Abstract

Dynamic Coordination Games: Theory and Applications

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Market outcomes often appear to be less related to fundamentals than traditional economic theory would suggest. Examples include bubbles in financial markets, speculative currency crises, and the evolution of technological standards. This dissertation emphasizes the role of strategic interaction and social learning by rational agents in modeling such phenomena. Toward this goal, we first develop the theory of dynamic coordination games with private information and Bayesian social learning. In the first two chapters of the dissertation we provide a group of theoretical results to characterize the equilibrium set of these games. In the third and fourth chapters we apply the theory thus developed to examine two specific economic problems. The first of these investigates the role of a large investor in precipitating a speculative attack on a currency. The second provides a model of contagious bank failure.

Dynamic Coordination Games: Theory and Applications

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Introduction

Even casual observation of booms and busts in financial markets suggests that observed economic outcomes may have less to do with "fundamentals" than traditional economic theory would suggest. "Self-fulfilling"crises, "irrational exuberance", and "herd behavior" are but a few of the terms that have been used to informally describe this decoupling.

A common example of self-fulfilling crises is the sudden collapse of a fixed exchange rate peg under speculative pressure. Consider a mass of speculators deciding whether to attack a fixed currency peg. Since even weak currency pegs do not fall in a day, a successful attack takes several periods to mount. To complicate matters, the exact amount of reserves that a central bank is willing to spend to defend the peg is not known with certainty. Short-selling has increasing returns to scale. It takes a certain mass of speculators to mount a successful attack. Finally, since it takes time to mount a successful attack on the peg, those who choose to wait before attacking are often able to observe some noisy aggregate statistic summarizing the state of the market which incorporates past choices (e.g. "short" or "long"). Assuming that each speculator has access to private research, such observation can productively increase the knowledge of the observer.

The stylized features of situations such as these are incomplete and private information, strategic complementarities, and multiple periods with observable actions. Information is incomplete in that agents do not know the true state of the world. However, agents typically have access to proprietary research or personal intuition about the state of the world. Thus information is also *private*. There are strategic complementarities because individual payoffs depend on the actions of others. Finally, since the situation lasts several time periods, agents may be able to at least noisily observe the actions of others. Other examples of such settings include bank runs and panics, bubbles in stock markets, and the adoption of new untested technologies in the presence of network externalities.

There is a rich tradition of formal and informal economic models that seek to explain such an apparent decoupling of fundamentals and outcomes, dating back to Keynes (1936). The more recent game theoretic tradition has given rise to at least two distinct strands of the literature that analyze such situations. The first strand utilizes static coordination games. Two leading early examples of these are Diamond and Dybvig's (1983) analysis of bank runs, and Obstfeld's (1986) examination of currency crises. These games take the coordination problem of agents seriously, but ignore the dynamic component to their decisions, and the social learning inherent in the original problem. The second, more recent, strand consists of the so-called "herding models". Leading early examples of these are Banerjee (1992) and Bikhchandani, Hirshleifer, and Welch (1992). These models analyze sequential social learning problems but ignore the strategic elements embodied in the payoff complementarities that are common to many such settings.

It is of interest to take both the learning (backward-looking) and strategic (forward-looking) elements in this class of problems seriously, and to examine how they interact. The essays of this dissertation do so. To achieve this, we model these problems using dynamic coordination games with private information and observable actions. *En route* to our goal, we have to address a theoretical problem. Coordination games, both static and dynamic, can have multiple equilibria. This can render comparative statics exercises difficult, and thus reduce the usefulness of the models in analyzing applied problems. Well-developed theories of equilibrium selection exist for a large class of static coordination games (see Carlsson and van Damme 1993, and Morris and Shin 2000), and for a subset of dynamic coordination games without learning (see Frankel and Pauzner 2000, and Burdzy, Frankel, and Pauzner 2001). The question of uniqueness of equilibria for coordination games with learning has not received systematic treatment. In the essays of this dissertation, we propose some tractable first steps towards extending the Carlsson-van Damme equilibrium selection techniques to a class of dynamic coordination games with social learning.

The essays of the dissertation can be divided into two groups. Chapters 1 and 2 investigate the theory of dynamic coordination games with learning. Chapters 3 and 4 apply this theory to two specific economic problems. In what follows we briefly summarize each of the chapters.

The first chapter studies how the introduction of social learning with costs to delay affects the equilibrium set of coordination games with incomplete information. We consider a continuum of agents who choose whether and when to invest in a risky project. The project succeeds if a sufficiently large proportion of agents participate, given the state of the world, which is indexed by a smooth variable. All agents receive private signals about this state variable, and agents who choose to invest late also receive a second signal that noisily summarizes the proportion of early investors. When the risky project succeeds, it pays a return that increases in the term of participation. Hence, there is a cost to delay in participation.

We show that this game has a unique monotone equilibrium. A comparison of the equilibrium of the dynamic game with the equilibria of analogous static coordination games explicates the role of social learning. In the limit as noise vanishes, social welfare is strictly ranked in the games we consider, with the highest welfare achieved in the dynamic game with endogenous ordering. We demonstrate that exogenous asynchronicity is not a substitute for endogenous asynchronicity. The latter can lead to *strict* improvement in the efficiency with which agents can coordinate, relative to any game where the order of actions is predetermined. We also show that under endogenous ordering, as noise vanishes, the efficiency of coordination is maximized at *intermediate* costs to delay. Our results have implications for the initial public offerings of debt, as well as for the adoption of new technology under incomplete information.

The next chapter, written prior to chapter 1, analyzes a similar problem in a more discrete set-up. Instead of using a continuum of players, and a continuous state variable, we start with a countable number of players and a pair of states, good and bad. N agents enter in exogenous order in N periods, and choose whether to invest or not in sequence. Investment is costly, and pays a positive return if and only if the state is good, and all other players also choose to invest. Agents receive private signals, which are conditionally *i.i.d.* and satisfy a monotone likelihood ratio property, and can perfectly observe the actions of all their predecessors. This model subsumes the canonical herding model of Bikhchandani, Hirshleifer, and Welch (1992) as a special case when the payoff externality is eliminated.

In our setting we demonstrate that agents may exhibit either strong herd behavior (complete imitation) or weak herd behavior (overoptimism) and characterize the informational requirements for these distinct outcomes. We also characterize the informational requirements for coordinated risk-taking in games with finite but unboundedly large numbers of players. By using discrete players we are able to incorporate a strategic element that is missing from the model in chapter 1. Discrete agents, knowing that their actions will be observed by successors, take this account in making their decisions, i.e., they *signal.* Unfortunately, however, this set-up does not possess the tractability of the model of chapter 1. It is difficult, for example, to tractably endogenize the order in which agents make their choices. In addition, for $N > 2$ there may be a multiplicity of equilibria even in the exogenous order game.

The remaining chapters of this dissertation examine two distinct applications of dynamic coordination games. The first of these is contained in Chapter 3, which is jointly authored with Giancarlo Corsetti, Stephen Morris, and Hyun Song Shin. The question we address is: To what extent do large investors increase the vulnerability of a country to speculative attacks in foreign exhange markets? To address this issue, we build a model of currency crises where a single large investor and a continuum of small investors decide whether to attack a currency based on their private information about fundamentals. We examine two versions of the model: a static version which abstracts from signalling, and a dynamic version which allows the large trader to send signals to small traders using her visible short position in the currency. Even abstracting from signalling, the presence of the large investor does make all other traders more aggressive in their selling. Relative to the case in which there is no large investors, small investors attack the currency when fundamentals are stronger. Yet, the difference can be small, or null, depending on the relative precision of private information of the small and large investors. Adding signalling makes the influence of the large trader on small traders' behavior much stronger. In particular, when the large trader is much better informed than the small traders, she can completely solve the coordination problem in the market. Small investors exhibit herd behavior: they attack if and only if the large trader attacks. This effect is independent of the size of the large trader.

Finally, in chapter 4, we examine the question of financial contagion. Financial contagion is the phenomenon by which a crisis in one financial institution may affect a different insitution through a variety of channels. Leading examples of such contagion, in the context of commercial banks, were widespread episodes of bank panics in the United States in the late 19th and early 20th centuries. We model financial contagion as an equilibrium phenomenon in a noisy dynamic coordination game with multiple banks. The probability of bank failure is uniquely determined. We explore how the cross holding of deposits motivated by imperfectly correlated regional liquidity shocks can lead to contagious effects conditional on the failure of a financial institution.

We show that contagion is possible in the unique monotone equilibrium of the economy and characterize exactly when it may exist. At the same time, we identify a direction of flow for contagious effects, which provides a rationale for localized financial panics. Simulations identify the optimal level of interbank deposit holdings in the presence of contagion risk. Our results suggest that when the probability of bank failure is low, maximal levels of interbank holdings are optimal. When cross holding of deposits is complete, we demonstrate that the intensity of contagion is increasing in the size of regionally aggregate liquidity shocks.

Chapter 1

Coordination, Learning, and Delay

In many applied problems in economics an agent's payoff from taking an action depends on some underlying (unknown) state of the world and increases in the mass of other agents taking similar actions.[1] Many such settings are also inherently dynamic, encompassing several time periods. Thus agents are presented with several occasions to act, and may be able to (noisily) observe the actions of others who make choices before them. In the presence of private information, such observation may help agents improve their knowledge of payoffs. However, there may be a cost to delay in making choices. We begin by providing a leading example of such a problem.

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[1] I am grateful to Stephen Morris, Ben Polak, Dirk Bergemann, David Pearce, Don Brown, Sanjeev Goyal, Johannes Horner, and Yianis Sarafidis for valuable discussions. I thank seminar participants at the Federal Reserve Bank of Minneapolis, Erasmus Universiteit Rotterdam, the Indian Statistical Institute (Calcutta), and Yale University for useful comments. An early version of this paper was circulated under the title "Global Equilibrium Selection in a Dynamic Coordination Game with Social Learning". Financial support for this project was provided by Yale University in the form of a Robert M. Leylan Dissertation Fellowship.

Consider a group of investors deciding whether to invest their resources in a safe domestic venture or in a risky emerging market project. The risky project requires the participation of a critical mass of speculators to succeed. Potential investors have private information (from a combination of publicly available information and personal research) about the underlying exogenous value of the project. Since investors operate in a common market, those who choose to wait before investing can at least noisily observe the actions of others who made their choices earlier. This can provide them with better information. But delay comes at a cost of remaining vested in the project for a shorter period of time, and therefore enjoying less of its benefits.[2] Similar examples include the adoption of new technologies with network externalities[3], currency crises[4], and bank runs[5].

At the heart of the example we have just described lies a coordination problem. An individual wishing to invest in the risky project must be convinced that enough fellow investors will also participate. Thus, we might expect equilibrium underinvestment relative to the social optimum. However, the standard coordination problem is complicated here

[2]Chari and Kehoe (2000) study this problem. However, they do not take into account strategic complementarities. In their model, the success of the project depends only on an exogenous state variable. Two related models that analyze settings with endogenous timing and private information in the absence of strategic complementarities are Caplin and Leahy (1993) and Chamley and Gale (1994).

[3]This problem is analyzed by Choi (1997). We discuss this paper further in Section 1.5.5.

[4]This problem is analyzed by Morris and Shin (1998), in a static model, ignoring learning. They extend the equilibrium selection techniques of Carlsson and van Damme (1993) to analyze the problem in a unique-equilibrium setting.

[5]Analyzed by Goldstein and Pauzner (2000), also ignoring learning, and also in the tradition of Carlsson and van Damme (1993).

by dynamics and social learning. Individuals have multiple periods to act. There may be an incentive to wait and collect more information. This may delay, or even discourage, investment. On the other hand, observing investment by some investors may encourage others to join in, and thus lead to greater investment. Finally, the additional information produced by observational learning may help agents make better choices. Given these complications, it is natural to ask: How might the presence of dynamics and social learning affect the level of equilibrium investment relative to benchmark cases?

In order to address this question we model the stylized features outlined above using a noisy dynamic coordination game with Bayesian social learning and costs to delay. To establish a benchmark for comparison we also analyze the same coordination problem in the absence of dynamic elements and learning, i.e., using *static* coordination games. To avoid the usual difficulties created by multiple equilibria in static coordination games we rely on the work of Carlsson and van Damme (1993) and Morris and Shin (1998, 2000). These authors demonstrate that in the presence of private information, a unique equilibrium is selected in a large class of static coordination games, commonly referred to as *global games.* The static global games analysis provides a tractable benchmark for our results. Comparing our results to these earlier analyses would address the question raised above. Formally stated: *How does the introduction of social learning with delay costs affect the probability of coordinated risk-taking in noisy supermodular games?* Does learning make a difference? If so, how? In order to complete the comparison, however, we must first overcome a theoretical hurdle.

Dynamic coordination games, like their static counterparts, have multiple equilibria when payoffs are common knowledge.[6] Many such games can have multiple equilibria even in the absence of complete information, e.g. Dasgupta (1999), Marx (2000), Chamley (2001). This makes it difficult to compare these games with their static counterparts, and therefore obscures the role of dynamics and learning.

The theory of equilibrium selection in dynamic supermodular games has received recent scholarly attention. In an important set of papers, Burdzy, Frankel, and Pauzner (2001), Frankel and Pauzner (2000), and Frankel (2000) have established that when payoffs are affected by a stochastic parameter with sufficient stationarity and frequent innovations, a unique equilibrium is selected in a class of dynamic coordination games where agents are offered random opportunities to switch between actions.[7] However, agents in these models cannot use the observed actions of others to Bayes update their beliefs about the state of the world. The evolving state variable is observed publicly, and the current value incorporates all available information about future values. Their results are, therefore, not directly applicable to the class of problems of interest to us.

To facilitate a precise comparison with static benchmarks, we propose a tractable model in which the Carlsson-van Damme/Morris-Shin equilibrium selection arguments can be partially extended to allow for social learning. In our model, a mass of agents choose whether to invest in a safe project or in a risky project of uncertain underlying value. The underlying state of the world is indexed by a variable θ, which becomes known at the end of

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[6] For an analysis of dynamic coordination games with complete information, see Gale (1995), who demonstrates multiplicity. In particular, for a continuum player version of Gale's model, which shares features with the model we present below, he demonstrates that there is a *continuum* of possible equilibria.

[7] In recent work, Levin (2000) extends their analysis to study overlapping generations games.

the game, when consumption occurs. Investors have two opportunities to invest before the end of the game, at times t_1 and t_2. The receipts of an agent who invests in the risky project depend on whether the project is successful. When the project succeeds, it generates a high payoff which is continuously compounded between the time of investment and the end of the game. Thus, there is a cost associated with investing later in the game. When the project fails, it pays nothing. The project succeeds at θ if a critical mass of agents participate, and this required mass is inversely related to the value of θ. Agents do not know θ for sure, but receive informative private signals about it. Agents who act in the later period, observe a noisy signal of an aggregate statistic based on the proportion of investors in the earlier period. While this statistic provides them with further information on the underlying state, as we have noted already, there is a cost associated with delayed investment.

We first establish the benchmark static analysis for this model. The static models, which enforce simultaneous moves, are, by definition, devoid of social learning. We define two natural static games, based upon the time (t_1 or t_2) at which they are played. Following these static models we examine dynamic extensions, which allow us to incorporate learning. There are two natural ways to do this. We may allow for asynchronicity while prespecifying exogenously the order in which the different agents must act. Alternatively we may consider a model where agents are allowed to choose both their actions and the time at which they act We consider each of these four models in turn.

1.0.1 Summary of Results

To explore the implications of our dynamic analysis, we first establish some existence and uniqueness results. We show that as long as noise is small enough there is a unique monotone equilibrium in each of the dynamic games we consider. This is true regardless of whether the order of actions is specified exogenously (Proposition 1.3), or chosen endogenously (Proposition 1.4). In the limit as noise vanishes, we can solve for these equilibria in closed form. We show that these dynamic equilibria are "well behaved": As we vary model parameters to bring the dynamic games "close" to the limiting static games, the dynamic equilibria converge smoothly to their static counterparts. These convergence results are shown in Corollaries 1.1 and 1.2 and discussed below in Sections 1.3 and 1.4.

The comparison of the equilbria of the static and dynamic games of exogenous and endogenous order provides insights into the role of dynamics and learning. Our results address two related but distinct issues. The first pertains to the equilibrium probability of coordinated investment. The second pertains to social welfare. We deal with these in turn.

As noise vanishes, we show that there exists a *strict ranking of the probability of coordinated investment across the different games.* The endogenous order dynamic game maximizes the probability of investment. This is followed by the first period static game, which in turn is followed by the exogenous order dynamic game. The lowest probability of investment is achieved in the second period static game. This is summarized in Corollary 1.3 and discussed below in Sections 1.4 and 1.5.

In a related finding, we show that *exogenous ordering cannot substitute for endogenous ordering.* As noise vanishes, for almost all parameter values, there exists no ex ante exogenous ordering of agents that can replicate the probability of coordinated achieved by the endogenous order dynamic game. This is because the endogenous coordination game utilizes the *revealed preference* of a group of agents to invest early, while the exogenous order game does not. We illustrate that this result is robust to the presence of significant amounts of private information in the games. This is discussed further in Section 1.5.2.

We demonstrate that as noise vanishes, the *probability of coordinated investment in the endogenous order dynamic game is maximized for intermediate cost to delay.* An intuition for this follows from the observation that the efficiency of coordination depends on the total mass of agents who can be persuaded to invest during the course of the game. The cost of delay has opposite effects on the masses of agents who invest early or late. A large cost to delay persuades more agents to invest early. But at the same time it dissuades agents who did not invest early from doing so later based on their updated information. We call this non-monotone relationship the *coordination effect* of introducing a costly delay option. We illustrate numerically that it is robust to the presence of significant amounts of private information. This is discussed further in Sections 1.5.3 and 1.5.4.

We show that the effect of introducing learning with exogenous ordering can be given a particularly clean characterization. The relationship between the equilibria of the exogenous order dynamic game and the two benchmark static games is essentially determined by the exogenous parameter specifying the division of players between the two periods. However, we show that later players in the dynamic game are able to use the additional information obtained by observing their predecessors to make more accurate choices.

We now turn to the question of social welfare. In the limit as noise vanishes, social welfare is a monotone increasing function of the equilibrium probability of coordinated investment. Thus, in the noise-free limit, social welfare is ranked as above: highest in the endogenous order game, followed by the first period static game, followed by the exogenous order game, and finally the second period static game. However, away from the limit, in addition to the coordination effect, the introduction of the costly option to delay has two other effects. When the option to delay is exercised, it leads to better information and higher welfare (the *learning effect*), but since the option is costly, leads to lower payoffs and therefore lower welfare (the *direct payoff effect*). The total welfare effect of introducing a costly delay/learning option into a coordination game results from the interaction of these three effects.

We illustrate that for low levels of private information, the coordination effect dominates the learning effect and social welfare is maximized at intermediate levels of delay costs. However, for high levels of private information, the learning effect dominates the coordination effect, and thus, for sufficiently noisy endogenous order dynamic coordination games, social welfare is maximized at minimal cost of delay. The interaction of the coordination, learning, and direct payoff effects is summarized in Section 1.5.4.

1.0.2 Applications

Our model has implications for at least two classes of applied problems. We outline them here. A more detailed discussion is provided in Section 1.5.5. First, consider a government financing an uncertain project by offering a debt contract, in a setting in which secondary markets for the debt contract may be missing or illiquid. Under these circumstances, our model suggests that it may be beneficial to "stagger" the initial offering to allow investors multiple opportunities to invest, and sort themselves over time. Then, under the results outlined above, the coordination effect will ensure that the project will succeed with higher probability than if the entire debt package was offered simultaneously.

Second, our results provide a fresh perspective on the question of whether it is beneficial or harmful to allow firms who are switching between technologies to have the option to delay or not. The so called "penguin effect"[8] can lead to socially suboptimal delay in this context. Choi (1997) suggests that in settings with incomplete information and network externalities, it may be socially optimal for agents to forfeit their option to wait and learn and to make choices simultaneously. In direct contrast with Choi, we find that introducing a costly option to delay and learn can enable agents to sort themselves efficiently over time, and lead to strict gains in the efficiency of coordination. Thus, the penguin effect, while present in our model, enhances rather than diminishes social welfare.

The rest of the paper is organized as follows. In the next section we describe the investment problem. In section 1.2 we analyze the problem using the static approach of Morris and Shin. Sections 1.3 and 1.4 extend the analysis to include dynamic elements. In section 1.3 the problem is analyzed using a dynamic coordination game with exogenously specified order of actions. Section 1.4 relaxes the exogeneity of order. Sections 1.5 and 1.6 discuss and conclude.

1.1 The Investment Project

The economy is populated by a continuum of risk neutral agents, indexed by $[0, 1]$, each of whom has one unit of resources to invest. They must choose between investing in a safe

[8]The tendency for agents in strategic settings to wait to act second, in order to gain more information, avoid intermediate or final miscoordination costs from temporary or permanent "stranding" in a ex post suboptimal technology. See Farrell and Saloner (1986)

project, which gives a gross payoff of 1, and a risky project of uncertain value. Uncertainty is summarized by a state variable θ which is distributed $N(0,1)$ and is revealed at time T, when consumption occurs. There are two periods in which an agent might be able to invest in the risky project: $t \in \{t_1, t_2\}$. We require that $T > t_2 > t_1$, i.e. at the times when agents have opportunities to invest, the value of the project is unknown.

Proceeds from investing in the risky project depend on whether the project succeeds or not. The success of the project, in turn, depends on the actions of the agents and the realized value of θ. In particular, if p denotes the total mass of agents who invest at the times when opportunities are available, then investment succeeds if $p \geq 1 - \theta$. Payoffs from the risky project can be summarized as follows:

- When the project fails, it pays 0.
- When the project succeeds, it pays an instantaneous rate of return $R > 0$, which is continuously compounded over the length of time that an agent has held the investment. Thus, for an agent who invests at time t_i, returns are $e^{R(T-t_i)}$, conditional on the success of the project.

Our payoffs are motivated by, and very similar to, those of Chari and Kehoe (2000). The major difference is that we incorporate strategic complementarities, i.e., we allow the success or failure of the project to depend not only on the exogenous state, but also on the endogenous number of agents who choose to invest.[9]

[9]A second, minor, difference between our models is that Chari and Kehoe (2000) allow agents $T_1 \geq 2$ occasions to invest, where $T_1 < T$, while we set $T_1 = 2$. The generalization of our model to include more than two periods presents no conceptual difficulties, but comes at great algebraic cost, given the strategic

We now perform some useful normalizations. We recast the game in terms of payoffs to *switching* from the safe project to the risky one, and divide through by $e^{R(T-t_1)}$. Let $c = \frac{1}{e^{R(T-t_1)}}$, and let $k = 1 - e^{R(t_1-t_2)}$. We label the act of switching at $t = t_1$ by I_1, and at $t = t_2$ by I_2. The act of never switching is denoted N. Thus, we may now represent agent's utilities by the following schedule:

$$u(I_1, p, \theta) = \begin{cases} 1-c & \text{if } p \geq 1-\theta \\ -c & \text{otherwise} \end{cases} \tag{1.1}$$

$$u(I_2, p, \theta) = \begin{cases} (1-k)-c & \text{if } p \geq 1-\theta \\ -c & \text{otherwise} \end{cases} \tag{1.2}$$

$$u(N, p, \theta) = 0 \tag{1.3}$$

Note that $k \in (0, 1-c)$, because $t_2 \in (t_1, T)$. Thus, k represents a cost to delay, the payoff forfieted by an agent due to her delay in switching.

