is endowed with a certain amount of consumption and decide how much to consume in each period as a function of current wealth. Thus a *strategy* for the period t decision maker is a consumption function

$$c_t : \mathbb{R}_+ \to \mathbb{R}_+,$$

with

$$0 \leq c_t(w) \leq w$$

for all $w \in \mathbb{R}_+$. Thus if the period t decision maker is born with wealth w, he will consume $c_t(w)$ and save $w - c_t(w)$.

Given the non-exponential weights, decision makers at different dates have different preferences over future consumption streams. Thus this situation is naturally modelled as a game. A natural solution concept is Markov perfect equilibrium, where the decision maker at each date chooses a consumption rule that depends only on her current wealth and maximizes her utility, given the strategies of other players. In what follows, we will refer to "equilibria" as shorthand for such Markov perfect equilibria.

This is the simplest example of a class of dynamic choice problems that were first studied in the economics literature by Strotz (1956). The last decade has seen very considerable progress in the theory, application and empirical testing of such models. This work is reviewed in Harris and Laibson (2001b).

Unfortunately, in the cake eating problem and the dynamic choice problems more generally, the set of such equilibria is badly behaved. There may exist many equilib-

ria[1] and those equilibria may be discontinuous and non-monotonic.[2] In two special cases, we know that the cake eating problem has an equilibrium with continuous and monotonic consumption functions: if there is exponential discounting - i.e., $\delta_t = \delta^t$ for some $\delta \in (0,1)$;[3] and if there is constant relative risk aversion utility function - i.e.,for some $c > 0$, $-\frac{xu''(x)}{u'(x)} = c$ for all x.[4] There has been only limited progress in finding equilibria without pathological behavior away from these two special cases.[5] The possibility of discontinuous consumption is obviously problematic: it is hard to take such equilibria seriously and hard to test them empirically. While there is no evidence that such discontinuities are of practical importance in applications of these models,[6] it would be nice to have some theoretical understanding of when equilibrium consumption is continuous away outside the neighborhood of the two special cases mentioned above.[7] This

[1] Krussell and Smith (2001).

[2] Kohlberg (1976) described an algorithm for finding a utility function that did not give rise to continuous, monotonic consumption rules. O'Donaghue and Rabin (1999) show that the consumption rule is non-monotonic in a discrete choice cake eating problem. Harris and Laibson (2001a) study the non-monotonic and discontinuous consumption rules that arise in a model with hyperbolic discounting, constant relative risk aversion, stochastic income and liquidity constraints. Morris and Postlewaite (1997) give a closed form example with a unique equilibrium that is non-monotonic and discontinuous in a standard cake-eating problem with continuous consumption and no liquidity constraints.

[3] Standard maximization techniques imply the existence of an optimal consumption path, continuous in initial wealth; given the intertemporal consistency of preferences, no future decision maker will have an incentive to deviate from that path.

[4] The optimal consumption rule will be linear in wealth in every period.

[5] Harris and Laibson (2001a) provide an elegant proof of existence and characterization of (continuous and discontinuous) pure strategy equilibria in a general class of intertemporal choice models with "hyperbolic" discounting. They are able to establish the existence of continuous equilibria for discount functions in the neighborhood of exponential discount functions. However, they do not have sufficient conditions for continuous consumption rules away from the neighborhood of the two special cases noted above.

[6] Harris and Laibson (2001b) report simulations for a dynamic choice model with hyperbolic discounting, constant relative risk aversion, liquidity constraints and stochastic income and argue that discontinuities are not important for empirically relevant parameters.

[7] A number of alternative models of control problems in intertemporal choice have been proposed that do not lead to discontinuous equilibrium behavior. Gul and Pesendorfer's (2001) model of temptation and self-control has dynamically consistent decision makers. O'Donaghue and Rabin (1999) focus on "naive" decision makers who do not anticipate that future decision makers will deviate from their plans. Harris and Laibson (2001c) consider a continuous time model where the only inconsistency between decision makers at different dates is an extra utility that a current decision maker places on consumption

note reports sufficient joint conditions on the utility function and discount weights for continuous consumption rules in the three period cake eating problem.