At the beginning of $t = t_1$ agents observe the state of fundamentals with idiosyncratic noise. In particular, each agent i receives the following signal at the beginning of the game:

$$x_i = \theta + \sigma\epsilon_i \tag{1.4}$$

where ϵ is distributed Standard Normal in the population and independent of θ.

We now present a sequence of (progressively more complex) games that can be used to study this investment problem. We begin with the benchmark static case analyzed by Morris and Shin, and then extend by introducing dynamic elements.

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complementarities. We conjecture that the results will be very similar.

1.2 The Benchmark Static Game

To analyze this investment problem within the framework of static global games in the style of Morris and Shin requires that we place an ad hoc restriction on the actions of players: they must all either move at $t = t_1$, or they must all move at $t = t_2$. This defines two natural static global games, which are mutually exclusive.

The first such game is one in which all players act at $t = t_1$ and payoffs are given by (1.1) and (1.3). We label this game $\Gamma_{st,1}$. We label the other game, in which players all move at $t = t_2$ and payoffs are given by (1.2) and (1.3), by $\Gamma_{st,2}$. We analyze $\Gamma_{st,1}$ and extend our results by symmetry to $\Gamma_{st,2}$.

It is useful to begin with a preliminary definition. Note that in these games, agents' strategies map from their private information into their action spaces.

Definition 1.1. *An agent i is said to follow a* monotone *strategy if her chosen actions are increasing in her private information, i.e., if her strategy takes the form:*

$$\sigma_i(x_i) = \begin{cases} I & \textit{when } x_i \geq x^* \\ N & \textit{otherwise} \end{cases}$$

We shall call equilibria in monotone strategies *monotone equilibria.* Monotone equilibria can be given a natural economic interpretation: when an agent chooses to invest, she correctly believes (in equilibrium) that all agents who have more optimistic beliefs than her also choose to do so.

If a continuum of players follow monotone strategies, a threshold level emerges naturally in the underlying state variable of the game. Therefore, we look for monotone equilibria which take the form $(x^*_{st,1}, \theta^*_{st,1})$ where agent i invests iff $x_i \geq x^*_{st,1}$ and investment is successful iff $\theta \geq \theta^*_{st,1}$. Now we may state:

Proposition 1.1 (Morris and Shin). [10] *If $\sigma < \sqrt{2\pi}$, there is a unique monotone equilibrium in $\Gamma_{st,1}$. As $\sigma \to 0$, it is given by the pair:*

$$x^*_{st,1} = c, \quad \theta^*_{st,1} = c$$

Proof: The following are necessary for the equilibrium:

The marginal agent, who receives signal $x^*_{st,1}$ must be indifferent between investing or not, i.e.

$$Pr(\theta \geq \theta^*_{st,1} | x^*_{st,1}) = c$$

Since $\theta | x \sim N(\frac{x}{1+\sigma^2}, \frac{\sigma^2}{1+\sigma^2})$, the indifference condition can be written as:

$$1 - Pr(\theta < \theta^*_{st,1} | x^*_{st,1}) = 1 - \Phi(\frac{\theta^*_{st,1} - \frac{x^*_{st,1}}{1+\sigma^2}}{\frac{\sigma}{\sqrt{1+\sigma^2}}}) = c$$

Thus,

$$x^*_{st,1} = (1+\sigma^2)\theta^*_{st,1} + \sigma\sqrt{1+\sigma^2}\Phi^{-1}(c) \tag{1.5}$$

The critical mass condition requires that:

$$Pr(x \geq x^*_{st,1} | \theta^*_{st,1}) = 1 - \theta^*_{st,1}$$

Substituting the indiffirence condition into the critical mass condition we get

$$\Phi(\sigma\theta^*_{st,1} + \sqrt{1+\sigma^2}\Phi^{-1}(c)) = \theta^*_{st,1} \tag{1.6}$$

Consider the function

$$F(\theta^*_{st,1}) = \Phi(\sigma\theta^*_{st,1} + \sqrt{1+\sigma^2}\Phi^{-1}(c)) - \theta^*_{st,1}$$

[10] This result is a special case of Morris and Shin (2000c): Proposition 3.1. It can be obtained by setting the precision of the public signal to 1.

Clearly as $\theta^*_{st,1} \to 1$, $F(\cdot) < 0$, and as $\theta^*_{st,1} \to 0$, $F(\cdot) > 0$. Differentiating yields

$$F'(\theta^*_{st,1}) = \sigma\phi(\cdot) - 1$$

If $\sigma < \sqrt{2\pi}$, then $F'(\theta^*_{st,1}) < 0$ for all $\theta^*_{st,1}$, which establishes the first part of the result. Letting $\sigma \to 0$ in (1.5) establishes the second part. ■

The corresponding result for $\Gamma_{st,2}$ follows immediately:

Proposition 1.2 (Morris and Shin). *If $\sigma < \sqrt{2\pi}$, there is a unique monotone equilibrium in $\Gamma_{st,2}$. As $\sigma \to 0$ it is given by the pair:*

$$x^*_{st,2} = \frac{c}{1-k}, \quad \theta^*_{st,2} = \frac{c}{1-k}$$

We now extend our analysis to introduce dynamic elements. The simplest way to achieve this is to require that some exogenous proportion of agents have to choose their actions at $t = t_1$, and the rest must do so at $t = t_2$. Even in this simplest of dynamic frameworks, we are able to incorporate Bayesian social learning, as we show below.

1.3 The Dynamic Game with Exogenous Order of Actions

We now modify the game to last the length of the investment project: $t \in \{t_1, t_2\}$. The continuum of agents is divided up (exogenously) into two (possibly unequal) groups. Agents $i \in [0, \lambda]$ must choose their actions at $t = t_1$. Agents $i \in (\lambda, 1]$ must choose their actions at $t = t_2$. The payoffs to this game are given by (1.1 - 1.3).

We can now incorporate Bayes social learning. Agents who act in period 2 are able to observe a statistic based on the proportion of time 1 agents who chose to invest, which we denote by p_1. Hence, they effectively observe a "market share". However, agents observe such a market share statistic with some idiosyncratic noise, which may be small. We shall be particularly interested in the case where the observation becomes essentially public, i.e. in the limit as such idiosyncratic noise vanishes. Thus, agents $(\lambda, 1]$ receive an additional signal:

$$y_i = \Phi^{-1}(p_1) + \tau\eta_i \tag{1.7}$$

where η is Standard Normal in the population, and independent of ϵ. The specific transformation of p_1 by the inverse standard normal CDF is an algebraic simplification only (to obtain closed forms), and serves no other purpose in the arguments that follow. As is apparent, the standard case of perfect observation of the past (as is common in the literature on herds and cascades, see Bikhchandani, Hirshleifer, and Welch 1992 for example) is obtained in the limit as $\tau \to 0$. We label this game Γ_{ex} and look for Bayes Nash equilibria of this game.

Players $(\lambda, 1]$ observe two noisy signals, x and y. Let $s(x, y)$ denote a sufficient statistic for (x, y). We look for monotone equilibria which take the form $(x^*_{ex}, s^*_{ex}, \theta^*_{ex})$, such that:[11]

1. Players $[0, \lambda]$ invest iff $x_i \geq x^*_{ex}$
2. Players $(\lambda, 1]$ invest iff $s_i \geq s^*_{ex}$
3. Investment is successful iff $\theta \geq \theta^*_{ex}$

Necessary conditions for such an equilibrium are as follows. Conditional upon receiving

[11] When a sufficient statistic exists, as it does in our problem, restricting attention to monotone equilibria where second period agents condition upon their sufficient statistics is without loss of generality.

signal x_{ex}^*, player i for $i \in [0, \lambda]$ must be indifferent between investing and not investing:

$$Pr(\theta \geq \theta_{ex}^* | x_{ex}^*) = c \tag{1.8}$$

Conditional upon receiving signals that lead to sufficient statistic s_{ex}^*, player i for $i \in (\lambda, 1]$ must be indifferent to investing and not investing:

$$Pr(\theta \geq \theta_{ex}^* | s_{ex}^*) = \frac{c}{1-k} \tag{1.9}$$

Finally, at state θ_{ex}^* just the correct proportion of agents must choose to invest for investment to be successful:

$$\lambda Pr(x \geq x_{ex}^* | \theta_{ex}^*) + (1-\lambda) Pr(s \geq s_{ex}^* | \theta_{ex}^*) = 1 - \theta_{ex}^* \tag{1.10}$$

Note that $\theta | x$ is distributed $N(\frac{x}{1+\sigma^2}, \frac{\sigma^2}{1+\sigma^2})$. The mass of people who invest in period 1 at state θ is

$$p_1 = \Phi(\frac{\theta - x_{ex}^*}{\sigma})$$

Substituting into the definition of the second period signal, y, we get:

$$y_i = \frac{\theta - x_{ex}^*}{\sigma} + \tau \eta_i$$

Defining $z_i = \sigma y_i + x_{ex}^*$ we get

$$z_i = \theta + (\sigma\tau)\eta_i$$

and thus $z_i | \theta$ is distributed $N(\theta, \sigma^2\tau^2)$. Then, using Bayes's Rule we know that

$$\theta | x_i, z_i \sim N \left[\frac{\frac{1+\sigma^2}{\sigma^2} \frac{x_i}{1+\sigma^2} + \frac{1}{\sigma^2\tau^2} z_i}{\frac{1+\sigma^2}{\sigma^2} + \frac{1}{\sigma^2\tau^2}}, \frac{1}{\frac{1+\sigma^2}{\sigma^2} + \frac{1}{\sigma^2\tau^2}} \right]$$

Substituting for z_i,

$$\theta | x_i, y_i \sim N \left[\frac{x_i + \frac{\sigma}{\tau^2} y_i + \frac{1}{\tau^2} x_{ex}^*}{1 + \sigma^2 + \frac{1}{\tau^2}}, \frac{\sigma^2}{1 + \sigma^2 + \frac{1}{\tau^2}} \right]$$

Thus if we define:

$$s_i = \frac{x_i + \frac{\sigma}{\tau^2} y_i + \frac{1}{\tau^2} x^*_{ex}}{1 + \sigma^2 + \frac{1}{\tau^2}} \tag{1.11}$$

then

$$\theta | x, y \equiv \theta | s \sim N \left[s, \frac{\sigma^2}{1 + \sigma^2 + \frac{1}{\tau^2}} \right] \tag{1.12}$$

Since s_i is a linear function of two conditionally Normal variables x and y, it is easy to see that:

$$s_i | \theta \sim N \left[\frac{1 + \tau^2}{1 + \tau^2 + \sigma^2 \tau^2} \theta, \frac{\sigma^2 \tau^2 (1 + \tau^2)}{(1 + \tau^2 + \sigma^2 \tau^2)^2} \right] \tag{1.13}$$

Now we can rewrite the necessary conditions for the equilibrium as follows.

Equation (1.8) can be re-written as:

$$x^*_{ex} = (1 + \sigma^2) \theta^*_{ex} + \sigma \sqrt{1 + \sigma^2} \Phi^{-1}(c) \tag{1.14}$$

Using (1.12), equation (1.9) can be rewritten as:

$$s^*_{ex} = \theta^*_{ex} + \frac{\sigma}{\sqrt{1 + \sigma^2 + \frac{1}{\tau^2}}} \Phi^{-1}(\frac{c}{1 - k}) \tag{1.15}$$

Finally, substituting from (1.14) and (1.15) into equation (1.10) we get:

$$\lambda(1 - \Phi(\sigma \theta^*_{ex} + \sqrt{1 + \sigma^2} \Phi^{-1}(c))) + (1 - \lambda)(1 - \Phi(\frac{\sigma \tau}{\sqrt{1 + \tau^2}} \theta^*_{ex} + \frac{\frac{\sigma \tau}{\sqrt{1 + \tau^2 + \sigma^2 \tau^2}} \Phi^{-1}(\frac{c}{1-k})}{\frac{\sigma \tau \sqrt{1 + \tau^2}}{1 + \tau^2 + \sigma^2 \tau^2}})) = 1 - \theta^*_{ex}$$

Rearranging, we get:

$$\lambda \Phi(\sigma \theta^*_{ex} + \sqrt{1 + \sigma^2} \Phi^{-1}(c)) + (1 - \lambda) \Phi(\frac{\sigma \tau}{\sqrt{1 + \tau^2}} \theta^*_{ex} + \frac{\sqrt{1 + \tau^2 + \sigma^2 \tau^2}}{\sqrt{1 + \tau^2}} \Phi^{-1}(\frac{c}{1 - k})) = \theta^*_{ex} \tag{1.16}$$

Equations (1.8), (1.9), and (1.16) are the dynamic counterparts of equations (1.5) and (1.6) in the proof of Proposition 1.1. Thus, taking the derivative, and re-utilizing methods used above, we note that if

$$\lambda\sigma + (1-\lambda)\frac{\sigma}{\sqrt{1+\frac{1}{\tau^2}}} < \sqrt{2\pi}$$

then there is a unique solution to this equation. Letting $\sigma \to 0$ enables us to obtain closed forms, and we can state:

Proposition 1.3. *If $\sigma < \frac{\sqrt{2\pi}}{\lambda+(1-\lambda)\frac{\tau}{\sqrt{1+\tau^2}}}$ there is a unique monotone equilibrium in Γ_{ex}. In the limit as $\sigma \to 0$, it is given by the triple:*

$$x^*_{ex} = \lambda c + (1-\lambda)\frac{c}{1-k} \quad s^*_{ex} = \lambda c + (1-\lambda)\frac{c}{1-k} \quad \theta^*_{ex} = \lambda c + (1-\lambda)\frac{c}{1-k}$$

We note in passing that as we let $\tau \to \infty$, and thus eliminate learning in this game, the condition for uniqueness converges to the usual static condition for uniqueness: $\sigma < \sqrt{2\pi}$.

Now that we have demonstrated the existence and uniqueness of monotone equilibria in Γ_{ex}, we can compare the selected equilibrium to those of $\Gamma_{st,1}$ and $\Gamma_{st,2}$. A clean comparison can be obtained by comparing the selected threshold levels in the fundamentals in the different games. The findings are summarized in:

Corollary 1.1. *As $\sigma \to 0$:*

- $\theta^*_{st,1} < \theta^*_{ex} < \theta^*_{st,2}$
- *As $\lambda \to 1$, $\theta^*_{ex} \to \theta^*_{st,1}$*
- *As $\lambda \to 0$, $\theta^*_{ex} \to \theta^*_{st,2}$*

Thus, the outcome in the dynamic game with Bayes learning when the order is specified exogenously is not fundamentally different from the outcomes in the individual static games. The differences are driven solely by the parameter determining the exogenous ordering. As all agents are forced to act in the first or second periods the selected equilibrium converges smoothly to the selected equilibria of the corresponding static games.

Intuitively, by making players act according to the exogenous division parameterized by λ, we are effectively forcing them to play two static coordination games, but with different payoffs. The outcome is simply a convex combination of the outcomes in the two static games, weighted by the mass of agents that play each of them.

However, period 2 players in Γ_{ex} have access to more precise information than players in $\Gamma_{st,2}$. Thus, we would expect them to do better on average than players in $\Gamma_{st,2}$. It turns out that they do. The relevant question is: When the project succeeds (fails) what proportion of later players choose to invest (not invest) in Γ_{ex} versus players in $\Gamma_{st,2}$? To answer this, we note that the proportion of players who choose a particular action in an arbitrary game Γ at any level of fundamentals θ is determined by the difference between θ and θ^*_Γ. In Γ_{ex}, the proportion of period 2 agents who choose to invest at state θ is given by $Pr(s \geq s^*_{ex}|\theta)$. Using the definitions and results above, this can be rewritten to be $\Phi\left(\frac{\frac{\theta-\theta^*_{ex}}{\sigma}}{\sqrt{1+\frac{1}{\tau^2}}} - \frac{\sigma\tau}{\sqrt{1+\tau^2}}\theta^*_{ex} - \frac{\sqrt{1+\tau^2+\sigma^2\tau^2}}{\sqrt{1+\tau^2}}\Phi^{-1}(\frac{c}{1-k})\right)$. The proportion of agents who choose to invest in $\Gamma_{st,2}$ at state θ is given by $Pr(x \geq x^*_{st,2}|\theta)$. This can similarly be rewritten as $\Phi\left(\frac{\theta-\theta^*_{st,2}}{\sigma} - \sigma\theta^*_{st,2} - \sqrt{1+\sigma^2}\Phi^{-1}(\frac{c}{1-k})\right)$. Label $\delta_\Gamma = \theta - \theta^*_\Gamma$. It is easy to see that there exists $\bar{\tau} > 0$ such that for all $\tau \leq \bar{\tau}$, if $\delta_{\Gamma_{st,2}} = \delta_{\Gamma_{ex}} > 0$, then

$$\Phi\left(\frac{\frac{\delta_{\Gamma_{ex}}}{\sigma}}{\sqrt{1+\frac{1}{\tau^2}}} - \frac{\sigma\tau}{\sqrt{1+\tau^2}}\theta^*_{ex} - \frac{\sqrt{1+\tau^2+\sigma^2\tau^2}}{\sqrt{1+\tau^2}}\Phi^{-1}(\frac{c}{1-k})\right) >$$

$$\Phi\left(\frac{\delta_{\Gamma_{st,2}}}{\sigma} - \sigma\theta^*_{st,2} - \sqrt{1+\sigma^2}\Phi^{-1}(\frac{c}{1-k})\right)$$

In words, when investment is successful, and learning is accurate enough, a larger proportion of period 2 agents choose to invest (thus, choose the right action) in Γ_{ex} than in $\Gamma_{st,2}$. If $\delta_{\Gamma_{st,2}} = \delta_{\Gamma_{ex}} < 0$, then the inequality is reversed. Thus, when investment fails, a larger proportion of period 2 agents choose not to invest in Γ_{ex} than in $\Gamma_{st,2}$. In other words, on average later agents may be able to improve their welfare in the dynamic game. We shall return to a more detailed discussion of welfare in Section 1.5.[12]

1.4 The Dynamic Game with Endogenous Order of Actions

We now further augment the original game to allow agents to endogenize the order of actions. The payoffs of the game are still given by (1.1-1.3) and the information structure is summarized as in the previous section by (1.4) and (1.7). However, now agents may also choose when to invest, if at all. In particular, in period 1, agents have the choice to invest or not. If they invest, then their choice is final. If they choose not to invest, however, they get another opportunity in period 2 to make the same choice, based on the additional information they receive at that time. As we have noted earlier, the payoffs to the investment project given in (1.1-1.3) induce an endogenous cost to delay in investing. Now that they may choose both their actions and the timing of their actions, agents will rationally trade off the possible excess gains to acting early against the option value of

[12] Readers familiar with the literature on global games may have noticed that the uniqueness results proved thus far are restricted to monotone strategy equilibria. For static global games Carlsson and van Damme (1993, later generalized by Frankel, Morris, and Pauzner 2000) prove a stronger result: the unique monotone equilibrium is also the unique strategy profile surviving the iterated deletion of dominated strategies. Existing arguments for this stronger result do not generalize to our dynamic game due to Bayesian learning. The existence of non-monotone equilibria, which are complex objects in this setting, remains an open question.

waiting and collecting more information in period 2. We call this game Γ_{en} and look for Bayes Nash equilibria.

As in the game with exogenous ordering, we look for equilibria in which agents choose monotone strategies with thresholds (x^*_{en}, s^*_{en}), such that:

1. Invest at $t = t_1$ iff $x_i \geq x^*_{en}$. Otherwise choose to wait.

2. Conditional on reaching $t = t_2$ with the option to invest, invest iff $s_i \geq s^*_{en}$

In $\Gamma_{st,i}$ and Γ_{ex} it was apparent that when agents followed monotone strategies there were corresponding equilibrium thresholds in the fundamentals above which investment would be successful, and below which it would fail. This characterization is not immediate in the current game (since the decisions to invest or not in the two periods are not independent) and requires closer examination.

When agents follow monotone strategies as outlined above, at any θ, a mass $Pr(x \geq x^*_{en}|\theta) + Pr(x < x^*_{en}, s \geq s^*_{en}|\theta)$ will choose to invest. Thus, investment is successful at θ if and only if:

$$Pr(x \geq x^*_{en}|\theta) + Pr(x < x^*_{en}, s \geq s^*_{en}|\theta) \geq 1 - \theta$$

Is there a critical θ^* above which investment is successful and below which it is not? The answer is in the affirmative, as we show below:

Lemma 1.1. *Fix any* (x^*, s^*)*. Let*

$$G(\theta) = Pr(x \geq x^*|\theta) + Pr(x < x^*, s \geq s^*|\theta) - 1 + \theta$$

There is a unique solution to

$$G(\theta) = 0$$

The proof is in the appendix.

Given Lemma 1.1, we can now look for monotone equilibria of the form $(x^*_{en}, s^*_{en}, \theta^*_{en})$ where x^*_{en} and s^*_{en} are defined as above, and investment is successful if and only if $\theta \geq \theta^*_{en}$.

Necessary conditions for such equilibria are as follows:

The indifference equation for those players who arrive at period 2 with the option to invest:

$$Pr(\theta \geq \theta^*_{en} | s^*_{en}) = \frac{c}{1-k} \tag{1.17}$$

The critical mass condition is:

$$Pr(x \geq x^*_{en} | \theta^*_{en}) + Pr(x < x^*_{en}, s \geq s^*_{en} | \theta^*_{en}) = 1 - \theta^*_{en} \tag{1.18}$$

We can rewrite equation (1.17) as

$$s^*_{en} = \theta^*_{en} + \frac{\sigma}{\sqrt{1+\sigma^2+\frac{1}{\tau^2}}} \Phi^{-1}(\frac{c}{1-k}) \tag{1.19}$$

Substituting this into equation (1.18) gives us:

$$Pr(x \geq x^* | \theta^*_{en}) + Pr(x < x^*, s \geq \theta^*_{en} + M | \theta^*_{en}) = 1 - \theta^*_{en}$$

where $M = \frac{\sigma}{\sqrt{1+\sigma^2+\frac{1}{\tau^2}}} \Phi^{-1}(\frac{c}{1-k})$. Now we note:

Lemma 1.2. *Fix any x^*. Let $\hat{\theta}$ be defined by $G(\hat{\theta}, x^*) = 0$ where*

$$G(\theta, x^*) = Pr(x \geq x^* | \theta) + Pr(x < x^*, s \geq \theta + M | \theta) - 1 + \theta$$

If $\sigma < \frac{\sqrt{2\pi}}{1+\frac{\tau}{\sqrt{1+\tau^2}}}$,

1. *For each x^*, there is a unique $\hat{\theta}$.*

2. *$\frac{d\hat{\theta}}{dx^*} \in (0, \frac{1}{1+\sigma^2})$.*

The proof is in the appendix. Now consider the third equation characterizing the monotone equilibrium, the indifference condition of players in period 1. In period 1, agents trade off the expected benefit of investing in period 1 against the expected benefit of retaining the option value to wait. Thus the marginal period 1 investor who receives signal x^*_{en} must satisfy:

$$Pr(\theta \geq \theta^*_{en}|x^*_{en}) - c = Pr(\theta \geq \theta^*_{en}, s \geq s^*_{en}|x^*_{en})[(1-k)-c] + Pr(\theta < \theta^*_{en}, s \geq s^*_{en}|x^*_{en})(-c) \tag{1.20}$$

Lemma 1.2 implies that we can write $\theta^*_{en} = \theta^*_{en}(x^*_{en})$ where $0 < \frac{d\theta^*_{en}(x^*_{en})}{dx^*_{en}} < \frac{1}{1+\sigma^2}$. Using this, and substituting from equation (1.19) into equation (1.20), we can express the period 1 indifference condition purely in terms of x^*_{en}, as $L(x^*_{en}) = R(x^*_{en})$, where

$$L(x^*_{en}) = Pr(\theta \geq \theta^*_{en}(x^*_{en})|x^*_{en}) - c$$

$$R(x^*_{en}) = (1-k-c)Pr(\theta \geq \theta^*_{en}(x^*_{en}), s \geq \theta^*_{en}(x^*_{en})+M|x^*_{en}) - cPr(\theta < \theta^*_{en}(x^*_{en}), s \geq \theta^*_{en}(x^*_{en})+M|x^*_{en})$$

Given the posterior distribution of θ given x, and Lemma 1.2, we know that $L(\cdot)$ is monotone increasing in x^*_{en}. Since s is positively but imperfectly correlated with θ conditional on x, intuitively the first term in $R(\cdot)$ also increases in x^*_{en} but at a slower rate than $L(\cdot)$.[13] In addition, the rate of increase of this term is "dampened" because it is multiplied by $1-k-c < 1$. The second term the $R(\cdot)$ has an ambiguous rate of change with x^*_{en}, since it represents the intersection of two events, one of which becomes more likely as x^*_{en} increases, while the other becomes *less* likely under the same circumstances. Heuristically, therefore,

[13] As we let $\tau \to 0$, $s \to \theta$, $M \to 0$, and thus the first term in $R(\cdot)$ becomes identical to the first term in $L(\cdot)$ while the second term vanishes.

the rate of change of the second term of $R(\cdot)$ due to x^*_{en} is small. Thus, based on this informal argument, we would expect that $L(x^*_{en})$ increases *faster* in x^*_{en} than $R(x^*_{en})$, which implies that there is a unique x^*_{en} which solves $L(\cdot) = R(\cdot)$. A more formal argument, given in the appendix, establishes that this is true, and we can state:

Proposition 1.4. *If* $\sigma < \frac{\sqrt{2\pi}}{1+\frac{\tau}{\sqrt{1+\tau^2}}}$, *there exists a unique monotone equilibrium in* Γ_{en}.

The proof is in the appendix.[14]

While we cannot give closed form to the equilibrium thresholds in general, a clean characterization emerges as we let noise become small. Observe that as we let $\tau \to 0$, equation (1.18) reduces to:

$$x^*_{en} = \theta^*_{en} + \sigma\Phi^{-1}(\frac{1-k}{c}\theta^*_{en}) \tag{1.21}$$

At the same time, equation (1.20) becomes:

$$\Phi\left(\frac{\frac{x^*_{en}}{1+\sigma^2} - \theta^*_{en}}{\frac{\sigma}{\sqrt{1+\sigma^2}}}\right) = \frac{c}{c+k}$$

Combining these two, we get:

$$\Phi\left(\frac{\frac{\theta^*_{en}+\sigma\Phi^{-1}(\frac{1-k}{c}\theta^*_{en})}{1+\sigma^2} - \theta^*_{en}}{\frac{\sigma}{\sqrt{1+\sigma^2}}}\right) = \frac{c}{c+k}$$

Which simplifies to:

$$\Phi\left(\frac{\Phi^{-1}\left(\frac{1-k}{c}\theta^*_{en}\right)}{\sqrt{1+\sigma^2}} - \frac{\sigma\theta^*_{en}}{\sqrt{1+\sigma^2}}\right) = \frac{c}{c+k}$$

Clearly, as $\sigma \to 0$, the unique solution to this is given by

$$\theta^*_{en} = \frac{c^2}{(c+k)(1-k)}$$

Thus we can now summarize:

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[14]This condition reduces to the familiar condition $\sigma < \sqrt{2\pi}$ as $\tau \to 0$.

Proposition 1.5. *In the limit as $\tau \to 0$, $\sigma \to 0$, the unique equilibrium thresholds of Γ_{en} can be written as:*

$$x^*_{en} \to \frac{c^2}{(c+k)(1-k)}, \quad s^*_{en} \to \frac{c^2}{(c+k)(1-k)}, \quad \theta^*_{en} \to \frac{c^2}{(c+k)(1-k)}$$

Two important sets of properties about these limiting thresholds are immediate. First, as the cost to delay gets arbitrarily large or small (i.e. as k tends to the boundaries of its feasible range), the thresholds converge smoothly to the unique thresholds of the corresponding static games.[15]

Corollary 1.2. *Convergence to static games:*

- *As $\tau \to 0$, $k \to 0$, $\sigma \to 0$, $x^*_{en} \to \infty$, $s^*_{en} \to c$, $\theta^*_{en} \to c$.*
- *As $\tau \to 0$, $k \to 1-c$, $\sigma \to 0$, $x^*_{en} \to c$, $s^*_{en} \to \infty$, $\theta^*_{en} \to c$*

Thus, as the cost of delay becomes small, nobody invests in the first period, and the entire mass of agents play the static game (with vanishing noise) in the second period. Similarly, as the cost of delay becomes large, nobody who waits till the second period ever invests, and the entire mass of agents play a static coordination game in the first period.

A more interesting conclusion emerges upon comparison of the thresholds of the endogenous order dynamic game with those of the static games and exogenous order dynamic game as noise vanishes. In particular, a clean and economically important result is apparent when comparing the threshold levels of the fundamentals in the unique monotone equilibria of these games.

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[15] To understand the behavior of x^*_{en} and s^*_{en} as it pertains to Corollary 1.2, it is easiest to use equations (1.21) and (1.17) respectively.

Corollary 1.3. *As noise vanishes, for all* $c \in (0, \frac{1}{2}), k \in (0, 1-c), \lambda \in (0,1)$*:*

$$\theta^*_{en} < \min[\theta^*_{st,1}, \theta^*_{st,2}] < \theta^*_{ex}$$

Thus, when θ is in $[0,1]$ coordinated investment becomes *more probable* when we let agents choose both how to act and when to act. The endogenous sorting of agents acts as an implicit coordination device which makes it more likely that they shall coordinate efficiently for any given level of the fundamentals. We shall return to discuss this in further detail later in the paper.

Note that θ^*_{en} has a non-monotonic relationship with k. It is minimized at $k = \frac{1-c}{2}$. We shall return to discuss this property further in Section 1.5.3. Figure 1.1 plots the limiting thresholds in the different games for $c = 0.3$, $\lambda = 0.5$, over different values of k.

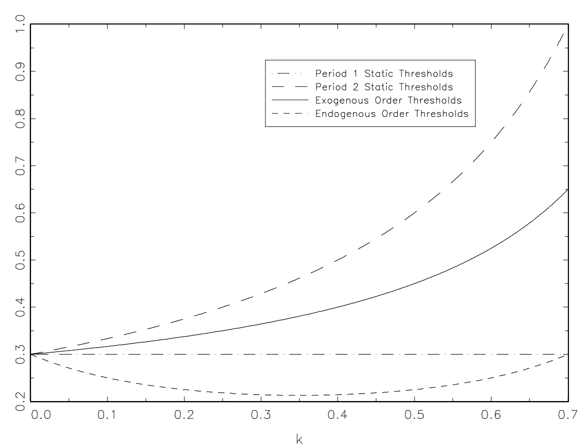

Figure 1.1: Limiting Thresholds: $c = 0.3, \lambda = 0.5$

Finally, we consider whether there is "excess delay" in this equilibrium. In the unique monotone equilibrium of Γ_{en}, a proportion of agents choose to postpone their investment decision until period t_2. Such waiting can be socially costly, since there is a cost to delay. Loosely adopting Bolton and Harris's (1999) terminology, we may be tempted to ask whether there is "too little experimentation" in our model: whether "too few" agents choose to invest early in equilibrium.[16] While Corollary 1.3 ensures that there is no decentralized solution that leads to a higher probability of investment than in Γ_{en}, it remains of interest to examine whether a informationally constrained social planner may be able to do better.[17] Is there a rule of behavior that could be implemented by an informationally constrained planner that increases the level of experimentation and also the probability of coordinated investment?

The answer turns out to be in the affirmative, as we show below:

Proposition 1.6. *There exist monotone decision rules parameterized by the threshold pair* $(\hat{x}, \hat{s})$ *such that* $\forall(\sigma, \tau, c, k)$, *the induced state-variable threshold* $\hat{\theta}(\hat{x}, \hat{s})$ *satisfies:* $\hat{\theta} \leq \theta^*_{en}$.

When (σ, τ) is sufficiently small, this decision rule improves on social welfare relative to Γ_{en}. The result follows immediately upon the construction of a variant of Γ_{en} by eliminating the payoff externality. The proof is presented in the appendix.

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[16]There are important differences between the role of experimentation in Bolton and Harris (1999) and in our model. In their model, agents are discrete, and therefore individual experimentation leads to better social information. This creates a free-rider problem in the production of information. This informational externality is missing in our model with a continuum of agents. The actions of individual agents are invisible to players in the game.

[17]For our purposes, an informationally constrained planner is a planner who has no information herself, but may specify the strategies of agents as a function of their own information.

1.5 Discussion

We have presented a sequence of models to study a multi-period investment problem characterized by incomplete information, strategic complementarities, social learning, and costs to delayed decision-making. It is useful to compare the results obtained from these different analyses. We begin by comparing welfare across the different models.

1.5.1 Welfare

We are particularly interested in welfare comparisons in the limit as social learning becomes public and accurate. It is useful to explicitly write down expressions corresponding to ex ante social welfare in the different games.

In the limit as $\tau \to 0$, we denote ex-ante social welfare in the first-period static coordination game $\Gamma_{st,1}$ by $W_{st,1}(c, \sigma)$. It is given by:

$$Pr(\theta \geq \theta^*_{st,1}, x \geq x^*_{st,1})(1 - c) + Pr(\theta < \theta^*_{st,1}, x \geq x^*_{st,1})(-c)$$

Similarly, for $\Gamma_{st,2}$, welfare $W_{st,2}(c, k, \sigma)$ is given by:

$$Pr(\theta \geq \theta^*_{st,2}, x \geq x^*_{st,2})([1 - k] - c) + Pr(\theta < \theta^*_{st,2}, x \geq x^*_{st,2})(-c)$$

For the exogenous order dynamic game, Γ_{ex}, welfare $W_{ex}(c, k, \lambda, \sigma)$ is defined as:

$$\lambda[Pr(\theta \geq \theta^*_{ex}, x \geq x^*_{ex})(1 - c) + Pr(\theta < \theta^*_{ex}, x \geq x^*_{ex})(-c)] + (1 - \lambda)Pr(\theta > \theta^*_{ex})([1 - k] - c)$$

Finally, for the endogenous order dynamic game, Γ_{en}, ex-ante social welfare $W_{en}(c, k, \sigma)$ is given by:

$$Pr(\theta \geq \theta^*_{en}, x \geq x^*_{en})(1 - c) + Pr(\theta < \theta^*_{en}, x \geq x^*_{en})(-c) + Pr(\theta > \theta^*_{en}, x < x^*_{en})([1 - k] - c)$$

Note that as we let noise vanish in the games, i.e., as $\sigma \to 0$, the product probability terms simplify and we get the following clean welfare ranking:

Remark 1.1. *As noise vanishes, for all* $c \in (0, \frac{1}{2}), k \in (0, 1-c), \lambda \in (0,1)$*:*

$$W_{en}(c,k) > W_{st,1}(c) > W_{ex}(c,k,\lambda) > W_{st,2}(c,k)$$

As $\sigma \to 0$, ex-ante welfare in each game becomes a monotone decreasing function of its unique equilibrium fundamental threshold. The lower the threshold, the higher is ex-ante social welfare. Thus, Remark 1.1 follows immediately upon inspection of Corollary 1.3.

It is also interesting to perform a welfare comparison away from the limit, i.e., for strictly positive σ. Figures 1.2 through 1.5 demonstrate this comparison for a representative set of parameter values. In each case, we set $c = 0.3$, and vary k over its permissible range. For the exogenous order dynamic game, we set $\lambda = 0.5$. We then plot social welfare for small (Figures 1.2 and 1.3) and large (Figures 1.4 and 1.5) levels of private information. We omit plotting $W_{st,2}$ as it is of little interest. As is apparent upon inspection of the figures, over wide ranges of parameter values, the welfare rankings summarized in Remark 1.1 are robust to the presence of private information. In particular, welfare in Γ_{en} is always higher than in $\Gamma_{st,1}$ and Γ_{ex}. For low values of k, and for high levels of noise, however, welfare under Γ_{ex} can occasionally be greater than welfare under $\Gamma_{st,1}$. The intuition for this is straightforward. Since we let $\tau \to 0$, learning becomes complete in period 2. Thus, when the cost of delay is sufficiently small, enforcing a large number of agents to act in period 2 provides them with greater information. Since agents make mistakes for large σ in period 1, welfare can be higher in Γ_{ex} than in $\Gamma_{st,1}$. For further discussion of related issues see Section 1.5.4.

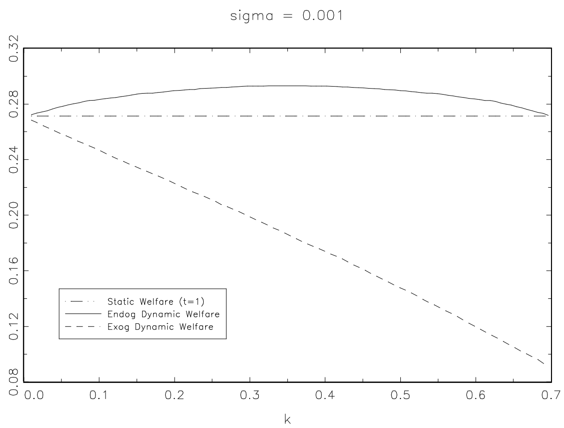

Figure 1.2: Welfare Comparisons: $\sigma = 0.001$

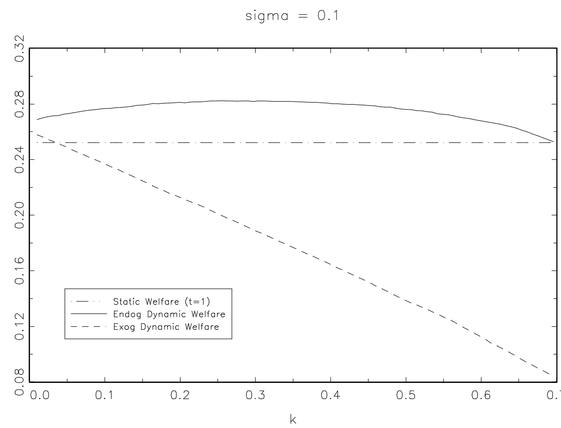

Figure 1.3: Welfare Comparisons: $\sigma = 0.1$

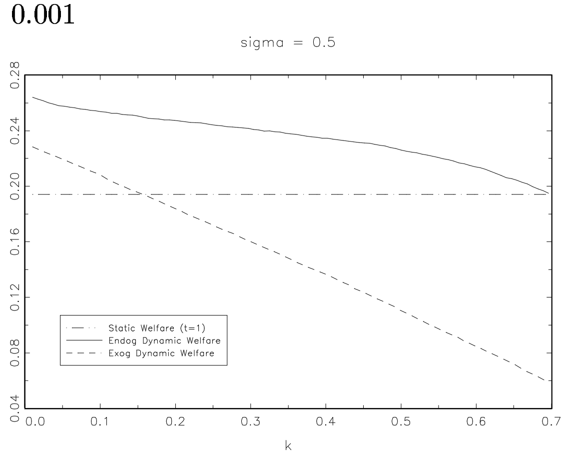

Figure 1.4: Welfare Comparisons: $\sigma = 0.5$

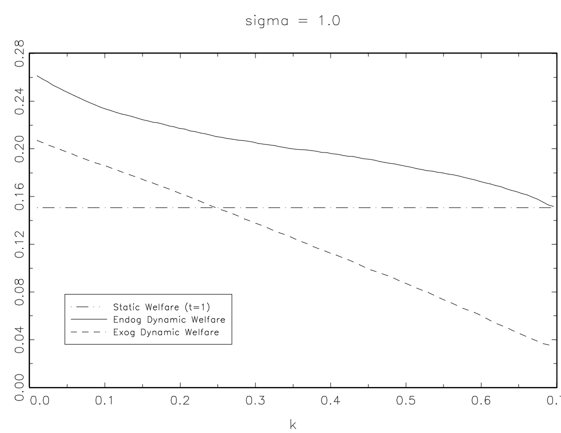

Figure 1.5: Welfare Comparisons: $\sigma = 1.0$

We now consider two properties that emerge from Remark 1.1 and upon inspection of figures 1.2 to 1.5. The first of these is the marked difference between the welfare properties of the games with endogenous and exogenous ordering of agents.

1.5.2 Exogenous vs. Endogenous Ordering

The dynamic games with exogenous and endogenous order are apparently quite similar. In both games subsets of agents move in each period, and late movers learn from the actions of early movers by paying a delay cost. It may seem, therefore, that there may be some way to parametrize the game with exogenous order of moves to match the equilibrium outcomes of the game with endogenous order. Remarkably, the answer turns out to be no.

Remark 1.2. *Exogenous asynchronicity is not a substitute for endogenous asynchronicity. As noise vanishes, there exists no* $\lambda \in (0,1)$ *such that* $W_{ex}(c,k,\lambda) \geq W_{en}(c,k)$.

The intuition behind this apparently surprising conclusion is as follows. The exogenous order game is parametrized by λ: each value of λ corresponds to a specific ex ante ordering of agents. However, any ex ante sorting of agents involves selecting a *homogenous* subsample of agents to make early choices. By definition, only *some* of these agents will choose to invest. The others wont invest in period one, and due to exogenous sorting, lose their investment option forever. However, these same agents who did not invest early in Γ_{ex} *may have invested* ex post in Γ_{en}, where they would have have another chance to do so. Thus, for any given mass of early investors, there is always a larger pool of second period investors under endogenous ordering than under exogenous ordering. In other words, endogenous ordering is more efficient, since it exploits the *revealed preference* of a subgroup of agents to make early decisions.

1.5.3 Efficiency Gains At Intermediate Costs of Delay

From inspection of Figures 1.2 and 1.3 it is apparent that when noise is small in Γ_{en}, the welfare of agents is maximized for *intermediate* costs of delay. In the limiting case as noise vanishes, the formal result is implied by Proposition 1.5:

Remark 1.3. *As noise vanishes, welfare in the endogenous order game is maximized for* *intermediate costs to delay. In particular, there exists* $k^* \in (0, 1-c)$ *such that* $k^* = argmax_k W_{en}(c,k)$.

This conclusion too may seem surprising, but it is simple to explain. When θ is in $[0,1]$, ex ante social welfare increases in the total proportion of agents who invest in the game (adding up the proportions in periods 1 and 2). A *high* cost of delay makes it unattractive for agents to wait. Thus, increasing the cost of delay persuades more agents to invest early. However, ex post in period two, a high cost of delay makes it *unattractive* for the remainder of agents to invest. Thus, increasing the cost of delay has opposite influences on the mass of agents who choose to invest in periods one and two. As a result, to maximize the total mass of agents who invest, it is natural that an intermediate cost of delay would be optimal. It is important to note here, that this argument does not depend on the size of the noise. The formal result is shown only for the case where noise vanishes, as we can obtain closed forms only in this case. However, there is no reason to suspect that the phenomenon of improved coordination at intermediate costs of delay is affected by the size of σ or τ. We shall illustrate this point numerically in Section 1.5.4.

Another puzzle remains. Careful readers may have noticed that while welfare in the endogenous order game is maximized at intermediate ranges of k for small noise (figures 1.2 and 1.3), at higher levels of noise (figures 1.4 and 1.5), welfare is maximized for *low* costs of delay. To understand this dichotomy, we must understand precisely how the costly option to wait makes a difference in our dynamic coordination games.

1.5.4 The Costs and Benefits of the Option to Wait

The introduction of a costly option to wait into a dynamic coordination game has three effects. First, when the option is exercised, it leads to better information, and therefore higher welfare. We call this the *learning effect*. However, since the option is costly, its use leads to lower payoffs, and therefore lowers welfare. Let us call this the *direct payoff effect*. Finally, the option to wait and the resultant endogenous asynchronicity improves coordination, by lowering (for intermediate values of k) the threshold above which investment is successful (and therefore the ex ante probability of successful coordinated risk taking). We call this the *coordination effect*. The total welfare gains for different levels of k in Γ_{en} result from the interaction of these three effects.

Note that the learning effect is independent of the size of k, since the informativeness of observational learning is independent of the measure of agents who choose to invest early (as long as the measure is strictly positive). The direct payoff effect is clearly increasing in the size of k. Thus, for low levels of k, the positive learning effect dominates the negative direct payoff effect. However, the coordination effect has a non-monotonic relationship with k. As noise vanishes, this relationship is clearly demonstrated by Figure 1.1. We plot the effect of k on the coordination threshold (along with welfare plots) in figures 1.6 through 1.9. As we have noted above in Section 1.5.3, and as is apparent upon inspection of figures 1.6 through 1.9, the coordination effect is *noise-independent*. We can now explain the shape of the welfare functions in the endogenous order game by appealing to the intuition that we have just built up.