To explain these sufficient conditions, consider the two parameter class of discount weights, with

$$\delta_0 = 1 \text{ and } \delta_t = \alpha\delta^{t-1}.$$

In the case where $0 < \alpha < \delta < 1$, we have the "hyperbolic" discounting of Phelps and Pollak (1968) and Harris and Laibson (2001a). In this case, each decision maker would like the next period decision maker to consume less and save more. At the other extreme, we have the case where $\delta = 0$ and $0 < \alpha \leq 1$. This case was extensively studied in the "intergenerational altruism" literature of the 1970s and 1980s (see, e.g., Kohlberg (1976), Leininger (1986) and Bernheim and Ray (1987)).[8] In this case, each decision maker would like the next period decision maker to consume more and save less. These opposite cases give rise to different sufficient conditions.

In the three period cake eating problem, all remaining wealth will be consumed in the third period. In the second period, there is a standard optimization problem with no dynamic inconsistency. The possibility of discontinuities arises only in the first period. Discontinuities will arise precisely if the first period valuation of wealth entering the second period is not concave. In the intergenerational altruism case, the marginal value of wealth entering the second period is simply the marginal utility of equilibrium consumption times the marginal propensity to consume. If the second period marginal propensity to consume is decreasing in wealth, we are guaranteed a concave first period valuation of second period wealth. Concave absolute risk tolerance is a sufficient condition for decreasing marginal propensity to consume. In the hyperbolic case, the argument is reversed: an increased marginal propensity to consume in the second period is required to ensure that the first period problem has a continuous solution, and convex absolute risk tolerance generates that property. The concavity or convexity of absolute risk tolerance is a fourth derivative property of the utility function.[9]

at the current moment; this model is not the continuous time limit of the discrete time model with more and more opportunities to change the rate of consumption. Each of these three elegant approaches have independent rationales and have the pleasing side-effect that they dispense with the discontinuities.

[8] In this literature, decision makers at different dates are interpreted as different generations and so the assumption that $\delta = 0$ corresponds to the assumption that a decision maker cares about his child's consumption, but not his grandchild's consumption.

[9] See section 11.3.1 of Gollier (2001) for a discussion of the plausibility of convex or concave absolute

The good news, then, is that it is possible to find sufficient conditions for continuous consumption behavior away from the two special cases, and those sufficient conditions depend in an intuitive way on the nature of the dynamic inconsistency. The bad news is that it is hard to extend this result. Alternative strategies for extending the results beyond the three period case are reviewed, but no results are available to date. Allowing a positive interest rate in the three period case *reverses* the results: if the equilibrium consumption is increasing through time then concave absolute risk tolerance gives a decreasing marginal propensity to consume, and thus continuity in the intergenerational altruism case; convex absolute risk tolerance gives an increasing marginal propensity to consume, and thus continuity in the hyperbolic case. Finally, we know from the work of Carroll and Kimball (1996) that adding uncertainty makes the consumption function more convex. Thus unfortunately, in the empirically most relevant case with hyperbolic discounting, increasing expected consumption through time and uncertainty, the uncertainty leads to a decreasing marginal propensity to consume out of wealth, which works against the existence of continuous consumption rules.[10]

2. Three Period Model

Let $\delta_0 = 1$, $\delta_1 = \alpha$ and $\delta_2 = \alpha\delta$, for some α, $\delta \in (0, 1]$.