With endogenous ordering, the proportions of early and late investors are sensitive to

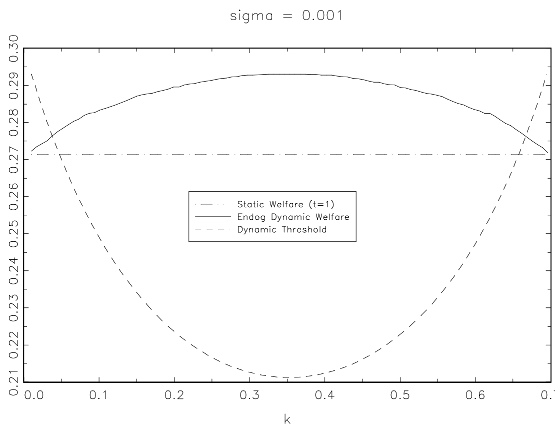

Figure 1.6: Effects: $\sigma = 0.001$

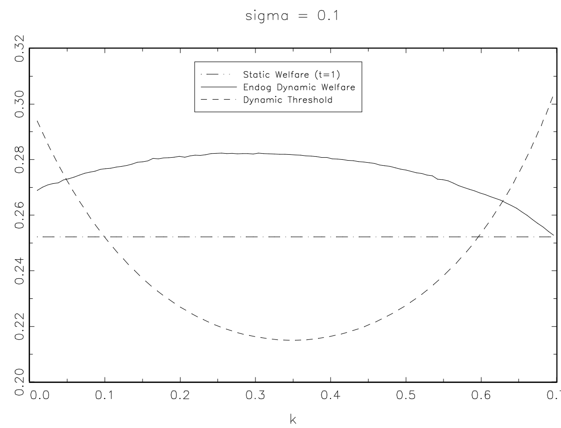

Figure 1.7: Effects: $\sigma = 0.1$

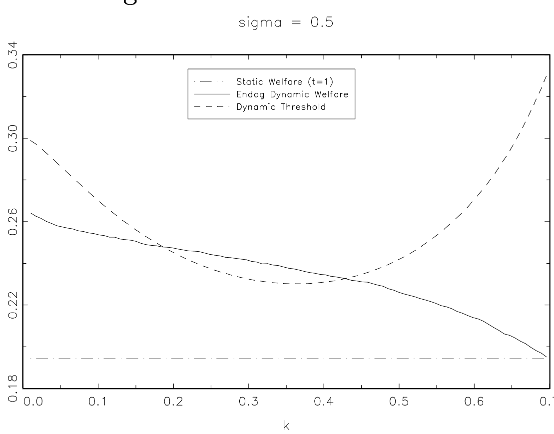

Figure 1.8: Effects: $\sigma = 0.5$

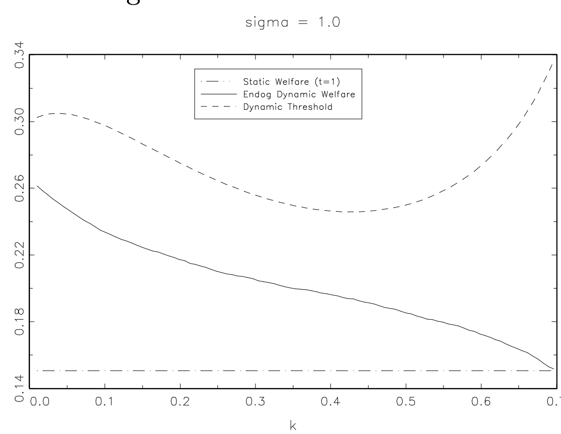

Figure 1.9: Effects: $\sigma = 1.0$

k. As Corollary 1.2 indicates, the outcomes of the endogenous order dynamic coordination game converge smoothly to those of the two limiting static games as k tends to the limits of its permissible range. As k gets very small, most investors choose to wait and the game resembles very closely the the second period static coordination game. As k becomes very large, essentially all agents choose to act early or not at all, so that the game resembles the first period static coordination game. For intermediate levels of k agents sort themselves over time.

When social learning becomes public and perfect, agents who choose to wait do not make errors in period 2. With positive noise, agents who decide to act in period 1 may still make mistakes. When noise is small, the chances that agents investing in period one will make mistakes is small. Thus, what matters for social welfare is how well agents are able to coordinate, i.e., the coordination effect dominates the learning effect. Thus, welfare in Γ_{en} tends to track the coordination threshold as a function of k. Welfare follows a bell-shaped curve as a function of k. This is best seen in figure 1.6. However, when noise gets large, the value of the learning effect becomes much larger. For small values of k, most agents choose to wait, get the benefit of the learning effect, and welfare is high. But for somewhat larger values of k, a significant proportion of agents choose to invest early. These agents tend to make many mistakes, since σ is large. Thus, welfare can be significantly reduced, even though the positive contribution of the coordination effect is maximized at intermediate values of k. Thus, when noise is large, welfare in Γ_{en} can be a monotonic decreasing function of k.

1.5.5 Applications

The sequence of models outlined in this paper contain the stylized features observed in at least two large classes of applied problems. The first of these is the financing of risky projects where there are increasing returns to scale from participation. The second is the adoption of new technologies in the presence of uncertainty and network externalities. The welfare results presented above have implications for both of these problems. We consider them in turn.

Staggered Debt Offerings

Consider an emerging market government that wants to float a bond to finance a long-term investment project using foreign investment. In addition, suppose that for reasons that we do not model, a secondary market in such bonds is likely to be absent or highly illiquid, with high transaction costs. Our results imply that when uncertainty about the state of the emerging economy is not overly large, it may be better for the government to float the bond in two pieces over time, and to provide information about initial rates of participation. In a nutshell, it may be optimal to "stagger" the initial offering of debt.

Let θ represent the underlying value of the emerging economy. It is natural to assume that if the underlying fundamentals of the economy are realized to be very good ($\theta > 1$) then domestic government can be wealthy enough to unilaterally finance the project, and the project succeeds even without foreign participation. On the other hand, if the economy ends up in a very bad state ($\theta < 0$) the project may fail even if all available foreign investors participated. Under these circumstances, we have demonstrated that in all our models, there is some threshold, θ^*, above which the project succeeds endogenously and below which it fails. Assuming that the emerging market government cares only about the success or failure of the project, its goal must be to make θ^* as low as possible. Then, Corollary 1.3 implies that it is best to offer investors at least two opportunities to invest, and let them choose endogenously between the two.

The Penguin Effect

Now consider a group of firms choosing whether to switch between a safe current technology and a risky unknown technology characterized by network externalities. In this context, it is unclear whether offering firms the option to delay switching is beneficial. While the option to delay can lead to the provision of more information, it can also lead to "too much" waiting, which can be socially suboptimal. There may be a tendency for players to delay making choices because doing so lets them make more informed choices, avoid interim payoff losses, and avoid being "stranded" in a suboptimal technology by later adopters who do not conform. Farrell and Saloner (1986) term this general phenomenon the "penguin effect", by analogy to penguins who often delay entering the water, hoping that others might do so first to test for the presence of predators. In a complete information model with multiple equilibria, they identify parameter ranges in which the option to wait can be harmful, because it leads to socially suboptimal delay. In a more recent paper, Choi (1997) provides a model of technology adoption under incomplete information, in which the penguin effect reappears. In his model, the use of technology by one user reveals its value to other users. Thus, the fear of being stranded in an ex post inferior technology may lead people to always want to choose second, which can produce socially suboptimal delay. Under certain ranges of parameters, Choi (1997) shows that forfeiting the option to wait and learn may be socially optimal.

Our results provide a different perspective on the penguin effect. We show that when a large number of firms are allowed the option to delay switching to obtain more information at some cost, they will sort themselves over time efficiently. In particular, for intermediate costs of delay, such endogenous sorting can improve efficiency, and lead to strict welfare improvements. This is true no matter how small the level of private information and therefore how small the benefit from waiting and learning. Thus, even though the penguin effect is present in our model for the same reasons as in Choi (1997)[18], its presence leads to *improved*, not diminished social welfare.

1.6 Concluding Remarks

In this paper we have explored the role of learning and delay in coordination problems under incomplete information. En route, we have established a simple template for partially extending the Carlsson-van Damme equilibrium selection technique for static coordination games to dynamic coordinations games with learning. We conclude with some remarks on the generality of these results. We begin with theoretical considerations.

For tractability and closed forms, we have made two main simplifying assumptions in this model. The first is the assumption of Gaussian noise. This assumption, taken together with our choice of market statistic function, $\Phi^{-1}(\cdot)$, allows us to construct a simple one-dimensional sufficient statistic. We conjecture that the results shall not change substantively by relaxing these assumptions. As $\tau \to 0$, it makes no difference what market statistic function we choose, as long as it is monotone increasing. We conjecture that the ordinal properties of our results will hold true for models with noise generated from any one-dimensional exponential family and for any choice of monotone increasing market statistic

[18]Note that as $\tau \to 0$, the use of one technology by a positive measure subset of agents fully reveals the value of the technology to agents who wait, just as in Choi (1997).

function. We are currently considering such generalizations.

From the perspective of applications, a natural extension would allow agents to choose the cost of delay, rather than pre-specify it in the model. It would also be desirable to let the cost of delay depend on the actions of agents in the early period. We conjecture that neither of these modifications would significantly modify our arguments, and hold promise for further interesting results.

1.7 Appendix

1.7.1 Proofs

Lemma 1.1 *Fix any* (x^*, s^*). *Let*

$$G(\theta) = Pr(x \geq x^*|\theta) + Pr(x < x^*, s \geq s^*|\theta) - 1 + \theta$$

There is a unique solution to

$$G(\theta) = 0$$

Proof: Since $s = \frac{\tau^2 x + \sigma y + x^*}{1+\tau^2+\sigma^2\tau^2}$, writing $x = \theta + \sigma\epsilon$, $y = \frac{\theta - x^*}{\sigma} + \tau\eta$, and substituting, we get $s = \frac{1+\tau^2}{1+\tau^2+\sigma^2\tau^2}\theta + \frac{\sigma\tau}{1+\tau^2+\sigma^2\tau^2}(\tau\epsilon + \eta)$. Then $s \geq s^* \equiv \gamma \geq \frac{1+\tau^2+\sigma^2\tau^2}{\sigma\tau}s^* - \frac{1+\tau^2}{\sigma\tau}\theta$, where $\gamma = \tau\epsilon + \eta$. Thus, we can rewrite:

$$G(\theta) = 1 - \Phi(A(\theta)) + \int_{-\infty}^{A(\theta)} \int_{B(\theta)}^{\infty} f(\epsilon, \gamma) d\gamma d\epsilon - 1 + \theta$$

where $A(\theta) = \frac{x^* - \theta}{\sigma}$ and $B(\theta) = \frac{1+\tau^2+\sigma^2\tau^2}{\sigma\tau}s^* - \frac{1+\tau^2}{\sigma\tau}\theta$. Differentiating under the double integral:

$$G'(\theta) = -A'(\theta)\phi(A(\theta)) + A'(\theta)\int_{B(\theta)}^{\infty} f(A(\theta, \gamma)d\gamma - B'(\theta)\int_{-\infty}^{A(\theta)} f(\epsilon, B(\theta)d\epsilon + 1$$

Writing the joint densities as products of conditionals and marginals:

$$f(\epsilon = A(\theta), \gamma) = \phi(A(\theta))f(\gamma|\epsilon = A(\theta))$$

$$f(\epsilon, \gamma = B(\theta)) = \hat{\phi}(B(\theta))f(\epsilon|\gamma = B(\theta))$$

writing $\phi(\cdot)$ to denote the standard normal PDF of ϵ, and $\hat{\phi}(\cdot)$ to denote the (non-standard) Normal PDF for γ. Finally,

$$A'(\theta) = -\frac{1}{\sigma}, B'(\theta) = -\frac{1+\tau^2}{\sigma\tau}$$

Now we can rewrite $G'(\theta)$ as:

$$\frac{1}{\sigma}\phi(A(\theta))\left[1-\int_{B(\theta)}^{\infty} f(\gamma|\epsilon = A(\theta)d\gamma\right] + \frac{1+\tau^2}{\sigma\tau}\hat{\phi}(B(\theta))\int_{-\infty}^{A(\theta)} f(\epsilon|\gamma = B(\theta)d\epsilon + 1$$

i.e. $G'(\theta) > 0$. Note that $\lim_{\theta\to\infty} G(\theta) = \infty$, and $\lim_{\theta\to-\infty} G(\theta) = -\infty$. Thus there exists a unique solution to $G(\theta) = 0$. ■

Lemma 1.2 *Fix any* x^* *Let* $\hat{\theta}$ *be defined by* $G(\hat{\theta}, x^*) = 0$ *where*

$$G(\theta, x^*) = Pr(x \geq x^*|\theta) + Pr(x < x^*, s \geq \theta + M|\theta) - 1 + \theta$$

If $\sigma < \frac{\sqrt{2\pi}}{1+\frac{\tau}{\sqrt{1+\tau^2}}}$,

1. *For each* x^*, *there is a unique* $\hat{\theta}$
2. $\frac{d\hat{\theta}}{dx^*} \in (0, \frac{1}{1+\sigma^2})$

Proof: As above, we know that $s = \frac{1+\tau^2}{1+\tau^2+\sigma^2\tau^2}\hat{\theta} + \frac{\sigma\tau}{1+\tau^2+\sigma^2\tau^2}(\tau\epsilon + \eta)$. Since $s^* = \hat{\theta} + M$, $s \geq s^* \equiv \gamma \geq \sigma\tau\hat{\theta} + \frac{1+\tau^2+\sigma^2\tau^2}{\sigma\tau}M$. Let

$$B(\hat{\theta}) = \sigma\tau\hat{\theta} + \frac{1+\tau^2+\sigma^2\tau^2}{\sigma\tau}M$$

Note that $B'(\hat{\theta}) = \sigma\tau$, and so, using the proof of Lemma 1.1,

$$\frac{\partial G(\hat{\theta}, x^*)}{\partial\hat{\theta}} = \frac{1}{\sigma}\phi(A(\hat{\theta}, x^*))\left[1-\int_{B(\hat{\theta})}^{\infty} f(\gamma|\epsilon = A(\hat{\theta}, x^*)d\gamma\right] - \sigma\tau\hat{\phi}(B(\hat{\theta}))\int_{-\infty}^{A(\hat{\theta}, x^*)} f(\epsilon|\gamma = B(\hat{\theta})d\epsilon + 1$$

where $\hat{\phi}(\cdot)$ denotes the non-standard Normal pdf of γ. Let

$$P_1 = \int_{B(\hat{\theta})}^{\infty} f(\gamma|\epsilon = A(\hat{\theta}, x^*)d\gamma$$

$$P_2 = \int_{-\infty}^{A(\hat{\theta}, x^*)} f(\epsilon|\gamma = B(\hat{\theta})d\epsilon$$

Since the variance of γ is $1+\tau^2$, $\hat{\phi}(\cdot) < \frac{1}{\sqrt{2\pi}\sqrt{1+\tau^2}}$, and $P_2 \leq 1$, clearly if $\sigma < \frac{\sqrt{2\pi}}{\frac{\tau}{\sqrt{1+\tau^2}}}$, $\frac{\partial G(\hat{\theta},x^*)}{\partial \hat{\theta}} > 0$. Similarly,

$$\frac{\partial G(\hat{\theta}, x^*)}{\partial x^*} = -\frac{1}{\sigma}\phi(A(\hat{\theta}, x^*))\,[1 - P_1] < 0$$

By the implicit function theorem

$$\frac{d\hat{\theta}(x^*)}{dx^*} = -\frac{\frac{\partial G(\hat{\theta},x^*)}{\partial x^*}}{\frac{\partial G(\hat{\theta},x^*)}{\partial \hat{\theta}}}$$

Let $Q = -\frac{\partial G(\hat{\theta},x^*)}{\partial x^*}$, where $Q > 0$. Then,

$$\frac{d\hat{\theta}(x^*)}{dx^*} = \frac{Q}{Q - \sigma\tau\hat{\phi}(\cdot)P_2 + 1}$$

It is easy to check, that when $\sigma < \frac{\sqrt{2\pi}}{1+\frac{\tau}{\sqrt{1+\tau^2}}}$

$$\frac{1}{1+\sigma^2} - \frac{d\hat{\theta}(x^*)}{dx^*} > 0$$

Since $\sigma < \frac{\sqrt{2\pi}}{1+\frac{\tau}{\sqrt{1+\tau^2}}}$ implies that $\sigma < \frac{\sqrt{2\pi}}{\frac{\tau}{\sqrt{1+\tau^2}}}$, we are done. ■

Proposition 1.4 *If* $\sigma < \frac{\sqrt{2\pi}}{1+\frac{\tau}{\sqrt{1+\tau^2}}}$, there exists a unique monotone equilibrium in Γ_{en}.

Proof: Initially, agents trade off the expected benefit of investing in period 1 against the expected benefit of retaining the option value to wait. Thus the marginal period 1 investor who receives signal x^*_{en} must satisfy:

$$Pr(\theta \geq \theta^*_{en}|x^*_{en}) - c = Pr(\theta \geq \theta^*_{en}, s \geq s^*_{en}|x^*_{en})[(1-k) - c] + Pr(\theta < \theta^*_{en}, s \geq s^*_{en}|x^*_{en})(-c) \tag{1.22}$$

Since $\theta^*_{en} = g(x^*_{en})$, we can rewrite equation (1.19) as:

$$s^*_{en} = g(x^*_{en}) + M \tag{1.23}$$

Write x for x^*_{en} and let

$$G(x) = Pr(\theta \geq \theta^*_{en}|x) - c - (1-k-c)Pr(\theta \geq \theta^*_{en}, s \geq s^*_{en}|x) + cPr(\theta < \theta^*_{en}, s \geq s^*_{en}|x)$$

Note that

$$Pr(\theta \geq \theta^*_{en}|x) = 1 - \Phi(\frac{\theta^*_{en} - \frac{x}{1+\sigma^2}}{\frac{\sigma}{\sqrt{1+\sigma^2}}})$$

Let $A(x) = \frac{\theta^*_{en} - \frac{x}{1+\sigma^2}}{\frac{\sigma}{\sqrt{1+\sigma^2}}}$. Given x,

$$s = \frac{\tau^2 x + \theta + \sigma\tau\eta}{1+\tau^2+\sigma^2\tau^2}$$

Rearranging terms, we can write this as

$$s = \frac{x}{1+\sigma^2} + \frac{\sigma}{1+\tau^2+\sigma^2\tau^2}\left[\frac{z}{\sqrt{1+\sigma^2}} + \tau\eta\right]$$

where $z = \frac{\theta - \frac{x}{1+\sigma^2}}{\frac{\sigma}{\sqrt{1+\sigma^2}}}$. is distributed $N(0,1)$ conditional on x. Let $\gamma = \frac{z}{\sqrt{1+\sigma^2}} + \tau\eta$. Then, $s \geq s^*$ is equivalent to

$$\gamma \geq \frac{1+\tau^2+\sigma^2\tau^2}{\sqrt{1+\sigma^2}}A(x) + \tau\sqrt{1+\tau^2+\sigma^2\tau^2}\Phi^{-1}(\frac{c}{1-k})$$

Let

$$B(x) = \frac{1+\tau^2+\sigma^2\tau^2}{\sqrt{1+\sigma^2}}A(x) + \tau\sqrt{1+\tau^2+\sigma^2\tau^2}\Phi^{-1}(\frac{c}{1-k})$$

Now, we may rewrite:

$$G(x) = 1 - \Phi(A(x)) - (1-k-c)Pr(z \geq A(x), \gamma \geq B(x)) + cPr(z < A(x), \gamma \geq B(x))$$

Differentiating under the double integral and rearranging we get:

$$G'(x) = -\phi(A(x))A'(x)\left[1-(1-k)P_1\right] + B'(x)\hat{\phi}(B(x))\left[(1-k)P_2 - c\right]$$

where by $\hat{\phi}(\cdot)$ we denote the non-standard normal density of γ, and P_1 and P_2 are defined as follows:

$$P_1 = \int_{B(x)}^{\infty} f(\gamma|z = A(x))d\gamma$$
$$P_2 = \int_{A(x)}^{\infty} f(z|\gamma = B(x))dz$$

Using standard formulae for computing conditional distributions of Normal random variables (see, for example, Greene 1996), we know that:

$$z|\gamma = B(x) \sim N(A(x) + \frac{\tau\sqrt{1+\sigma^2}}{\sqrt{1+\tau^2+\sigma^2\tau^2}}\Phi^{-1}(\frac{c}{1-k}), \frac{\tau^2(1+\sigma^2)}{1+\tau^2+\sigma^2\tau^2})$$

Thus,

$$P_2 = \int_{A(x)}^{\infty} f(z|\gamma = B(x))dz = \frac{c}{1-k}$$

and therefore

$$G'(x) = -\phi(A(x))A'(x)[1 - (1-k)P_1]$$

Under the conditions of the theorem $A'(x) < 0$ and therefore the proof is complete. ∎

Lemma 1.3. *Fix any $\hat{\theta} \in [0, 1]$. Let $\hat{x}$ be defined by $H(\hat{x}, \hat{\theta}) = 0$, where*

$$H(\hat{\theta}, x) = Pr(\theta \geq \hat{\theta}|x) - c - Pr(s \geq \hat{\theta} + M, \theta \geq \hat{\theta}|x)(1-k-c) + Pr(s \geq \hat{\theta} + M, \theta < \hat{\theta}|x)(c)$$

Then

$$\frac{d\hat{x}(\hat{\theta})}{d\hat{\theta}} > 0$$

Proof: Using the proof of Proposition 1.4 above, we can write

$$H(\hat{\theta}, x) = 1 - \Phi(A(x, \hat{\theta})) - (1-k-c)Pr(z \geq A(x, \hat{\theta}), \gamma \geq B(x, \hat{\theta})) + cPr(z < A(x, \hat{\theta}), \gamma \geq B(x, \hat{\theta}))$$

where

$$A(x,\hat{\theta}) = \frac{\hat{\theta} - \frac{x}{1+\sigma^2}}{\frac{\sigma}{\sqrt{1+\sigma^2}}}$$

$$B(x,\hat{\theta}) = \frac{1+\tau^2+\sigma^2\tau^2}{\sqrt{1+\sigma^2}}A(x,\hat{\theta}) + \tau\sqrt{1+\tau^2+\sigma^2\tau^2}\Phi^{-1}(\frac{c}{1-k})$$

Note that

$$\frac{\partial A}{\partial x} < 0, \quad \frac{\partial A}{\partial \hat{\theta}} > 0$$

Then, using the same analysis as above, we have:

$$\frac{\partial H}{\partial x} = -\phi(A(x))\frac{\partial A}{\partial x}[1-(1-k)P_1]$$

where $P_1 = \int_{B(x,\hat{\theta})}^{\infty} f(\gamma|z = A(x,\hat{\theta}))d\gamma$, where $z|x \sim N(0,1)$ and $\gamma = \frac{1}{\sqrt{1+\sigma^2}}z + \tau\eta$ as above.

Similarly,

$$\frac{\partial H}{\partial \hat{\theta}} = -\phi(A(x))\frac{\partial A}{\partial \hat{\theta}}[1-(1-k)P_1]$$

Thus, $\frac{\partial H}{\partial x}$ and $\frac{\partial H}{\partial \hat{\theta}}$ are of opposite sign and bounded away from zero. Thus, by the implicit function theorem:$\frac{d\hat{x}(\hat{\theta})}{d\hat{\theta}} > 0$. ■

Proposition 1.6 *There exist monotone decision rules parameterized by the threshold pair $(\hat{x},\hat{s})$ such that $\forall(\sigma,\tau,c,k)$, the induced state- variable threshold $\hat{\theta}(\hat{x},\hat{s})$ satisfies: $\hat{\theta} \leq \theta^*_{en}$.*

Proof: We construct the game $\hat{\Gamma}_{en}$ in which payoffs are still given by 1.1 through 1.3, but the condition for the success of the project $p \geq 1-\theta$ is replaced by the condition $\theta \geq 0$. This eliminates the payoff externality.

In this game, agents who arrive at t_2 with the option to invest will choose to invest if $s \geq \hat{s}$ where

$$Pr(\theta \geq 0|\hat{s}) = \frac{c}{1-k}$$

i.e.,

$$\hat{s} = \frac{\sigma\tau}{\sqrt{1+\tau^2+\sigma^2\tau^2}}\Phi^{-1}(\frac{c}{1-k})$$

Agents at t_1 will choose to invest if $x \geq \hat{x}$ where:

$$Pr(\theta \geq 0|\hat{x}) - c = Pr(\theta \geq 0, s \geq \hat{s}|\hat{x})(1-k-c) + Pr(\theta < 0, s \geq \hat{s}|\hat{x})(-c)$$

Using methods similar to those given above, we can rewrite this to be:

$$0 = H(\hat{x},\hat{\theta}) = 1-\Phi(A(\hat{x},\hat{\theta}))-(1-k-c)Pr(z \geq A(\hat{x},\hat{\theta}),\gamma \geq B(\hat{x},\hat{\theta}))+cPr(z < A(\hat{x},\hat{\theta}),\gamma \geq B(\hat{x},\hat{\theta}))$$

where z, γ, $A(\cdot)$, and $B(\cdot)$ are defined as in the proof of Proposition 1.4 and $\hat{\theta} = 0$. Now, appealing to Lemma 1.3, and because $\theta^*_{en} \geq 0 = \hat{\theta}$, we conclude that $\hat{x} \leq x^*_{en}$. In addition, it is clear that $\hat{s} \leq s^*_{en}$.

Now, returning to Γ_{en}, let a social planner force agents to play according to $(\hat{x},\hat{s})$. Let $\hat{\theta}$ be the level of θ above which $p(\theta) \geq 1-\theta$ under $(\hat{x},\hat{s})$. Since $\hat{x} \leq x^*_{en}$ and $\hat{s} \leq s^*_{en}$, it follows that $\hat{\theta} \leq \theta^*_{en}$. ■

Chapter 2

Social Learning with Payoff Complementarities

Observers of financial market booms and busts, both casual and experienced, will often note that behavior in the market is characterized by an excess of optimism or pessimism. There appears to be a tendency for market participants to "jump on the bandwagon."[1] They get so carried away by the decisions of others around them that they simply imitate their predecessors, paying no attention to any information about fundamentals that they may receive, or making no effort to gather such information. It is as though they were all moving in a herd. When the market tanks, traders tend to exit the market quicker then

[1]Stephen Morris provided invaluable guidance for this project. I also gratefully acknowledge useful conversations with David Pearce, Ben Polak, Giuseppe Moscarini, Dirk Bergemann, Hyun Song Shin, Jonathan Levin, Felix Kubler, and participants at the Yale seminar on game theory. Part of this project was supported by the NSF and the Cowles Foundation. A previous version of this chapter was circulated under the title: "Learning, Signalling, and Coordinating: A Rational Theory of "Irrational Exuberance.""

they would have if they took into account the fundamentals of the economy. When the market booms, traders get excessively optimistic compared to levels that are justified by the underlying fundamentals. This, perhaps, is what Alan Greenspan was referring to in his now famous "irrational exuberance" speech of December 5, 1996, at the heart of the stock market boom of the 1990s. Given the pertinence of such market herd behavior in both good times and bad, there is clearly a need to analyze the probem carefully. To begin, let us try to separate the central stylized characteristics of situations such as stock market booms and panics, currency crises, or bank runs.

The salient features of such situations are as follows. A number of market participants are called upon to make similar decisions (buy/sell, long/short, withdraw/remain etc.) at about the same time. Since they are all in the same market, they can observe each other's actions. Each participant has non-trivial private information (ideas, intuition, acquired knowledge) about the fundamentals of the situation. These are, after all, educated financial traders. In order to make their decisions, partipants may use either their private information, or the public information generated by observing their predecessors' actions, or both. However, participants also have to worry about their successors, because each person's payoff depends upon the actions of everybody else. Even if a few predecessors have chosen to go short on the market, a trader may worry that his successors will not, thus preventing a market downturn and leaving him stranded. In short, there are *strategic complementarities*[2]. The observation that agents seem to herd, then, amounts to noting that later agents

[2] A term coined by Bulow, Geanakoplos, and Klemperer (1985), otherwise refered to as positive payoff externalities,network externalities, supermodularities etc. in various specific contexts.

pay “too much” attention to the choices of their predecessors and “too little” attention to their own private information.

In this paper, we propose a game theoretic model to study such situations. There are n risk neutral agents in our model who act in an exogenous sequence and choose either to invest or not. There are two states of the world, a state that is conducive to investment, and a state that is not. Agents receive signals that are informative about the state of the world in a stochastic sense: very roughly, higher signals increase the likelihood of the state being good. Conditional on the state, the signal generating process is independent and identical across agents. At the point when they have to choose their actions, agents are able to observe the choices of their predecesors and their own private signals (but not the signals of their predecessors). Finally, there are strong strategic complementarities. Investment leads to a positive net payoff only if the state is good and all other agents also choose to invest. Otherwise, it generates negative net return. Not investing costs and pays nothing, independent of the state of the world.

In this set up, we show that it is inevitable that agents shall become progressively more optimistic as more and more predecessors choose to invest (Proposition 2.5). This is natural and to be expected. However, it turns out that such optimism can take excessive forms, depending on the properties of the information system of the game. If the information system has the property that likelihood ratios for individual agents are bounded (i.e. agents can exhibit only limited amounts of personal skepticism based upon their available information), then agents may literally start to imitate others and ignore their own payoff relevant information (*strong herd behavior*). Indeed, under these circumstances, such “irrational exuberance" is the *only* outcome of rational behavior (Proposition 2.6). We are able to tightly characterize the informational requirements that would lead to such strong herd behavior for linear information systems (Propositions 2.7 and 2.8). However, we also show that if the information system is rich enough to allow agents to exhibit unbounded personal skepticism, i.e., possesses the unbounded likelihood ratio property, such extreme forms of exuberance are ruled out. The exclusion of strong herd behavior does not mean that overoptimism vanishes. In fact it is quite possible that agents do not ignore their private information but are overoptimistic in comparison to the case where information is aggregated efficiently in the market. We call such phenomena *weak herd behavior* and lay down informational conditions necessary and sufficient for weak herding to occur. We show that for the important class of Gaussian information systems weak herding occurs with positive probability.

It is apparent from the structure of the model that if players do not exhibit strong herd behavior, it shall be harder and harder to persuade the first player to invest as the number of agents gets progressively larger. In order to address these concerns, we characterize the informational requirements that shall create the possibility of coordinated investment even in games with unbounded likelihood ratios when the number of players is arbitrarily large (Propositions 2.9 and 2.10).

The study of situations where people's decisions are influenced by those of others around them is not new. Stylized versions of situations similar to ours have been extensively studied in the literature. The pioneering papers are by Banerjee and by Bikhchandani, Hershleifer, and Welch, both in 1992. Variations, generalizations, and applications have also been studied. Lee (1993) provides conditions on the action choices of agents in a generalized herding model that guarantee herding. Gul and Lundholm (1995), Chamley and Gale (1994), and Chari and Kehoe (2000) examine similar models but allow the order of action to be endogenous. Froot, Scharfstein, and Stein (1992), Chari and Kehoe (1997), Avery and Zemsky (1998), and Lee (1998), among others, apply herding models to study various financial situations. For a recent selective survey of this literature see Bikhchandani, Hershleifer, and Welch (1998). However, all these models are characterized by a common feature: *individual payoffs are unaffected by the actions of others.* The only externality present in these models is an informational one. Agents are concerned about each other's choices only to the extent that prior actions generate information about the state of the world. There are no strategic complementarities. Therefore, agents in these models exhibit only backward-looking behavior. As a result, it becomes much harder to apply these models to real financial situations.

In many settings, in addition to the informational externality, it is essential to incorporate direct payoff externalities. The situations discussed above are but a few of a plethora of possible examples. When payoff complementarities exist, agents must be concerned not only with the actions of their predecessors but also with those of their successors. Thus, in situations such as these, agents would exhibit both backward-looking (learning) and forward-looking (strategic) behavior. This strategic component complicates the arguments in the models of Banerjee, and Bikhchandani, Hershliefer, and Welch. Games with payoff complementarities that capture strategic behavior by agents have been studied in the literature with the goal of explaining situations similar to the ones above. For example, Obstfeld

(1986), Cole and Kehoe (1996), and Morris and Shin (1998) model currency crises in various degrees as static coordination games under uncertainty with payoff complementarities. However, the static nature of these games excludes the learning behavior seen in sequential action models. Finally, the interaction of sequential action with strategic complementarities creates signalling behavior in our model, an effect that is missing from both herding models and static coordination games. Agents are concerned about the signals that their action choices send to their successors. We are, therefore, able to capture the learning behavior of herding models, the strategic behavior of static coordination games, and the signalling behavior absent from both of these previous classes of models in one unified framework.

Our analysis also helps to understand better the way in which information plays a role in creating strong herding in markets. We provide two versions of the model featuring qualitatively different information systems, one in which the private information of agents is rich enough to allow them to exercise unlimited personal skepticism (unbounded likelihood ratios) and one in which this is not possible. We demonstrate that that latter is necessary (but not sufficient) for strong herd behavior and characterize the precise conditions under which strong herding takes place under additional assumptions. This provides a foundation upon which to build a theory of optimal information structure in such games, paving the way for mechanism design in situations where market participants must be prevented from herding or persuaded to herd upon risky but socially productive alternatives.

In an important recent contribution, Smith and Sorensen (2000) provide similar characterizations of the informational prerequisites for herd behavior. Their model generalizes the traditional herding literature by allowing for heterogeneous preferences and makes explicit the conditions under which Bayesian learning may be incomplete as opposed to confounded. Our setting retains the identical preferences of traditional herding models, but adds in payoff complementarities with the goal of capturing the other relevant strategic aspects of a market boom or bust, thereby unifying the literature on herding with static models of coordination.

Choi (1997) builds network externalities into a model of sequential action under uncertainty. His model is one of strategic technology choice by firms. Firms choose between two competing technologies with unknown values. It is beneficial for firms to choose the technology that shall be adopted by most other firms because of network externalities. While this model is ostensibly similar to ours, it is significantly different in spirit. First, once a technology is used by a firm, its true value becomes common knowledge amongst participants in the game. Thus, after the first player has chosen a technology, the rest of the game is effectively one of complete information. Second, since firms receive no private signals about the alternative technologies, there is no private information in Choi's model. Herding happens purely due to the network effect and risk aversion. Herding, in the traditional sense, is simply the phenomenon by which followers may pregressively (suddenly or gradually) disregard their private information in favour of already available public signals. A proper analysis of herding requires a fully-specified model that explicitly distinguishes between private and public information. Our model provides such a framework.

Two other recent papers that contain elements of strategic complementarities and herding are Jeitschko and Taylor (2001) and Corsetti, Dasgupta, Morris, and Shin (2000). In the former, agents play pairwise coordination games due to random matching, but learning is not "social" since agents observe only their own private histories. In the latter, a sequential coordination game is set up to explore the influence of a large trader in a model of speculative currency attacks with private information. When the large trader is arbitrarily better informed in comparison to the rest of the market, smaller traders exhibit strong herd behavior in the sense of our model.

The rest of the paper is organized as follows. In section 2.1 we lay out the model. Section 2.2 demonstrates two important properties of the equilibria of this game. Section 2.3 defines strong and weak herding in our setting and provides informational requirements for their occurence. In section 2.4 we characterize the informational requirements to ensure the possibility of coordination in numerous-player versions of our game. Section 2.5 discusses and concludes.

2.1 The Model

2.1.1 The Structure of the Game

There are n agents who choose whether to invest (I) or not (N). We write $a_i \in A_i = \{I, N\}$ for $i = 1, 2, ..., n$, and $A = \times_{i=1}^{n} A_i$. There are two states of the world: a state G which is good for investment, and a state B which is bad for investment. Nature selects which state of the world occurs. Investing is risky. For an agent to get positive net return (of 1) from investing, it is necessary that the state is conducive to investment, i.e., G, and that all other agents also choose to invest. If even one of these conditions are violated, then investment generates negative net return of $-c$. Not investing is safe. It generates a constant return of 0 independent of the actions of other agents and the state of the world. Agents' payoffs can thus be represented by the mappings $(u_i : \{G, B\} \times A \to \mathbb{R})_{i=1}^{n}$ defined for each i by:

$$u_i(G, a_i, a_{-i}) = \begin{cases} 1 & \text{when } a_i = I \text{ and } a_j = I \text{ for all } j \neq i, \\ -c & \text{when } a_i = I \text{ and } a_j = N \text{ for some } j \neq i, \\ 0 & \text{when } a_i = N \end{cases}$$

$$u_i(B, a_i, a_{-i}) = \begin{cases} -c & \text{when } a_i = I \\ 0 & \text{when } a_i = N \end{cases}$$

Agents act sequentially, in the order $1, 2, ..., n$. Each agent observes the actions of those who have preceded her. In addition, each agent receives a private signal (her type), which summarizes her private information about the state of the world. In particular, agent i receives signal $s_i \in S = [\underline{s}, \bar{s}] \subset \mathbb{R}$ or $S_i = \mathbb{R}$ for all i.[3] Conditional on the state, the signals are *independent* and *identically distributed.* For each i, s_i is distributed according to some *continuous, state-dependent* density, $f(.)$. We require that in state G, private signals have full support.[4] $f(.)$ satisfies the following (strict) *monotone likelihood ratio property* (MLRP): $\frac{f(s|B)}{f(s|G)}$ is *strictly* decreasing in s. We shall sometimes refer to these stochastic processes as making up the *information system* for the game, and write $f = \{f(.|G), f(.|B)\}$ to denote it.

Agents share a common prior over the state of the world: $Pr(G) = 1 - Pr(B) = \pi \in [0, 1]$. They are expected utility maximizers.

[3]For notational convenience, we shall use $\underline{s}$ and $\bar{s}$ below to denote lower and upper bounds for S even when $S = \mathbb{R}$, assuming implicitly that $\bar{s} = \infty$ and $\underline{s} = -\infty$ when this is the case.

[4]This is done to eliminate the trivial case where an agent may discover for sure that the state is B. In such a case, there is no strategic content left in the game.

For future reference, we shall denote the game we have just described by $\Gamma(n)$, where the argument refers to the number of players in the game. Unless otherwise stated, we shall assume that $n \in Z_{++}$, i.e. the number of players is finite. In what follows, we consider Weak Perfect Bayesian Equilibria of $\Gamma(n)$ which are defined below. We preface our analysis by some brief remarks about the information system.

2.1.2 A Note on Likelihood Ratios

As we have noted above, the signals in $\Gamma(n)$ can be generated either from some closed subinterval of $\mathbb{R}$, or from $\mathbb{R}$itself. This distinction is made to explicitly distinguish between two versions of the game: the case with *bounded likelihood ratios* and the case with *unbounded likelihood ratios*.[5] We denote the likelihood ratio by $r(s) = \frac{f(s|B)}{f(s|G)}$. The full support assumption on $f(s|G)$ ensures that $r(s)$ is well defined on S. When $S = [\underline{s}, \bar{s}] \subset \mathbb{R}$, the MLRP property and the boundedness of probability density functions implies that there exist bounds $B \geq 0$ and $T < \infty$ such that $r(s) \in [B, T]$ for $s \in S$. $B = 0$ when $f(s|B)$ is not full support. When $S = \mathbb{R}$, the MLRP property implies that $r(s)$ is unbounded above and asymptotes to 0 below. Conversely, when $r(s)$ is unbounded above or below, the boundedness of probability density functions that $S = \mathbb{R}$

Intuitively, the case with unbounded likelihood ratios can be thought to be the version of $\Gamma(n)$ when players exhibit unbounded personal skepticism, i.e. may observe some private information that reverses any level of optimism they may have enjoyed ex ante. The case with bounded likelihood ratios is the reverse: players are only boundedly skeptical. A

[5]While the distinction is formal, i.e., represents alternative modelling strategies, it is useful in classifying the results of the game.

certain level of ex ante optimism cannot be reversed by any private information, however discouraging. The properties of $r(.)$ shall turn out to be crucial to our analysis of $\Gamma(n)$, and we shall return to this point again below.

2.1.3 Possible Strategy Profiles

How does an agent, say i, decide whether to invest or not? When agent i is called upon to act, she knows only what her predecessors have done and the value of her own signal. Hence, her strategies take the form of mappings from her predecessors' actions and her own private signal to her action set. Formally, for each i, $\sigma_i : (\times_{j<i} A_j) \times S \rightarrow \{I, N\}$. Given this notation, we first provide a useful definition:

Definition 2.1. *Player i follows a* trigger strategy *in $\Gamma(n)$ if she chooses her actions according to the map*

$$\sigma_i(s_i, (a_j)_{j<i}) = \begin{cases} I & \textit{when } s_i \geq t_i \textit{ and } a_j = I \ \forall j < i \\ N & \textit{otherwise} \end{cases}$$

for some $t_i \in \Re$ where $\Re = \mathbb{R} \cup \{-\infty\} \cup \{\infty\}$, the augmented real line.

We call t_i player i's trigger. An equilibrium in which each player follows a trigger strategy is called a *trigger equilibrium.*

It is important to note that while players' *signals* are drawn from some subset of the real numbers $\mathbb{R}$, *triggers* are drawn from the augmented real line, $\Re$, because players may follow strategies of "always invest" (corresponding to a trigger of $-\infty$ if $S = \mathbb{R}$) or of "never invest" (corresponding to a trigger of ∞ if $S = \mathbb{R}$).

Given the payoff complementarities, it is clear that in any equilibrium if $a_j = N$ for some $j < i$, $\sigma_i((a_j)_{j<i}, s_i) = N$ for all s_i, since investing is a strictly dominated action. Thus, agent i's decision problem is interesting only in the instance that $a_j = I$ for all $j < i$. In this instance, since we know by definition $(a_j)_{j<i} = (I, ..., I)$, an agent's equilibrium strategy is formally just some function of her private signal. Thus, for notational convenience, we can now drop the explicit dependence of the strategies σ_i on the observed history of actions $(a_j)_{j<i}$. When the argument is suppressed, it is tacitly assumed that $a_j = I$ for all $j < i$.

If she observes investment by all her predecessors, then agent i has some beliefs (posterior) about the state of the world, say $\pi_i \in [0, 1]$. Her expected utility from investing depends upon this posterior belief, her private signal, and the strategies of her successors. Formally, $EU_i(\pi_i, s_i, (\sigma_j)_{j>i}) = (1)P_i + (-c)(1 - P_i)$, where

$$\begin{aligned}
P_i &= Pr(G, (\sigma_j(s_j) = I)_{j>i} | s_i) \\
&= \frac{Pr(G)Pr((\sigma_j(s_j) = I)_{j>i}, s_i | G)}{Pr(G)Pr(s_i|G) + Pr(B)Pr(s_i|B)} \\
&= \frac{\pi_i Pr((\sigma_j(s_j) = I)_{j>i} | G) f(s_i|G)}{\pi_i f(s_i|G) + (1 - \pi_i) f(s_i|B)} \\
&= \frac{\pi_i Pr((\sigma_j(s_j) = I)_{j>i} | G)}{\pi_i + (1 - \pi_i)\frac{f(s_i|B)}{f(s_i|G)}}
\end{aligned}$$

where the third equality follows from the conditional independence of the signals.[6] Given this notation, we define a Weak Perfect Bayesian Equilibrium for $\Gamma(n)$.

Definition 2.2. *A* Weak Perfect Bayesian Equilibrium *of $\Gamma(n)$ is a tuple of strategies $(\sigma_1, ..., \sigma_n)$ and a tuple of posterior beliefs $(\pi_1, ..., \pi_n)$ where for each i, $\pi_i : (a_j)_{j<i} \to [0, 1]$*

[6] Note that we could divide by $f(s|G)$ above because of our assumption of full support in state G.

which satisfy the following conditions:

1. *Given π_i, σ_i is a best response to σ_{-i} after every possible history.*

2. *If the observed history of play can happen with positive probability in the equilibrium path prescribed by σ, then π_i is derived from the original priors by Bayesian updating. If not, then π_i is any member of* $[0, 1]$.

In the specific setting of our model, these conditions translate into the following:

1. For each i, if $a_j = N$ for any $j < i$, then $\sigma_i = N$. If $a_j = I$ for all $j < i$, then $\sigma_i = I$ if and only if

$$\frac{\pi_i Pr((\sigma_j(s_j) = I)_{j>i}|G)}{\pi_i + (1-\pi_i)\frac{f(s_i|B)}{f(s_i|G)}}(1+c) - c \geq 0$$

2. If $Pr(\sigma_j(s_j) = I \ \forall j < i) > 0$, then π_i is obtained by updating $\pi_1 = \pi$ using Bayes' rule. If not, then π_i is *any* member of the interval $[0, 1]$.

It turns out that in any Weak Pefect Bayesian Equilibrium, each player in $\Gamma(n)$ will follow a simple trigger strategy as we demonstrate below.

Proposition 2.1. *Any Weak Perfect Bayesian Equilibrium of* $\Gamma(n)$ *is a trigger equilibrium.*

Proof: Let $(\sigma_1, ..., \sigma_n)$ be any WPBE of $\Gamma(n)$. We shall show that each player follows a trigger strategy. We already know that each player i, conditional on having observed a history of investment, follows $\sigma_i = I$ if and only if

$$EU_i = \frac{\pi_i Pr((\sigma_j(s_j) = I)_{j>i}|G)}{\pi_i + (1-\pi_i)\frac{f(s_i|B)}{f(s_i|G)}}(1+c) - c \geq 0$$

Since $EU_i(s_i)$ is clearly increasing and continuous in s_i, player i will adopt then invest only if $s_i \geq t_i$ where t_i is defined by $EU_i(t_i) = 0$. Upon not observing a history of investment, player i will not invest for sure. Thus, player i follows:

$$\sigma_i(s_i, (a_j)_{j<i}) = \begin{cases} I & \text{when } s_i \geq t_i \text{ and } a_j = I \ \forall j < i \\ N & \text{otherwise} \end{cases}$$

which is exactly a trigger strategy as defined above. But this immediately implies that any WPBE of $\Gamma(n)$ is a trigger equilibrium. ■

Proposition 2.1 allows us to restrict our attention to trigger equilibria. Thus, the Weak Perfect Bayesian Equilibria of $\Gamma(n)$ are n-tuples, $(t_1, ..., t_n) \in \Re^n$ where player i follows a trigger strategy with trigger t_i. Henceforth, we shall refer to equilibria of $\Gamma(n)$ simply as trigger equilibria.

2.1.4 The Existence of Trigger Equilibria

Case 1: Bounded Signal Support

To demonstrate the existence of pure strategy equilibria, we define the best response function:

Definition 2.3. *Consider a set of triggers $t \in [\underline{s}, \bar{s}]^n$. Denote the* best response mapping *by $\beta : [\underline{s}, \bar{s}]^n \to [\underline{s}, \bar{s}]^n$ and the ith component of $\beta(t)$ by $\beta_i(t)$. Then, $\beta_i(t) = r^{-1}(E_i)$, where*

$$E_i = \frac{\pi}{1-\pi}\left[\frac{1+c}{c}\prod_{j>i} Pr(s \geq t_j|G) - 1\right]\prod_{j<i}\frac{Pr(s \geq t_j|G)}{Pr(s \geq t_j|B)}$$

if $E_i \in [B, T]$. If $E_i < B$, $\beta_i(t) = \bar{s}$ and if $E_i > T$, $\beta_i(t) = \underline{s}$.

The strict MLRP property implies that $\beta(.)$ is single valued, and the continuity of $f(.|G)$ and $f(.|B)$ imply that $\beta(.)$ is continuous. Thus, $\beta(.)$ is a continuous function that maps $[\underline{s}, \bar{s}]^n$, a compact and convex set, into itself. Therefore, by Brouwer's Fixed Point Theorem, there is a $t^* \in [\underline{s}, \bar{s}]^n$ such that $\beta(t^*) = t^*$. Thus a trigger equilibrium of $\Gamma(n)$ exists.