In period 3, the individual will consume all his wealth. Thus we will have period 3 equilibrium consumption rule

$$c_3(w) = w.$$

In period 2, there will not be a consistency problem. Write $c_2 : \mathbb{R} \to \mathbb{R}$ for the optimal second period consumption rule, i.e.,

$$c_2(w) = \underset{x \in [0,w]}{\arg\max} \; u(x) + \alpha u(w - x).$$

By standard arguments, $c_2(\cdot)$ will be a continuous function with marginal propensity to consume (i.e., first derivative of consumption with respect to wealth) strictly between 0 and 1. It is characterized by the first order condition

$$u'(c_2(w)) = \alpha u'(w - c_2(w)). \tag{2.1}$$

risk tolerance, and the relation to other higher derivative properties of the utility function.

[10] In the simulations of Harris and Laibson (2001b), with hyperbolic discounting, constant relative risk aversion, liquidity constraints and uncertain income, increased uncertainty helps *remove* discontinuities.

We are interested in how the marginal propensity to consume varies with wealth (i.e., the behavior of the second derivative of consumption with respect to wealth). This depends on *fourth* derivative properties of the utility function. An individual's absolute risk tolerance is the inverse of his absolute risk aversion

$$T(x) = -\frac{u'(x)}{u''(x)}.$$

Observe that

$$T'(x) = -1 + \frac{u'''(x)\, u'(x)}{[u''(x)]^2}$$

Linear absolute risk tolerance is equivalent to the assumption that $u(\cdot)$ has hyperbolic absolute risk aversion (HARA). HARA utility functions imply linear consumption functions in cake eating problems.[11] Thus it is the nature of the non-linearity in absolute risk tolerance that determines the shape of the consumption function.

Lemma 1. *If absolute risk tolerance is concave (i.e., $T''(\cdot) \leq 0$), then the marginal propensity to consume is decreasing in wealth ($c_2''(\cdot) \leq 0$); if absolute risk tolerance is convex (i.e., $T''(\cdot) \geq 0$), then the marginal propensity to consume is increasing in wealth ($c_2''(\cdot) \geq 0$).*

Essentially this result is reported as Proposition 58 in Gollier (2001). The following proof adapts arguments in Carroll and Kimball (1996) to show this result.

PROOF. Write $f(\cdot)$ and $\widehat{w}(\cdot)$ for the inverses of $u'(\cdot)$ and $c(\cdot)$, respectively, so that

$$z = u'(f(z)) \text{ and } x = c(\widehat{w}(x)).$$

Let $s(x) \equiv \widehat{w}(x) - x$; thus $s(x)$ is the quantity saved when quantity x is consumed. Since $\widehat{w}(x) = x + s(x)$, $\widehat{w}''(x) = s''(x)$. But $\widehat{w}''(x) \geq (\leq) 0$ if and only if $c''(\widehat{w}(x)) \leq (\geq) 0$. Observe that by the first order condition (2.1),

$$s(x) = f\left(\frac{u'(x)}{\alpha}\right)$$

[11] HARA utility functions include quadratic utility functions, constant absolute risk aversion utility functions and constant relative risk aversion utility functions. However, only the latter case satisfies the Inada condition that is maintained in this note. The Inada condition is important as it removes corner solutions that would otherwise arise.

Thus

$$s'(x) = \frac{1}{\alpha} f'\left(\frac{u'(x)}{\alpha}\right) u''(x)$$

and

$$s''(x) = \frac{1}{\alpha}\left[\frac{1}{\alpha} f''\left(\frac{u'(x)}{\alpha}\right)\left(u''(x)\right)^2 + f'\left(\frac{u'(x)}{\alpha}\right) u'''(x)\right]$$

Now

$$f'(z) = \frac{1}{u''(f(z))}$$

and

$$f''(z) = -\frac{u'''(f(z))}{\left(u''(f(z))\right)^2} f'(z) = -\frac{u'''(f(z))}{\left(u''(f(z))\right)^3}$$