However, since the argument above admits the possibility that t^* lies on the boundary of S, this leaves open the possibility that the only equilibrium of $\Gamma(n)$ is the trivial equilibrium in which $t_j = \bar{s}$ for all j, and thus nobody invests in equilibrium.

In fact, the extreme form of strategic complementarities embodied in $\Gamma(n)$ ensures that there is always a trivial trigger strategy equilibrium in which nobody ever invests. Let us construct such an equilibrium. Consider the problem of a player, say i, with posterior belief π_i upon observing investment by her predecessors, who is sure that all her successors (if any) will invest. Let $t(\pi_i)$ be the trigger selected by such a player. Clearly, $t(\pi_i)$ is decreasing in π_i. Let π^* be defined by $Pr(s \geq t(\pi^*)|G) = \frac{c}{1+c}$. Note that for some $\epsilon \in (0, \pi^*]$, $Pr(s \geq t(\pi^* - \epsilon)|G) < \frac{c}{1+c}$.

Now consider the problem of player $i-1$ with posterior beliefs π_{i-1}, who observes signal s_{i-1}. She knows that upon observing investment by her, player i will have posterior beliefs $\pi^* - \epsilon$. She will certainly not invest if $Pr(G, s \geq t(\pi^* - \epsilon)|s_{i-1}) < \frac{c}{1+c}$, i.e., if she does not assign sufficient probability to the event that the state is good and (at least) her immediate successor invests (if her immediate successor doesn't invest, it matters not to player $i-1$ what later player do). But notice that

$$Pr(G, s \geq t(\pi^* - \epsilon)|s) = \frac{\pi_{i-1} Pr(s \geq t(\pi^* - \epsilon)|G)}{\pi_{i-1} + (1 - \pi_{i-1}) \frac{f(s|B)}{f(s|G)}} \leq Pr(s \geq t(\pi^* - \epsilon)|G) < \frac{c}{1+c}$$

where the first inequality corresponds to the case where $s_{i-1} = \bar{s}$. This means that if player

$i-1$ knew that upon observing her invest player i would have beliefs $\pi^* - \epsilon$, then she would assign probability strictly less than $\frac{c}{1+c}$ to the event that the state is good and that player i will invest, *regardless of her own prior belief.* So, player $i-1$ will not invest.

Now it is easy to see that the strategy set $(\bar{s}, \bar{s}, ..., \bar{s})$ is a Perfect Bayesian Equilibrium if upon seeing investment by a predecessor each player has probabilistic beliefs given by $\pi^* - \epsilon$ for any $\epsilon \in (0, \pi^*]$. Given these beliefs off the equilibrium path, the first player will never find it profitable to deviate from her equilibrium strategy to "never invest." This is because even if she believed that players $3, ..., n$ would invest for sure conditional upon investment by their predecessors, she would still assign too low a probability to the event that the state is good and that player 2 (with beliefs given by $\pi^* - \epsilon$ upon seeing player 1 invest) will invest. Thus, the first player will not invest, and so all her successors will set their triggers optimally to infinity (i.e., never invest) [7].

However, this is not a very interesting equilibrium, and it is natural to wonder if there is a non-trivial trigger equilibrium of $\Gamma(n)$. In such an equilibrium players would choose interior triggers, and therefore allow for the possibility of coordinated investment. Formally, we refer to these equilibria as investment equilibria.

Definition 2.4. *Trigger equilibrium* $(t_1, ..., t_n) \in \Re^n$ *of* $\Gamma(n)$ *is an* investment equilibrium *if* $t_j < \bar{s}$ *for all* $j = 1, ..., n$.

Investment equilibria allow for the possibility of coordinated investment.

In order to ensure the existence of investment equilibria, we must lay down some suffi-

[7] These out of equilibrium beliefs are sufficient but not necessary to support the $(\bar{s}, ..., \bar{s})$ strategy profile as an equilibrium.

cient conditions on the information system of the game. This requires a preamble.

Consider the following situation. Players 1 through $n-1$ choose to invest blindly, i.e., $t_1 = ... = t_{n-1} = \underline{s}$. Consider player n's best response to such strategies (upon observing investment by all her predecessors), t_n. Given our definition of the best response function above,

In other words,

$$t_n = r^{-1}\left[\frac{\pi}{1-\pi}\frac{1}{c}\right]$$

This uniquely defines t_n in terms of the parameters (c), the prior (π), and the information system (f). We write $t_n = U_n(c, \pi, f)$, or U_n for short. Further, we require, that

$$Pr(s \geq U_n(c, f)|G) > \frac{c}{1+c}$$

Call this *Condition* Ψ_n.

Now consider the situation where players 1 through $n-2$ choose to invest blindly, i.e., $t_1 = ... = t_{n-2} = \underline{s}$, while player n plays according to trigger U_n. Now, player $n-1$ will choose her trigger according to:

$$t_{n-1} = r^{-1}\left[\frac{\pi}{1-\pi}\left(\frac{1+c}{c}Pr(s_n \geq U_n|G) - 1\right)\right]$$

Note that Condition Ψ_n ensures that t_{n-1} is well defined in terms of the parameters, and we write $t_{n-1} = U_{n-1}(c, \pi, f)$ or U_{n-1} for short. Now we require that

$$Pr(s \geq U_{n-1}|G)Pr(s \geq U_n|G) > \frac{c}{1+c}$$

Call this *Condition* Ψ_{n-1}.

We continue iteratively in this way, defining $U_{n-2}(c, \pi, f), ..., U_1(c, \pi, f)$, and conditions

$\Psi_{n-2}, ..., \Psi_1$. Now we are ready to define the useful properties of the information system promised above.

Definition 2.5. *Let* $U_1(c,\pi,f),...,U_n(c,\pi,f)$ *and the conditions* $\Psi_1,...,\Psi_n$ *be defined as above. We say* Property Ψ *holds if conditions* Ψ_2 *through* Ψ_n *hold simultaneously, i.e., if*

$$\prod_{j=2}^{n} \int_{U_j(c,\pi,f)}^{\infty} f(x|G)dx > \frac{c}{1+c}$$

Given c and π, the property defined above imposes a restriction on the *information system,* i.e., on the stochastic process generating the private information processes of the agents. Stated in words, Property Ψ simply says that in the *good* state, the information system must be *reliable* enough, i.e., generate signals above predetermined levels (the U_j's) with sufficient probability. This property turns out to be useful for the case with unbounded likelihood ratios.[8] However, for the present case with bounded likelihood ratios, we need a slightly stronger condition. Finally, therefore, a last definition.

Definition 2.6. *We say that* f *satisfies* Property $\Psi+$ *in* $\Gamma(n)$ *if it satisfies Property* Ψ *and if*

$$\frac{\pi}{1-\pi}\left[\frac{1+c}{c}\prod_{j>1} Pr(s \geq U_j|G) - 1\right] > B$$

Then the following result holds:

Proposition 2.2. *When Property* $\Psi+$ *holds, there exist* $L, U \in S$ *with* $L < U$ *and* $U < \bar{s}$ *such that for* $t \in [L,U]$, $\beta(t) \in [L,U]$.

---

[8] As we shall see below, Property Ψ turns out to be sufficient to guarantee existence of investment equilibria in $\Gamma(n)$ when $S = \mathbb{R}$

Proof: Let $L = (L_1, L_2, ..., L_n)$, where $L_i = \beta((U_1, ..., U_{i-1}), (L_{i+1}, ..., Ln))$ where U_i is defined as above. Let $U = (U_1, U_2, ..., U_n)$. Clearly, $L < U$. Let $t \in [L, U]$. Since Property $\Psi+$ holds, $U_1 < \bar{s}$, thus t is interior in $[\underline{s}, \bar{s}]$ and $\beta(t) \in (B, T)$. Thus,

$$\beta_i(t) = r^{-1}\left(\frac{\pi}{1-\pi}\left[\frac{1+c}{c}\prod_{j>i} Pr(s \geq t_j|G) - 1\right]\prod_{j<i}\frac{Pr(s \geq t_j|G)}{Pr(s \geq t_j|B)}\right)$$

Notice that

$$L_i = r^{-1}\left(\frac{\pi}{1-\pi}\left[\frac{1+c}{c}\prod_{j>i} Pr(s \geq L_j|G) - 1\right]\prod_{j<i}\frac{Pr(s \geq U_j|G)}{Pr(s \geq U_j|B)}\right)$$

and since $L \leq t \leq U$, $\beta_i(t) \geq L_i$. Similarly, notice that

$$U_i = r^{-1}\left(\frac{\pi}{1-\pi}\left[\frac{1+c}{c}\prod_{j>i} Pr(s \geq U_j|G) - 1\right]\right)$$

and thus $\beta_i(t) \leq U_i$. ■

Corollary 2.1. *When Property $\Psi+$ holds, there is an investment equilibrium in $\Gamma(n)$ with bounded signal support.*

Proof: When Property $\Psi+$ holds, there exists a compact and convex set $[L, U] \subset [\underline{s}, \bar{s}]$ with $L < \bar{s}$ such that $\beta(t) \in [L, U]$ for all $t \in [L, U]$. Observation of the best response mapping establishes immediately that $\beta(.)$ is continuous on $[L, U]$. Thus, by Brouwer's Fixed Point Theorem, $\beta(.)$ has a fixed point in $[L, U]$. ■

We now turn to the case for existence of trigger equilibria in the case with unbounded signal support (thus unbounded likelihood ratios).

Case 2: Signals drawn from $\mathbb{R}$

Since we do not have to worry about endpoint problems when $S = \mathbb{R}$, the best response mapping is more simply defined than above. Given a set of triggers $t \in \mathbb{R}^n$, the best response is defined to be $\beta(t) \in \mathbb{R}^n$, where

$$\beta_i(t) = r^{-1}\left[\frac{\pi}{1-\pi}\left[\frac{1+c}{c}\prod_{j>i} Pr(s \geq t_j|G) - 1\right]\prod_{j<i}\frac{Pr(s \geq t_j|G)}{Pr(s \geq t_j|B)}\right]$$

Given this definition, we are ready to examine the existence of trigger equilibria for the game with $S = \mathbb{R}$.

As in the case with bounded signal support, a trivial trigger equilibrium with infinite triggers exists. Such an equilibrium can be constructed with an argument identical to the one above. However, the no-investment equilibrium is not very interesting, and we naturally turn to the existence of investment equilibria. The question of existence of investment equilibria is more involved when signals are drawn from $\mathbb{R}$, since the underlying signal generating process has unbouded support, making it impossible to directly appeal to a fixed point theorem. However, a subtler argument establishes that if Property Ψ holds, then investment equilibria exist even in this case. The following result is crucial.

Proposition 2.3. *If Property Ψ holds, then there exist $\underline{t} \in \mathbb{R}^n$ and $\bar{t} \in \mathbb{R}^n$ such that for all $t \in \Re^n$, $\underline{t} \leq \beta(t) \leq \bar{t}$.*

The proof of this result is involved. It is relegated to the appendix.

Corollary 2.2. *When Property Ψ holds, there is an investment equilibrium in $\Gamma(n)$ with $S = \mathbb{R}$*

Proof: Proposition 2.3 tells us that there exists $[\underline{t}, \bar{t}\,] \subset \mathbb{R}^n$ such that for all $t \in \Re^n$, $\beta(t) \in [\underline{t}, \bar{t}\,]$. Thus, in particular, for all $t \in [\underline{t}, \bar{t}\,]$, $\beta(t) \in [\underline{t}, \bar{t}\,]$. Inspection of the best response mapping establishes that for all $t \in [\underline{t}, \bar{t}\,]$, $\beta(.)$ is single-valued and continuous. Clearly $[\underline{t}, \bar{t}\,]$ is compact and convex. Now, by a simple application of Brouwer's fixed point theorem, we note that there exists $t^* \in [\underline{t}, \bar{t}\,]$ such that $t^* = \beta(t^*)$. Thus, a bounded equilibrium of $\Gamma(n)$ exists. ■

Next we examine some pertinent properties of these investment equilibria.

2.2 Properties of Investment Equilibria

In this section we demonstrate two key structural properties of investment equilibria. The properties apply to investment equilibria of $\Gamma(n)$ in general, *regardless* of whether the underlying signals have bounded or unbounded support. Thus, in order to ensure the existence of these equilibria we tacitly assume that Properties Ψ or $\Psi+$ hold, depending on which version of $\Gamma(n)$ we are considering.

The first of these properties encapsulates a simple relation between the relative magnitudes of triggers in any investment equilibrium of $\Gamma(n)$.

Proposition 2.4. *Suppose $(t_1, t_2, ..., t_n)$ is any investment equilibrium of $\Gamma(n)$. Then, t_i is* decreasing *as a function of t_j for $j < i$, and* increasing *as a function of t_j for $j > i$. In other words, an agent's equilbrium triggers is increasing in the triggers of her successors, and decreasing in the triggers of her predecessors.*

Proof: The proof follows directly upon examination of the best response correspondence. Let $(t_1, t_2, ..., t_n)$ be any investment equilbrium of $\Gamma(n)$. Then, by the definition of the best response mapping:

$$t_i = r^{-1}\left[\frac{\pi}{1-\pi}\left[\frac{1+c}{c}\prod_{j>i} Pr(s \geq t_j|G) - 1\right]\prod_{j<i}\frac{Pr(s \geq t_j|G)}{Pr(s \geq t_j|B)}\right]$$

Now it is apparent that t_i is increasing in t_j for $j > i$ and decreasing in t_j for $j < i$ since $r(s)$ is decreasing and $\frac{Pr(s \geq x|G)}{Pr(s \geq x|B)}$ is increasing in x. ■

The intuition behind this result is simple. In equilibrium, conditional upon observing investment by a predecessor, the higher the predecessor's trigger, the higher the signal the predecessor must have observed. The MLRP property of the information system of $\Gamma(n)$ implies that higher private signals make players more optimistic about the state of the world. Large signals are "good news" for would-be investors. They make it likelier that the state is G, which in turn makes it likelier that other players will receive relatively high signals. Thus, observing investment by a predecessor with a high trigger conveys more "good news" for a player, and makes her more optimistic about the state of the world, and about the probability that her successors will also invest. Agents in this model have two sources of information, both of which affect their level of optimism: the public information encapsulated in the observed decisions of their predecessors, and the private information contained in their signals. Thus, when an agent observes more encouraging public information (investment by predecessors with high triggers), she requires less persuasive private information in order to choose to invest. Thus, she picks a lower trigger. Similarly, if an agent believes that her successors have extremely high triggers, then she may be concerned that they shall not invest with higher probability, and "leave her stranded" if she chooses to invest. Thus, she will be inclined in equilibrium to require more persuasive private evidence for the fact that the state is G before deciding to invest. In other words, she will choose a higher trigger.

The above proposition has an immediate consequence for the two player version of our game, as we note below:

Corollary 2.3. *There is a unique investment equilibrium in* $\Gamma(2)$.

In an investment equilibrium, the process by which agents become more optimistic (or pessimistic) about the state of the world is by Bayesian learning. Agents update their priors about the state of the world by Bayes' Rule upon observing their predecessor's actions. As we have just argued, observation of investment by a predecessor with a high trigger makes an agent more optimistic about the state of the world than the observation of investment by a predecessor with a low trigger. Intuitively, it seems also likely that observing investment by two predecessors makes an agent (at least weakly) more optimistic about the state of the world than observing investment by one predecessor.[9] Thus, upon observing investment by more and more predecessors, later players will require less and less persuasive private information in order to invest. In other words, the greater the mass of public evidence in favour of a good state, the lower the level of private evidence required to make investors take potentially productive but risky actions. The following result captures this intuition.

Proposition 2.5. *In any Investment Equilibrium of* $\Gamma(n)$, $(t_1, ..., t_n)$, $t_j \geq t_{j+1}$ *for* $j = 1, .., n-1$.

[9] If the second predecessor has a trigger of $\underline{s}$ or $-\infty$, then her decision to invest does not affect the optimism of succeeding players. Hence the relation if weak.

Proof: In equilibrium $(t_1, ..., t_n)$, consider the magnitudes of t_i and t_{i+1}. We know from the definition of the best response mapping:

$$\frac{f(t_i|B)}{f(t_i|G)} = \frac{\pi}{1-\pi}\left[\frac{1+c}{c}\prod_{j=i+1}^{n} Pr(s_j \geq t_j|G) - 1\right]\prod_{j=1}^{i-1}\frac{Pr(s_j \geq t_j|G)}{Pr(s_j \geq t_j|B)}$$

$$\frac{f(t_{i+1}|B)}{f(t_{i+1}|G)} = \frac{\pi}{1-\pi}\left[\frac{1+c}{c}\prod_{j=i+2}^{n} Pr(s_j \geq t_j|G) - 1\right]\prod_{i=1}^{i}\frac{Pr(s_j \geq t_j|G)}{Pr(s_j \geq t_j|B)}$$

Note that $Pr(s \geq t_{i+1}|G) \leq 1$ and $\frac{Pr(s \geq t_i|G)}{Pr(s \geq t_i|B)} \geq 1$ due to the MLRP property of f. Note that the equalities follow in both cases if and only if $t_{i+1} = \underline{s} or - \infty$ and $t_i = \underline{s} or - \infty$ respectively. Thus, $\prod_{j=i+1}^{n} Pr(s_j \geq t_j|G) \leq \prod_{j=i+2}^{n} Pr(s_j \geq t_j|G)$ and $\prod_{j=1}^{i-1} \frac{Pr(s_j \geq t_j|G)}{Pr(s_j \geq t_j|B)} \leq \prod_{j=1}^{i} \frac{Pr(s_j \geq t_j|G)}{Pr(s_j \geq t_j|B)}$. This means that $\frac{f(t_i|B)}{f(t_i|G)} \leq \frac{f(t_{i+1}|B)}{f(t_{i+1}|G)}$, and thus $t_i \geq t_{i+1}$. $\diamond$

This proposition implies that conditional upon observing investment by predecessors, later players shall tend to invest more easily, i.e., for larger ranges of private information. Very roughly speaking, this means that later agents are less concerned about the content of their private information than earlier agents. This is because the observation of investment by predecessors make later players progressively more optimistic.

Is it possible that Bayesian learning has made players "too optimistic" relative to some (as yet unspecified) superior social alternative? Could there be versions of $\Gamma(n)$ where successors completely ignore their private information upon observing predecessors invest, and thereby clearly act suboptimally in a social sense? These questions are addressed in the following section.

2.3 Herding

In settings with sequential decision making in the presence of uncertainty, private information, and observed public actions, agents are said to "herd" if they blindly imitate their predecessors' choices without heed to their own private information. In other words, herd behavior occurs when, in the words of Douglas Gale (1996) "imitation dominates information." Interpreting this concept literally in terms of our trigger equilibria, agents herd in equilibrium if one or more successors set their triggers to $\underline{s}$, the lower bound of the signal generating process. This means that conditional upon observing investment by their predecessors, some agents choose to invest for *all* possible values of their private signals. The strongest version of such herd behavior is if *all* successors choose to blindly imitate the first player. We shall call this type of behavior *strong herding.* For our purposes, we shall also define a much weaker form of herd behavior. When strong herding occurs, later agents become so optimistic that they pay no attention whatsoever to their private information. However, it is not difficult to imagine situations in which agents do not become optimistic enough that to imitate blindly, but still become overoptimistic compared to a situation where private information was aggregated efficiently in the market. We shall call such phenomena *weak herding.* The idea is formally defined later in the paper.

It turns out that strong herding can occur in $\Gamma(n)$ only if signals are drawn from a bounded support, i.e. the likelihood ratios are bounded. However, in the case where signals are drawn from $\mathbb{R}$ and likelihood ratios are unbounded, there can still be weak herding. In what follows, we lay down the informational prerequisites for strong and weak herding in $\Gamma(n)$.

2.3.1 Bounded Likelihood Ratios: Strong Herd Behavior

We begin with a definition.

Definition 2.7. *An investment equilibrium t of $\Gamma(n)$ exhibits* strong herding *if $\underline{s} < t_1 < \bar{s}$ and $t_j = \underline{s}$ for $j \geq 2$.*

In words, this simply means that all followers choose to completely ignore their private information. This definition, taken together with Proposition 2.4 leads to a very useful property of strong herding equilibria.

Proposition 2.6. *If investment equilibrium t of $\Gamma(n)$ exhibits strong herding, it is unique.*

Proof: Let $t = (t_1, \underline{s}, ..., \underline{s})$ where $\underline{s} < t_1 < \bar{s}$ be a strong herding equilibrium. Suppose it is not unique. Let z be an investment equilibrium, with $z \neq t$. Since $z \neq t$, clearly, there is a j, $j \geq 2$ such that $z_j > t_j$. Proposition 2.5 implies that if this is so, $z_2 > t_2$. For simplicity, let $z_j = t_j$ for $j \geq 3$. Let $z_2 > t_2$. Then, by Proposition 2.4 we know $z_1 > t_1$ (equilibrium triggers are increasing in those of successors). But this in turn implies, also by Proposition 2.4, that $z_2 < t_2$ (equilibrium triggers are decreasing in those of predecessors). This is a contradiction. ■

Proposition 2.6 tells us that if $\Gamma(n)$ has a strong herding equilibrium, it is unique in the class of investment equilibria. Corollary 2.3 tells us that the two player game has a unique equilibrium. Together the two imply that in order to analyze whether $\Gamma(n)$ has a unique strong herding equilibrium, it is sufficient to look at $\Gamma(2)$. In what follows, therefore, we lay down the conditions under which $\Gamma(2)$ has a strong herding equilibrium.

Let (t_1, t_2) be any investment equilibrium of $\Gamma(2)$. Recall that $r(s) \in [B, T]$. Then,

$$r(t_1) = \frac{\pi}{1-\pi}\left[\frac{1+c}{c} Pr(s \geq t_2|G) - 1\right]$$

$$r(t_2) = \frac{\pi}{1-\pi}\frac{1}{c}\frac{Pr(s \geq t_1|G)}{Pr(s \geq t_1|B)}$$

For t to be a strong herding equilibrium, we need $t_2 = \underline{s}$. So $t_1 = r^{-1}\left(\frac{\pi}{1-\pi}\frac{1}{c}\right)$. But if $t_2 = \underline{s}$, then $r(t_2) \geq T$. This implies

$$\frac{Pr(s \geq r^{-1}\left(\frac{\pi}{1-\pi}\frac{1}{c}\right)|G)}{Pr(s \geq r^{-1}\left(\frac{\pi}{1-\pi}\frac{1}{c}\right)|B)} \geq \frac{1-\pi}{\pi}\, c\, T$$

This condition defines precisely the informational requirements for strong herding in $\Gamma(2)$, and therefore for $\Gamma(n)$. It is apparent that there is no unique way of characterizing the information systems that satisfy such a condition. However, it is possible to provide tight characterizations over broad classes of information systems. Below, we provide such a characterization for all information systems where the signal generating processes are linear in the signals.

In considering the case for linear signal generating processes we limit attention without loss of generality to a support of $[0, 1]$. In addition, since it is the ratio of densities and not the individual densities that are important in our model, we normalize the density in state G to be uniform ($U[0, 1]$) also without loss of generality. Given the strict MLRP, this means that $f(s|B)$ is decreasing and linear in s. We are now ready to provide two results characterizing when strong herding will occur in $\Gamma(n)$.

Proposition 2.7. *Suppose $S = [0, 1]$. Let $f(s|G) = 1$. Consider the class of densities in state B that are linear in the signal and that do not have full support. Then, for a given c*

and for any prior π we can construct an information system such that $\Gamma(n)$ has a unique equilibrium with the strong herding property.

Proof: Let $f(s|B) = a - bs$, where $a > 0$ and $b > 0$. Since $f(s|B)$ is a density, it must integrate to 1 over its support. The support is given by $[0, \frac{a}{b}]$ where since $f(s|B)$ is not full support we require that $a \leq b$. Also, $\int_0^{a/b}(a - bs)ds = 1$ implies $b = \frac{a^2}{2}$. Putting this two relations together, we get $a \geq 2$. Thus, $f(s|B) = a - \frac{a^2}{2}s$, and since $f(s|G) = 1$, $r(s) = f(s|B)$. Thus, clearly, $0 \leq r(s) \leq a$. Let $k = \frac{\pi}{1-\pi}\frac{1}{c}$. By the conditions defining a strong herding equilibrium (t_1, t_2), we require that $r(t_1) = k$ which implies $t_1 = \frac{2}{a^2}(a - k)$. To ensure that this is an investment equilibrium, we require that $t_1 < 1$, i.e., $a - \frac{a^2}{2} < k$. To ensure that the equilibrium is not trivial, we require $t_1 > 0$, i.e., $a > k$. Finally, in order to make $r(t_2) \geq a$, we require $k\ \frac{Pr(s \geq t_1|G)}{Pr(s \geq t_1|B)} \geq a$, which, upon algebraic simplification implies that we require $\frac{a-2}{1-\frac{2}{a}} \geq k$. Thus we want $a \geq 2$, $a > k$, $a - \frac{a^2}{2} < k$, and $\frac{a-2}{1-\frac{2}{a}} \geq k$. Clearly, for any given k, by picking a large enough, we can satisfy these conditions, which are necessary and sufficient for the existence of a unique investment equilibrium with strong herding. ■

Proposition 2.8. *Suppose $S = [0, 1]$. Let $f(s|G) = 1$. Consider the class of densities in state B that are linear in the signal and full support. Then there is no equilibrium of $\Gamma(n)$ with the strong herding property.*

Proof: Let $f(s|B) = a - bs$ where $a > 0$ and $b > 0$. Since $f(s|B$ is a density it must integrate to 1 over it's support. The support in this case is given by $[0, 1]$ and in order to ensure this full support, we require that $a > b$. So, $\int_0^1(a - bs)ds = 1$ which implies $b = 2(a - 1)$. Together, these imply $a < 2$ and, since $b > 0$, $a > 1$. Thus, $1 < a < 2$.

So, $f(s|B) = a - 2(a-1)s$ and $r(s) = f(s|B)$. Clearly, $2 - a \leq r(s) \leq a$. Let $k = \frac{\pi}{1-\pi}\frac{1}{c}$. (t_1, t_2) is a strong herding equilibrium if and only if $t_1 = r^{-1}(k) = \frac{a-k}{2(a-1)}$. Also, since $0 < t_1 < 1$ by definition of strong herding, $a > k$ and $a > 2 - k$. Finally, $t_2 = 0$ if and only if $k \, \frac{Pr(s \geq t_1|G)}{Pr(s \geq t_1|B)} \geq a$. Upon algebraic simplification, this yields, $a \leq k$. But we have already required $a > k$. This is a contradiction. ■

Thus, in the class of linear information systems with bounded likelihood ratios, $\Gamma(n)$ can have a strong herding equilibrium if and only if the the signal generating process in state B is not full support. This provides a criterion for mechanism design in contexts where players need to be coordinated upon some socially productive risky action (or prevented from coordinating upon a socially unproductive one). If it was possible for the mechanism designer to provide private information via linear stochastic processes to the players, she would know exactly how to make all but one player ignore their own private information, or, by the same token, how to force players to pay more attention to their private information.

We now address the question of herd behavior when likelihood ratios are unbounded.

2.3.2 Unbounded Likelihood Ratios: Weak Herd Behavior

It is apparent by inspection of the best response mapping that when $r(s)$ is unbounded, i.e., when $S = \mathbb{R}$, it is impossible to have strong herding in $\Gamma(n)$. However, the lack of extreme informational inefficiencies in such instances does not mean that there aren't any. Inefficiencies in the aggregation of information can lead to "excessive optimism" in $\Gamma(n)$. How can we measure such excessive optimism?

Information about the state of the world in $\Gamma(n)$ is generated by the sequence of payoff relevant private signals that the players receive. The problem is that the signals are *private*, i.e., only the original recipient of the signal can observe its true value. Others must be satisfied with simply observing the *actions* chosen by the original recipient and guessing from this action what the recipient's signal may have been. If information was efficiently aggregated, each agent would be able to observe the equivalent of all signals that had been received (by herself or others) at the point of time they she is called upon to act. This could be achieved, for example, by a social planner, who could observe each agent's signal and announce it to the rest of the group.[10] Let us denote this variation of $\Gamma(n)$ with observed signals by $\hat{\Gamma}(n)$. We can now define weak herding in $\Gamma(n)$.

Definition 2.8. *An investment equilibrium t in $\Gamma(n)$ is said to exhibit* weak herding *if there exists $i > 1$ such that with positive probability $t_i < \hat{t}_i$, where $\hat{t}$ is the unique equilibrium in $\hat{\Gamma}(n)$.*

In other words, an investment equilibrium of $\Gamma(n)$ exhibits weak herding if at least one follower becomes excessively optimistic with positive probability. In what follows we present in brief the game with observed signals. For brevity, we simply consider the two-player case.

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[10]Note that it is not easily possible to simply get each agent to simply announce their signals, since there are significant credibility problems inherent in such announcements. Once an agent has chosen to invest, she has a clear incentive to get her successors to invest, *regardless* of the actual state of the world, and thus has motive to overstate her signal.

$\Gamma(2)$ with Observed Signals

Let π_2 denote player 2's updated prior after she has observed her predecessor's signal. Clearly,

$$\frac{\pi_2}{1-\pi_2} = \frac{\pi}{1-\pi}\frac{f(s_1|G)}{f(s_1|B)},$$

where s_1 is the realization of player 1's signal. Upon observing player 1 invest, player 2's expected utility from investing is given by

$$EU_2(I) = \frac{\pi_2}{\pi_2 + (1-\pi_2)\frac{f(s_2|B)}{f(s_2|G)}}(1+c) - c$$

if she observes private signal s_2. Her expected utility from not investing is 0. Since $EU_2(I)$ is clearly increasing and continuous in s_2, player 2 shall choose a trigger strategy, where her trigger $\hat{t}_2$ is defined by $\hat{t}_2 = r^{-1}(\frac{\pi_2}{1-\pi_2}\frac{1}{c})$, i.e.,

$$\hat{t}_2 = r^{-1}(\frac{\pi}{1-\pi}\frac{1}{c}\frac{f(s_1|G)}{f(s_1|B)})$$

.

We are now ready to provide a characterization of weak herding in $\Gamma(2)$. Recall that by definition of the best response mapping in $\Gamma(2)$, $t_2 = r^{-1}(\frac{\pi}{1-\pi}\frac{1}{c}\frac{Pr(s \geq t_1|G)}{Pr(s \geq t_1|B)})$. The investment equilbrium t possesses the weak herding property if with positive probability, $t_2 < \hat{t}_2$, i.e. with positive probability

$$\frac{Pr(s \geq t_1|G)}{Pr(s \geq t_1|B)} > \frac{f(s_1|G)}{f(s_1|B)}$$

It is apparent that this property shall hold for large classes of full support distributions on $\mathbb{R}$, particularly those with thin tails. A natural example of this is the Gaussian Distribution. In what follows, we present a few examples of how weak herding occurs with positive probability in $\Gamma(2)$ when the information system is Normal.

The results are presented in Table 2.1. We assume $c = 1$, $\pi = 0.6$, $f(\cdot|G) = N(5, \sigma^2)$, $f(\cdot|B) = N(0, \sigma^2)$, and vary the standard deviation of the signal generating processes. In each case, in the table below, we provide, the unique equilibrium triggers of $\Gamma(2)$, the ranges of signals for which weak herding occurs, as well as the corresponding ex ante probability of weak herd behavior. We call the upper and lower bounds of the weak herding range wh_U, and wh_L respectively.[11]

StDev	Equilibrium		WH Range		Pr(Herding)
σ	t_1	t_2	wh_L	wh_U	(*ex ante*)
1	2.419	1.449	2.419	2.908	1%
2	2.199	0.645	2.199	3.477	12%
3	1.910	-0.344	1.910	4.187	22%
4	1.559	-1.478	1.559	4.937	27%
5	1.123	-2.735	1.123	5.712	32%
10	-3.011	-10.471	-3.011	9.647	46%

Table 2.1: Signal Ranges for Weak Herding

The comparison of $\Gamma(2)$ with the game with observed signals raises an obvious related question. If inappropriate aggregation of information in $\Gamma(2)$ represents a source of potential inefficiency, so do the strategic complementarities built into the payoffs. Conditional upon

[11] Naturally, in each of the cases above, it is also possible that weak herd behavior shall not occur, since the signals received shall be high enough to justify, or even dwarf, the optimism inherent in the trigger equilibrium. The point of this exercise with unbounded likelihood ratios is to demonstrate that overoptimism is *likely*, not inevitable, as the outcome of rational behavior in $\Gamma(2)$.

investment by their predecessors, when agents choose whether to invest or not, they take into account only their personal gains and losses from investing, not the gains and losses to society as a whole. When agent 2 chooses not to invest (conditional upon agent 1 having already invested) she imposes an immediate cost of $-c$ on Agent 1. Agent 2 does not take this cost into account, and thereby may possibly be too conservative in her investment strategy relative to the social optimum. In order to compare $\Gamma(2)$ with the alternative game that incorporates *both* observed signals and awareness of social costs, we have to consider the single-agent decision problem. In this, one agent chooses successively to invest or not, and gains the sum of the payoffs to individual players in $\Gamma(2)$. This means that she earns 2 at the end if she chooses to invest twice and the state is G, $-2c$ if the state is B, $-c$ if she chooses to invest exactly once, and 0 if she does does not invest at all. The agent remembers her history of signals when choosing to invest. The comparison of the equilibrium of this single agent decision problem with the trigger equilibria of $\Gamma(2)$ turns out to be similar to the comparison with the case of observed signals above. The broad conclusions are that the trigger equilibrium of $\Gamma(2)$ is inefficient for sure, and can exhibit both overoptimism and overpessimism with positive probability. The single agent version of $\Gamma(2)$ is worked out in the appendix.

The lack of the strong herding property in equilibria of $\Gamma(n)$ when $S = \mathbb{R}$ raises another interesting question. When an equilibrium of $\Gamma(n)$ exhibits the strong herding property, it is possible to coordinate an infinite number of players upon risky investment. However, when the equilibria of $\Gamma(n)$ lack the strong herding property it is natural to wonder whether it is possible to coordinate larger and larger numbers of players upon investment in equilibrium.

Intuitively, since all players in the game choose finite triggers, as the number of players get larger and larger, it may be harder and harder to convince the first player to take a risk and invest, since there are more and more later players who could "leave him stranded" by choosing to not invest after he does so. What would happen in $\Gamma(n)$ with $S = \mathbb{R}$ as n grew larger and larger? Is it still possible to make Player 1 invest in equilibrium?

Recall that in our discussion to date, we have assumed that the information system in $\Gamma(n)$ satisfied Property Ψ, which guaranteed the existence of a investment equilibrium when $S = \mathbb{R}$. However, a cursory glance at the definition of Property Ψ might lead one to believe that as one increased the number of players, the information system may no longer satisfy Ψ. However, note that Ψ is only sufficient and not necessary for the existence of equilibria in this model. Whether we can satisfy Ψ in $\Gamma(n)$ as n grows larger, and whether an equilibrium might exist even if Ψ is violated, depends crucially on the information structure chosen for the game. Thus, in considering the effects of increasing n, we are effectively proposing an exercise in comparative information systems. In order to give some structure to such a comparative exercise, it is necessary to parametrize the information system. For this purpose, we henceforth consider Gaussian (Normal) information systems for arbitrary (general) parameter values. This is simply an analytical simplification. Several of our results will not be contingent on the precise functional form of the Gaussian distribution, and we shall point out generalizations in due course. With this in mind, we progress to characterizing the informational requirements for creating the possibility of coordinated investment in $\Gamma(n)$ when $S = \mathbb{R}$.

2.4 On the Possibility of Coordination

The existence of investment equilibria in $\Gamma(n)$ ensures the possibility of coordinated investment by the participants in the game. It is intuitive that it should be harder to coordinate progressively larger numbers of players upon a risky but socially productive action in our setting. In this section, we explore the informational conditions that will allow us to ensure that it is at least *possible* to coordinate a large number of players upon a productive risky action.

Our new setting specializes the original setting in one sense. The information system $f = \{f(.|G), f(.|B)\}$ is specified to be Gaussian. In the good state, signals are generated by some arbitrary Gaussian process with mean $\mu > 0$ and standard deviation $\sigma > 0$. In the bad state, signals are generated by a Gaussian process with mean 0 and standard deviation σ. Choosing the standard deviation to be identical in both states ensures the strict MLRP property of f. The choice of 0 as the mean of the signal generating process in the bad state is without loss of generality. It is easy to see that what matters is the *difference* between the two means. Thus, μ the mean in the good state could also be viewed as the difference of means between the two states: $\mu = \mu_G - \mu_B$. μ_B is set to 0 for notational simplicity. In sum, therefore, $f = \{N(\mu, \sigma), N(0, \sigma)\}$. We denote this information system by $f(\mu, \sigma)$.

In this new setting consider what happens in $\Gamma(n)$ as n gets bigger. It is easy to see that as the number of players gets large, it becomes harder for f to satisfy Property Ψ. Whether Ψ is violated or not turns out to depend on the initial level of optimism of the players. We shall show below that for initial priors above a certain cutoff point *determined solely by the parameters*, we can always find an information system to satisfy Ψ, and thus ensure the existence of an investment equilibrium. On the other hand, for priors below this cutoff point, there is some finite n for which Ψ is violated in $\Gamma(n)$. However, Ψ is only sufficient and not necessary for the existence of an investment equilibrium. Thus, the violation of Ψ does not exclude the possibility of investment equilibria. In fact, we shall show that for *any* nondegenerate prior on the states, we can find an information system that ensures the existence of an investment equilibrium for $\Gamma(n)$ where n is as large as we please in the set of integers. The following propositions explicate these points.

Proposition 2.9. *If $\pi > \frac{c}{1+c}$, then for any $n \in Z_{++}$ there exists a $f(\mu, \sigma)$ with σ large and finite such that Property Ψ is satisfied in $\Gamma(n)$. If $\pi < \frac{c}{1+c}$ there exists $n \in Z_{++}$ such that Ψ is violated for $\Gamma(n)$.*

Proof: Adapting the best response mapping to the specific context of the Gaussian information system $f(\mu, \sigma)$, we know that t_i is defined by:

$$\exp(\frac{\mu^2 - 2\mu t_i}{2\sigma^2}) = \frac{\pi}{1-\pi}\left(\frac{1+c}{c}\prod_{j=i+1}^{n} Pr(s_j \geq t_j|G) - 1\right)\prod_{j=1}^{i-1}\frac{Pr(s_j \geq t_j|G)}{Pr(s_j \geq t_j|B)}$$

This implies,

$$t_i = \frac{\mu}{2} - \frac{\sigma^2}{\mu}\ln\left[\frac{\pi}{1-\pi}\left(\frac{1+c}{c}\prod_{j=i+1}^{n} Pr(s_j \geq t_j|G) - 1\right)\prod_{j=1}^{i-1}\frac{Pr(s_j \geq t_j|G)}{Pr(s_j \geq t_j|B)}\right] \tag{2.1}$$

Now, recalling the definitions of $U_j(c, \pi, f)$ for $j = 2, ..., n$ from section 2.1.4, we can write

$$U_n = \frac{\mu}{2} - \frac{\sigma^2}{\mu}\ln\left[\frac{\pi}{1-\pi}\left(\frac{1+c}{c} - 1\right)\right] \tag{2.2}$$

For $i = 2, ..., n-1$

$$U_i = \frac{\mu}{2} - \frac{\sigma^2}{\mu} \ln \left[\frac{\pi}{1-\pi} \left(\frac{1+c}{c} \prod_{j=i+1}^{n} Pr(s_j \geq U_j | G) - 1 \right) \right] \tag{2.3}$$

Now consider the case where $\pi > \frac{c}{1+c}$. This means that $\ln \left[\frac{\pi}{1-\pi} \left(\frac{1+c}{c} - 1 \right) \right] > 0$. Equation (2.2) now implies that U_n is *decreasing* as a function of σ. In particular, $U_n = \frac{\mu}{2} - J_n \sigma^2$ for some $J_n \in \mathbb{R}_{++}$. This means that by picking σ high enough we can ensure that

$$\ln \left[\frac{\pi}{1-\pi} \left(\frac{1+c}{c} Pr(s \geq U_n | G) - 1 \right) \right] = \ln \left[\frac{\pi}{1-\pi} \left(\frac{1+c}{c} Pr(z \geq -J_n \sigma - \frac{\mu}{2}) - 1 \right) \right] > 1$$

where z is the standard Gaussian variable.

But from equation (2.3), this in turn ensures that U_{n-1} is also decreasing in σ. In particular, $U_{n-1} = \frac{\mu}{2} - J_{n-1}(\sigma)\sigma^2$ for some with $0 < J_{n-1}(\sigma) < J_n$ for $\sigma < \infty$. So we can pick σ high enough to ensure that

$$\ln \left[\frac{\pi}{1-\pi} \left(\frac{1+c}{c} Pr(z \geq -J_n \sigma - \frac{\mu}{2}) Pr(z \geq -J_{n-1} \sigma - \frac{\mu}{2}) - 1 \right) \right] > 1$$

where z is the standard normal variable. This ensures that U_{n-2} is decreasing in σ, $U_{n-2} = \frac{\mu}{2} - J_{n-2}(\sigma)\sigma^2$ with $J_{n-2}(\sigma) < J_{n-1}(\sigma) < J_n$ for any $\sigma < \infty$.. Notice that as σ gets arbitrarily large, the J_i's get arbitrarily close to each other. Therefore, continuing in this way, for any finite n, we can clearly choose σ high enough (but finite) to satisfy Property Ψ in $\Gamma(n)$.

Next consider the case where $\pi < \frac{c}{1+c}$. By analogy to the above, we know that this means that U_n is *increasing* as a function of σ. Thus, U_n is bounded below by $\frac{\mu}{2}$ (corresponding to $\sigma = 0$). But notice that U_{n-1}, U_{n-2},...etc. are all bounded below by $\frac{\mu}{2}$ because by equation (2.3) we observe that they are all of the form

$$U_i = \frac{\mu}{2} - \frac{\sigma^2}{\mu} \ln \left[\frac{\pi}{1-\pi} \left(\frac{1+c}{c} J - 1 \right) \right]$$

where $J < 1$. But clearly $\frac{1+c}{c}J - 1 < \frac{1+c}{c} - 1 < 1$. Thus, the product $\prod_{j=2}^{n} Pr(s_j \geq U_j|G)$ is bounded below by $[Pr(s \geq \frac{\mu}{2}|G)]^{n-1}$. This clearly vanishes as $n \to \infty$. Thus, there exists an $n \in Z_{++}$ for which Ψ is violated in $\Gamma(n)$. ■

This result is ostensibly counterintuitive but powerful. It says that if sufficiently optimistic players are offered sufficiently *garbled* information about the state of the world, then there exists the possibility for coordinated investment regardless of how large the number of players is, as long as the number is finite and known. What makes the result strong is that the level of initial optimism is *independent of the number of players.*

What if the beliefs of potential players fell below the range that guarantees the existence of an investment equilibrium from Proposition 2.9? It turns out that there is an alternative way to ensure the possibility of coordinated investment that is independent of the initial priors: to provide sufficiently accurate (instead of sufficiently garbled) information to players.

Proposition 2.10. *Fix $n \in Z_{++}$ and $\mu > 0$. Let $M > 0$ be such that $[1 - \Phi(-M-\mu)]^{n-1} > \frac{c}{1+c}$. Then we can find $(\epsilon, \sigma) > 0$ such that $t = (t_1, ..., t_n)$ where $t_1 \in [\frac{\mu}{2} - \epsilon, \frac{\mu}{2} + \epsilon]$, and $t_j \leq -M$ for $j = 2, ..., n$ is a trigger equilibrium of $\Gamma(n)$*

Proof: Recall from equation (2.1) that

$$t_1 = \frac{\mu}{2} - \frac{\sigma^2}{\mu} \ln \left[\frac{\pi}{1-\pi} \left(\frac{1+c}{c} \prod_{j=2}^{n} Pr(s_j \geq t_j|G) - 1 \right) \right]$$

Our choice of M above means that t_1 is well defined for $\sigma \leq 1$.

$$t_2 = \frac{\mu}{2} - \frac{\sigma^2}{\mu} \ln \left[\frac{\pi}{1-\pi} \left(\frac{1+c}{c} \prod_{j=2}^{n} Pr(s_j \geq t_j|G) - 1 \right) \frac{Pr(s \geq t_1|G)}{Pr(s \geq t_1|B)} \right]$$

Notice that as $\sigma \to 0$ and resultantly $t_1 \to \frac{\mu}{2}$, the thin tailed property of the Gaussian distribution ensures that $\frac{Pr(s \geq t_1|G)}{Pr(s \geq t_1|B)} \to \infty$. In particular, this term explodes much faster than $\sigma^2 \to 0$ so that $t_2 \to -\infty$. Thus, for a given $M > 0$, we can clearly find $(\epsilon, \sigma) > 0$ such that $t_1 \in [\frac{\mu}{2} - \epsilon, \frac{\mu}{2} + \epsilon]$ and $t_2 \leq -M$. But notice that in this setting $t_2 \geq t_3 \geq ... \geq t_n$. Thus clearly $t_j \leq -M$ for $j = 2, ..., n$ and we have found a $f(\mu, \sigma)$ to rationalize t as a bounded equilibrium of $\Gamma(n)$. ■

Proposition 2.9 and Proposition 2.10 lay down sufficient informational conditions to create the possibility of coordination in $\Gamma(n)$ for any $n \in Z_{++}$, no matter how large. Jointly they imply that for initially optimistic individuals, either very good or very bad quality information creates the possibility of coordinated investment. Intermediate quality information does not guarantee the possibility of coordination. Sufficiently good quality information always creates the possibility of coordinated investment regardless of whether agents are initially optimistic or not.

2.5 Discussion

The results presented in the preceding sections provide a general framework within which to view phenomena associated with herd behavior such as market panics, bank runs, and currency crashes. We unify two prior strands of the literature that address such phenomena: sequential choice models with Bayesian learning without payoff complementarities (herding models), and static coordination games with payoff complementarities without Bayesian learning. It is apparent that agents involved in bank runs and financial panics in general have to simultaneously solve coordination problems (captured by the static games literature), learn from their predecessors (captured by the herding literature), and send effective signals to other market participants (not captured by either prior class of models). This model, albeit in extremely stylized form, captures all these three aspects to the behavior of agents within one framework.