So

$$\begin{aligned} s''(x) &= \frac{1}{\alpha}\left[\frac{1}{\alpha} f''\left(\frac{u'(x)}{\alpha}\right)\left(u''(x)\right)^2 + f'\left(\frac{u'(x)}{\alpha}\right) u'''(x)\right] \\ &= \frac{\left(u''(x)\right)^2}{\alpha u'(x)\, u''(s(x))}\left[\frac{u'(x)\, u'''(x)}{u''(x)} - \frac{u'(s(x))\, u'''(s(x))}{\left(u''(s(x))\right)^2}\right] \\ &= \frac{\left(u''(x)\right)^2}{\alpha u'(x)\, u''(s(x))}\left[T'(x) - T'(s(x))\right]. \end{aligned}$$

If T is convex, then (since $x > s(x)$), the sign of $T'(x) - T'(s(x))$ is positive. The sign of

$$\frac{\left(u''(x)\right)^2}{\alpha u'(x)\, u''(s(x))}$$

is negative. So $s''(\cdot)$ is negative, $\widehat{w}''(\cdot)$ is negative and $c''(\cdot)$ is positive. ■

Now we move backwards to period 1. The value to the first period decision maker of savings w is

$$V(w) = \alpha\left[u(c_2(w)) + \delta u(w - c_2(w))\right].$$

Now

$$\begin{aligned} V'(w) &= \alpha\left[u'(c_2(w))\, c_2'(w) + \delta u'(w - c_2(w))\left(1 - c_2'(w)\right)\right] \\ &= \alpha\left[u'(c_2(w))\, c_2'(w) + \frac{\delta}{\alpha} u'(c_2(w))\left(1 - c_2'(w)\right)\right], \text{ by (2.1)} \\ &= u'(c_2(w))\left[\delta + (\alpha - \delta)\, c_2'(w)\right]. \end{aligned}$$

Write $c_1 : \mathbb{R} \to \mathbb{R}$ for the optimal first period consumption rule, i.e.,

$$c_1(w) = \underset{x \in [0,w]}{\arg\max} \; u(x) + V(w - x)$$

If V is concave, then c_1 will be continuous and increasing with a derivative between 0 and 1. If V is not concave, then c_1 will have discontinuous downward jumps at some values of w. Now observe that

$$V''(w) = u''(c_2(w))\, c_2'(w) \left[\delta + (\alpha - \delta)\, c_2'(w)\right] + u'(c_2(w))\,(\alpha - \delta)\, c_2''(w).$$

This expression will be negative as long as $(\alpha - \delta)\, c''(w) \leq 0$. Thus we have:

Proposition 2. *First period optimal consumption $c_1(\cdot)$ is an increasing continuous function of wealth in each of the following three cases: (1) $\alpha > \delta$ and concave absolute risk tolerance; (2) $\alpha = \delta$; and (3) $\alpha < \delta$ and convex absolute risk tolerance.*

Recall that $\alpha > \delta$ incorporates the "intergenerational altruism" case and $\alpha < \delta$ is the hyperbolic discounting case.

3. Extensions

3.1. Infinite Horizon

Can Proposition 2 be extended beyond the three period case? One strategy that might work for finite $T > 3$ is to work by backward induction, checking that the continuation valuation functions at each date inherit the appropriate fourth derivative properties. This strategy appears difficult but perhaps not impossible.

Another strategy is to exploit the stationarity of the infinite horizon problem. Suppose that in the infinite horizon, a stationary differentiable consumption rule $c(\cdot)$ is followed at all dates. A necessary condition for a differentiable equilibrium is that

$$u'(c(w)) = u'(w - c(w)) \left[\delta + (\alpha - \delta)\, c'(w - c(w))\right].$$

This was shown for the very special case where $\delta = 0$ by Kohlberg (1976). It was shown for the case of hyperbolic discounting ($0 < \alpha < \delta$) by Harris and Laibson (2001a).

Kohlberg (1976) noted that this equation can be thought of as a (complicated) differential equation: writing $f(\cdot)$ for the inverse of $u'(\cdot)$, we have

$$c(w) = f\left(u'(w - c(w))\left[\delta + (\alpha - \delta)\, c'(w - c(w))\right]\right).$$

If one could find fourth derivative properties on u that were sufficient for the differential equation to have a solution and verify second order conditions, one would have a proof of existence of a continuous stationary solution.

3.2. Positive Interest Rates, Stochastic Income and Liquidity Constraints

More realistic models should incorporate positive interest rates, stochastic income and liquidity constraints. Each is discussed in turn.

Suppose we added a positive rate of return R to the three period cake eating problem. If $R\alpha < 1$, the analysis will be qualitatively unchanged. If $R\alpha > 1$, then period 3 consumption will exceed period 2 consumption. So Lemma 1 is reversed: concave absolute risk tolerance implies increasing marginal propensity to consume and convex absolute risk tolerance implies decreasing marginal propensity to consume. Proposition 2 is also reversed: sufficient conditions for continuous consumption are (1) $\alpha > \delta$ and convex absolute risk tolerance; (2) $\alpha = \delta$; and (3) $\alpha < \delta$ and concave absolute risk tolerance.

Carroll and Kimball (1996) have shown that adding uncertainty tends to concavify the consumption function. Thus in the intergenerational altruism case and whenever $\alpha > \delta$, the sufficient conditions for continuous consumption will continue to hold when uncertainty is added. However, in the hyperbolic discounting case, when $\alpha < \delta$, adding uncertainty goes in the wrong direction and we no longer have a sufficient condition.[12]

Finally, we note that liquidity constraints will certainly change these results. Harris and Laibson (2001b) document the discontinuous consumption functions that arise with uncertainty and liquidity constraints, even in the constant relative risk aversion case.

References

[1] Bernheim, D. and D. Ray (1987). "Economic Growth with Intergenerational Altruism," *Review of Economic Studies* 54, 227-241.

[12] Gollier (2001) section 16.3.2 discusses how with both convex absolute risk tolerance and increasing expected consumption through time (i.e., $R\alpha > 1$), there is a decreasing marginal propensity to consume.

[2] Carroll, C. and M. Kimball (1996). "On the Concavity of the Consumption Function," *Econometrica* 64, 981-992.

[3] Gollier, C. (2001). *The Economics of Risk and Time.* Cambridge: MIT Press.

[4] Gul, F. and W. Pesendorfer (2001). "Temptation and Self-Control," *Econometrica* 69, 1403-1436.

[5] Harris, C. and D. Laibson (2001a). "Dynamic Choices of Hyperbolic Consumers," *Econometrica* 69, 935-958.

[6] Harris, C. and D. Laibson (2001b). "Hyperbolic Discounting and Consumption," forthcoming in the proceedings of the 8th World Congress of the Econometric Society.

[7] Harris, C. and D. Laibson (2001c). "Instantaneous Gratification."

[8] Kohlberg, E. (1976). "A Model of Economic Growth with Altruism between Generations," *Journal of Economic Theory* 13, 1-13.

[9] Krusell, P. and A. Smith (2001). "Consumption-Savings Decisions with Quasi-Geometric Discounting."

[10] Leininger, W. (1986). "The Existence of Perfect Equilibria in a Model of Growth with Altruism between Generations," *Review of Economic Studies* 53, 349-367.

[11] Morris, S. and A. Postlewaite (1997). "Observational Implications of Non-Exponential Discounting," available at `http://www.econ.yale.edu/~sm326/nonexp.pdf.`

[12] O'Donaghue, T. and M. Rabin (1999). "Doing It Now or Doing It Later," *American Economic Review* 89, 103-124.

[13] Phelps, E. and R. Pollak (1968). "On Second-Best National Saving and Game-Equilibrium Growth," *Review of Economic Studies* 35, 185-199.

[14] Strotz, R. (1956). "Myopia and Inconsistency in Dynamic Utility Maximization," *Review of Economic Studies* 23, 165-180.