An important message that emerges from our analysis is that information matters. In particular, the stochastic properties of the information that agents receive is relevant in determining outcomes. In the preceding sections, we have demonstrated that herd behavior emerges in equilibrium in varying degrees depending on the properties of the information system. In particular, if the information system is such that agents can exhibit only limited amounts of personal skepticism (bounded likelihood ratios), we have demonstrated that very extreme forms of herd behavior can emerge as the unique outcome of rational behavior. If information systems are such that agents exhibit unbounded personal skepticism (unbounded likelihood ratios), however, the outcomes of the model are less extreme, but by no means represent the efficient aggregation of information. Agents may still exhibit the excessive optimism (or pessimism) that is often observed in the market.

While our model takes an essential step towards appropriately modeling the strategic aspect of market booms and busts, and extends the herding literature by incorporating forward-looking behavior on the part of agents, it admits several caveats. One of these is that we require complete coordination on the part of agents to achieve positive payoff from investment. A richer model would allow for payoffs from investment to be a continuous and increasing function of the number of investors. The arguments in such a model would be complicated by combinatorial considerations, because the order of action matters in models such as ours. However, we conjecture that the broad results will be similar to ours. In particular, in a set-up similar to this, we antipate that agents will still follow trigger strategies, and that later agents will choose smaller triggers exhibiting strong or weak herding along the lines of this model. In particular, since requiring complete agreement encourages greater conservatism on the part of players in their choices of action, we conjecture that herd behavior would occur more easily in games where positive payoffs may be earned even without complete agreement.

Other potential extensions of the model would allow players to choose their time of entry into the market, i.e., endogenize the order of actions, or allow for imperfect observation of prior choices. Richer models such as these would more closely approximate the reality of financial market booms and panics. This model provides a benchmark against which to compare such future models.

2.6 Appendix

2.6.1 Proof of Proposition 2.3

Consider the largest and smallest possible triggers that can be chosen by the players. Suppose initially we allow players to choose triggers in an unrestricted way, i.e., anywhere in $\Re$. Call the bounds corresponding to this $\underline{t}^0$ and $\bar{t}^0$. We denote the jth component of $\underline{t}^0$ by $\underline{t_j}^0$ and of $\bar{t}^0$ by $\bar{t_j}^0$. So, $\underline{t}^0 = (-\infty, ..., -\infty)$ and $\bar{t}^0 = (\infty, ..., \infty)$.

Now consider best responses to triggers chosen in $[\underline{t}^0, \bar{t}^0]$. What range would these best responses lie in? Call this range $[\underline{t}^1, \bar{t}^1]$. What is $\underline{t_1}^1$? $\underline{t_1}^1$ is a best response to $(\underline{t_2}^0, ..., \underline{t_n}^0) = (-\infty, ..., -\infty)$. Thus, in the spirit of the computations above, $\underline{t_1}^1$ is defined by the solution to the equation, $Pr(G|x) = \frac{c}{1+c}$, which implies $\frac{f(\underline{t_1}^1|B)}{f(\underline{t_1}^1|G)} = \frac{1}{c}$. Thus, $\underline{t_1}^1 \in \mathbb{R}$[12] How about $\underline{t_2}^1$? It is a best response to $\bar{t_1}^0$ and $(\underline{t_3}^0, ..., \underline{t_n}^0)$. Since $\bar{t_1}^0 = \infty$ Player 2 assumes upon observing investment that Player 1 must have received an infinitely high signal, or, equivalently, that the state must be G.[13] So $\underline{t_2}^1 = -\infty$. By the same token $\underline{t_j}^1 = -\infty$ for $j = 3, ..., n$. How about $\bar{t_1}^1$? This is a best response to $(\bar{t_2}^0, ..., \bar{t_n}^0) = (\infty, ..., \infty)$. So, Player 1 knows that Players 2 through n will *never* invest. So, her best response must be to set her own trigger to ∞. Thus, $\bar{t_1}^1 = \infty$. Similarly, $\bar{t_j}^1 = \infty$ for $j = 2, ..., n-1$. However, $\bar{t_n}^1$ is a best response to $(\underline{t_1}^0, ..., \underline{t_{n-1}}^0) = (-\infty, ..., -\infty)$, and so, by reasoning identical to the case of $\underline{t_1}^1$ that $\bar{t_n}^1$ is defined as the solution to $Pr(G|x) = \frac{c}{1+c}$, and thus, $\bar{t_n}^1 = \underline{t_1}^1 \in \mathbb{R}$ Thus,

$$(\underline{t}^1, \bar{t}^1) = ((\underline{t_1}^1, -\infty, ..., -\infty), (\infty, ..., \infty, \bar{t_n}^1))$$

Now consider best responses to triggers in $[\underline{t}^1, \bar{t}^1]$. What range would these best responses lie in? Call this range $[\underline{t}^2, \bar{t}^2]$. Notice first that $\underline{t_1}^2 = \underline{t_1}^1 \in \mathbb{R}$ because they are best reponses to the same triggers. Also, by an argument identical to that constructed for computing $\underline{t_j}^1 = -\infty$ for $j = 2, ..., n$, we observe that $\underline{t_j}^2 = -\infty$ for $j = 2, ..., n$. Similarly, for $j = 1, ..., n-2$, each of $\bar{t_j}^2$ is a best response to at least one successor trigger of

[12] In particular, $\underline{t_1}^1 = U_n(c, \pi, f)$ defined above.

[13] This is an ad hoc refinement that we introduce for this iterative process. It is important to note that the refinement is irrelevant for investment equilibria.

∞. So, for $j = 1, ..., n-2$, $\bar{t_j}^2 = \infty$. However, $\bar{t_{n-1}}^2$ is a best response to predecessors $(\underline{t_1}^1, -\infty, ..., -\infty)$ and successor $\bar{t_n}^1 \in \mathbb{R}$ Thus, $\bar{t_{n-1}}^2$ is chosen to solve the equation $Pr(G, s_n \geq \bar{t_n}^1 | s_1 \geq \underline{t_1}^1, x) = \frac{c}{1+c}$, and thus clearly, $\bar{t_{n-1}}^2 \in \mathbb{R}$ Finally, note that since triggers are decreasing as best responses to their predecessors and since $\underline{t_1}^0 < \underline{t_1}^1$ while $\underline{t_j}^0 = \underline{t_j}^1$ for $j = 2, ..., n$, $\bar{t_n}^2 \leq \bar{t_n}^1$. Thus,

$$(\underline{t}^2, \bar{t}^2) = ((\underline{t_1}^2, -\infty, ..., -\infty), (\infty, ..., \infty, \bar{t_{n-1}}^2, \bar{t_n}^2))$$

By iterating this argument it is easy to see that

$$(\underline{t}^3, \bar{t}^3) = ((\underline{t_1}^3, -\infty, ..., -\infty), (\infty, ..., \infty, \bar{t_{n-2}}^3, \bar{t_{n-1}}^3, \bar{t_n}^3))$$

and so on, until finally

$$(\underline{t}^{n+1}, \bar{t}^{n+1}) = ((\underline{t_1}^{n+1}, ..., \underline{t_n}^{n+1}), (\bar{t_1}^{n+1}, ..., \bar{t_n}^{n+1}))$$

i.e., $(\underline{t}^{n+1}, \bar{t}^{n+1}) \in \mathbb{R}^n \times \mathbb{R}^n$.

It is clear that the iterative process defined above satisfies the property that $\underline{t}^j \geq \underline{t}^{j-1}$ for all j and $\bar{t}^j \leq \bar{t}^{j-1}$ for all j. Thus, since $(\underline{t}^{n+1}, \bar{t}^{n+1}) \in \mathbb{R}^n \times \mathbb{R}^n$ and $(\underline{t}^j, \bar{t}^j) \in [\underline{t}^{n+1}, \bar{t}^{n+1}]^2$ for all $j \geq n+1$, the trigger bound sequence is monotonic and bounded. So it must converge. Thus there exist $\underline{t} \in \mathbb{R}^n$ and $\bar{t} \in \mathbb{R}^n$ such that for all $t \in \Re^n$, $\underline{t} \leq \beta(t) \leq \bar{t}$. $\diamond$

2.6.2 The Single Player Version of $\Gamma(2)$

Consider a modification to $\Gamma(2)$ in which a single player makes all choices, in order, constrained by the same signal structure. She remembers her past signals. Her payoffs are given by the total payoffs at the end of the game contingent upon *both* her choices in periods 1 and 2, by summing the final payoffs from these choices. Thus, payoffs are given by the following:

$$u(G,a_1,a_2) = \begin{cases} 2 & \text{when } a_1 = I \text{ and } a_2 = I, \\ -c & \text{when } a_1 = I \text{ and } a_2 = N, \\ -c & \text{when } a_1 = N \text{ and } a_2 = I, \\ 0 & \text{when } a_1 = N \end{cases}$$

$$u(B,a_1,a_2) = \begin{cases} -2c & \text{when } a_1 = I \text{ and } a_2 = I, \\ -c & \text{when } a_1 = I \text{ and } a_2 = N, \\ -c & \text{when } a_1 = N \text{ and } a_2 = I, \\ 0 & \text{when } a_1 = N \end{cases}$$

We denote the decision maker's updated beliefs about the state after observing the signal in period 1 by π_2. Clearly

$$\frac{\pi_2}{1-\pi_2} = \frac{\pi}{1-\pi}\frac{f(s_1|G)}{f(s_1|B)}.$$

Now, the expected utility to the decision maker of investing in period 2, conditional upon having invested in the previous period is given by

$$EU_2(I) = \frac{\pi_2}{\pi_2 + (1-\pi_2)\frac{f(s_1|B)}{f(s_1|G)}}(2+2c) - 2c.$$

The expected utility to the decision maker of not investing in period 2 conditional upon having invested in period 1 is simply $-c$. Thus, by employing arguments made repeatedly above, the decision maker shall choose a trigger $t_2^{'}$ given by

$$t_2^{'} = r^{-1}\left(\frac{\pi}{1-\pi}\frac{f(s_1|G)}{f(s_1|B)}\frac{2+c}{c}\right).$$

Clearly then, the unique trigger equilibrium of $\Gamma(2)$ is socially optimal only if

$$\frac{Pr(s_1 \geq t_1|G)}{Pr(s_1 \geq t_1|B)} = (2+c)\frac{f(s_1|G)}{f(s_1|B)}.$$

This, of course, happens with zero probability given the assumptions on the signal generating processes. The trigger equilibrium shall be characterized by both overoptimism and underoptimism compared to the socially optimal case with positive probability (for finite c).

Chapter 3

A Theory of Currency Crises with Large and Small Traders

A commonly encountered view among both seasoned market commentators and less experienced observers of the financial markets is that large traders can exercise a disproportionate influence on the likelihood and severity of a financial crisis by fermenting and orchestrating attacks against weakened currency pegs.[1] The famously acrimonious exchange between the financier George Soros and Dr. Mahathir, the prime minister of Malaysia at the height of

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[1] This chapter is coauthored with Giancarlo Corsetti, Stephen Morris, and Hyun Song Shin. It is currently circulating under the title "Does One Soros Make a Difference? A Theory of Currency Crises with Large and Small Traders". We are grateful to the editor, James Dow, and four anonymous referees for their constructive comments. We have also benefited from comments and advice from Richard Clarida, Hélène Rey, Andres Velasco, Shang-Jin Wei and seminar participants at the Bank of Italy, the Board of Governors of the Federal Reserve Bank, Columbia University, Harvard University, the NBER IFM meeting, and the New York Federal Reserve Bank.

the Asian crisis is a prominent example in which such views have been aired and debated. The issues raised by this debate are complex, but they deserve systematic investigation.

At one level, the task is one of dissecting the problem in search of the possible mechanisms (if any) that may be at play in which a large trader may exercise such influence on the market outcome. What is it about the large trader that bestows such influence? Is it merely that this trader can bring to bear larger resources and hence take on larger trading positions? What if the information of the large trader is no better than the small traders in the market? Does the large trader still exercise a disproportionate influence? Finally, does it make a difference to the market outcome as to whether the trading position of the large trader is disclosed publicly to the market? If so, does such "transparency" of the trading position enhance financial stability or undermine it? This last question is especially important given the emphasis placed by policy makers on the public disclosures by the major market participants as a way of forestalling future crises.[2]

We propose to investigate these issues in a model of speculative attacks in which a large trader interacts with a continuum of small traders. The large trader is 'large' by virtue of the size of the speculative position that he can take on as compared to the small traders. The two types of traders face a monetary authority defending a currency peg, and stand to gain if their attack on the peg is successful, but stand to lose if the attack fails to break

[2]The response of the regulators and official bodies to the financial turbulence of 1998 has been to call for greater public disclosures by banks and hedge funds. The recent document from the Financial Stability Forum (2000) reiterates similar calls by the BIS, IOSCO, and the President's Working Group. In contrast, the private sector is more ambivalent towards the value of public disclosures. See, for instance, Counterparty Risk Management Policy Group (1999).

the peg. Both types of traders are well informed about the underlying fundamentals, but they are not perfectly informed. Moreover, we allow the possibility that the information precision of one type of trader is higher than another. We can examine the case in which the large trader is better informed than the small trader and contrast this with the case in which small trader is relatively better informed.

To anticipate our main conclusions to these questions, we can summarize our findings as follows.

- As a general rule, the presence of the large trader *does* increase the incidence of attack against a peg. The reason is not so much that the large trader's market power manufactures these crises, but rather that the presence of the large trader makes the small traders more aggressive in their trading strategies. In other words, the large trader injects a degree of strategic fragility to the market.
- However, within this broad general finding, the relative precision of information between the two types of traders matters.
  - When a typical small trader is better informed than the large trader, the influence of the latter on the market is moderate. His presence can make little or no difference on small traders' strategies.
  - But when the large trader is better informed than a typical small trader, his influence is much larger.
- Finally, the influence of the large trader is magnified greatly when the large trader's trading position is revealed to the small traders prior to their trading decisions. Thus,

when the large trader moves first, and his position is disclosed publicly to other traders before their trading decisions, the impact of the large trader is that much larger. The reason for this added impact lies in the signalling potential of the large trader's first move. To the extent that a speculative attack is the resolution of a coordination problem among the traders, the enhanced opportunity to orchestrate a coordinated attack helps to resolve this collective action problem.

The technical and modelling innovations necessary to reach our main conclusions deserve some attention by itself, and it is to these that we now turn. The theoretical framework employed in this paper is an extension of the incomplete information game formulation used in Morris and Shin (1998). In this earlier setting, the argument makes heavy use of the fact that the game is *symmetric* - that is, all the speculators are identical. This assumption is clearly not available to us in the current setting. It is not at all obvious that the argument used in Morris and Shin (1998) to prove uniqueness of equilibrium is applicable in asymmetric payoffs settings, and one of the contributions of our current paper is to demonstrate that this argument can be used with some modifications.

There is a more subtle, but important theoretical contribution. The incomplete information game approach of Morris and Shin (1998) is an instance of a more general approach to equilibrium selection pioneered by Carlsson and van Damme (1993), in which the type space underlying the game is generated by adding a small amount of noise in the signals of the players concerning some payoff relevant state. Carlsson and van Damme refer to such games as "global games", and the general class of such games turn out to have a rich and interesting structure. Morris and Shin (2000) discuss some general results and applications.

Analysis using global games should be seen as a particular instance of equilibrium selection through perturbations, but it is important to disentangle two distinct sets of results concerning global games. The first question is whether a unique outcome is selected in the game. A second, more subtle, question is whether such a unique outcome depends on the underlying information structure and the structure of the noise in the players' signals. One of the remarkable results for *symmetric* binary action global games is that the answer to the second question is 'no'. In other words, not only is a unique equilibrium selected in the limit as the noise becomes small, but the selected equilibrium is insensitive to the structure of the noise (see Morris and Shin (2000) section 2). However, our second bullet point above points to the fact that, in our model, the structure of the noise *does* make a difference. The equilibrium outcome depends on whether the large trader is relatively better or worse informed as compared to the small traders. Thus, in our *asymmetric* global game, although we have a unique equilibrium being selected, this unique equilibrium depends on the noise structure. It is this latter feature that allows us to draw non-trivial conclusions concerning the economic importance of information. Frankel, Morris and Pauzner (1999) explore the equilibrium selection question in the context of general global games.

Our examination of the sequential move version of the game necessitates a further extension the current state of the art. When moves occur sequentially in which the actions of the early movers are observable to the late movers, herding and signalling effects must be taken into consideration, as well as the usual strategic complementarities. Although a general analysis of sequential move variations of global games is rather intractable, the fact that small traders (individually) are of measure zero in our model allows us to focus attention on the signalling effects of the large trader. This simplifies the analysis sufficiently for us to derive explicit closed form solutions to the game.[3]

The paper is organized as follows. Section 2 lays out the basic framework and establishes two benchmark results in setting the stage for the general analysis. Section 3 characterizes the unique equilibrium in a simultaneous move trading game. Section 4 explores the comparative statics properties of the equilibrium to changes in the traders' information precision. The focus here is on the interaction between the size of the large trader and his information precision. Section 5 investigates the sequential move version of the game. Section 6 concludes.

3.1 The model

The focus of our analysis is on the mechanism by which a fixed exchange parity is abandoned as a result of a speculative attack on the currency. Consider an economy where the central bank pegs the exchange rate. There is a single "large" trader and a continuum of "small" traders. The distinguishing feature of the large trader is that he has access to a sufficiently large line of credit in the domestic currency to take a short position up to the limit of $\lambda < 1$. In contrast, the set of all small traders taken together have a combined trading limit of $1 - \lambda$.

We envisage the short selling as consisting of borrowing the domestic currency and selling it for dollars. There is a cost to engaging in the short selling, denoted by $t > 0$.

[3] Dasgupta (1999) has examined some of the issues that arise with many large players and multi-period signalling.

The cost t can be viewed largely as consisting of the interest rate differential between the domestic currency and dollars, plus transaction costs. This cost is normalized relative to the other payoffs in the game, so that the payoff to a successful attack on the currency is given by 1, and the payoff from refraining from attack is given by 0. Thus, the net payoff to a successful attack on the currency is $1 - t$, while the payoff to an unsuccessful attack is given by $-t$.

Each trader must decide independently, and (for now) simultaneously whether or not to attack the currency. The strength of the economic fundamentals of this economy are indexed by the random variable θ, which has the (improper) uniform prior over the real line.[4]

Whether the current exchange rate parity is viable depends on the strength of the economic fundamentals and the incidence of speculative attack against the peg. The incidence of speculative attack is measured by the mass of traders attacking the currency in the foreign exchange market. Denoting by ℓ the mass of traders attacking the currency, the currency peg fails if and only if

$$\ell \geq \theta \tag{3.1}$$

So, when fundamentals are sufficiently strong (i.e. $\theta > 1$) the currency peg is maintained irrespective of the actions of the speculators. When $\theta \leq 0$, the peg is abandoned even in

[4]Improper priors allow us to concentrate on the updated beliefs of the traders conditional on their signals without taking into account the information contained in the prior distribution. In any case, our results with the improper prior can be seen as the limiting case as the information in the prior density goes to zero. See Hartigan (1983) for a discussion of improper priors, and Morris and Shin (2000, section 2) for a discussion of the latter point.

the absence of a speculative attack. The interesting range is the intermediate case when $0 < \theta \leq 1$. Here, an attack on the currency will bring down the currency provided that the incidence of attack is large enough, but not otherwise. This tripartite classification of fundamentals follows Obstfeld (1996) and Morris and Shin (1998). Although we do not model explicitly the decision of the monetary authorities to relinquish the peg, it may be helpful to keep in mind the example of an economy endowed with a stock of international reserves, where the central bank is willing to defend the exchange rate as long as reserves do not fall below a predetermined critical level. The central bank predetermines this level based on its assessment of the economic fundamentals of the country. The critical level is low when fundamentals are strong (θ is high): the central bank is willing to use a large amount of (non-borrowed and borrowed) reserves in defending the exchange rate. Conversely, the critical level is high when fundamentals are weak (θ is low). Even a mild speculative attack can convince the central bank to abandon the peg.

3.1.1 Information

Although the traders do not observe the realization of θ, they receive informative private signals about it. The large trader observes the realization of the random variable

$$y = \theta + \tau\eta \tag{3.2}$$

where $\tau > 0$ is a constant and η is a random variable with mean zero, and with smooth symmetric density $g(\cdot)$. We write $G(\cdot)$ for the cumulative distribution function for $g(\cdot)$. Similarly, a typical small trader i observes

$$x_i = \theta + \sigma\varepsilon_i \tag{3.3}$$

where $\sigma > 0$ is a constant and the individual specific noise ε_i is distributed according to smooth symmetric density $f(\cdot)$ (write $F(\cdot)$ for the c.d.f.) with mean zero. We assume that ε_i is i.i.d. across traders, and each is independent of η.

A feature already familiar from the discussion of global games in the literature is that even if σ and τ become very small, the realization of θ will not be common knowledge among the traders. Upon receiving his signal, the representative trader i can guess the value of θ, and the distribution of signals reaching the other traders in the economy, as well as of their estimate of θ. He cannot, however, count on the other traders to know what he knows – and agree with his guesses. The other traders will have to rely exclusively on their own information to form their beliefs. This departure from the assumption of common knowledge of the fundamentals, no matter how small, is key to the results to follow. The relative magnitude of the constants σ and τ indexes the relative precision of the information of the two types of traders.

A trader's strategy is a rule of action which maps each realization of his signal to one of two actions - to attack, or to refrain. We will search for Bayes Nash equilibria of the game in which, conditional on each trader's signal, the action prescribed by this trader's strategy maximizes his conditional expected payoff when all other traders follow their strategies in the equilibrium.

3.1.2 Two benchmark Cases

Before proceeding to our main task of solving the game outlined above, we present a brief discussion of the coordination problem under two special cases to set a benchmark for our main results. The first is when all traders are small ($\lambda = 0$), the second is when the sole trader is the large trader himself ($\lambda = 1$).

Small traders only

The case when $\lambda = 0$ takes us into the symmetric game case of Morris and Shin (1998). We will conduct the discussion in terms of switching strategies in which traders attack the currency if the signal falls below a critical value x^*. We will show later that this is without loss of generality, and that there are no other equilibria in possibly more complex strategies. The unique equilibrium can be characterized by a critical value θ^* below which the currency will always collapse, and a critical value of the individual signal x^* such that individuals receiving a signal below this value will always attack. To derive these critical values, note first that, if the true state is θ and traders attack only if they observed a signal below x^*, the probability that any particular trader receives a signal below this level is

$$\text{prob}(x_i \leq x^* \mid \theta) = F\left(\frac{x^* - \theta}{\sigma}\right) \tag{3.4}$$

Since the noise terms $\{\varepsilon_i\}$ are i.i.d., the incidence of attack ℓ is equal to this probability. We know that an attack will be successful only if $\ell \geq \theta$. The critical state θ^* is where this holds with equality. Thus, the first equilibrium condition – a "critical mass condition" – is

$$F\left(\frac{x^* - \theta^*}{\sigma}\right) = \theta^*. \tag{3.5}$$

Given x^*, any realization of the fundamental $\theta \leq \theta^*$ is associated with a successful speculative attack on the currency.

Second, consider the optimal trigger strategy for an trader receiving a signal x_i, given θ^*. The trader has the conditional probability of a successful attack of

$$\text{prob}(\theta \leq \theta^* \mid x_i) = F\left(\frac{\theta^* - x_i}{\sigma}\right), \tag{3.6}$$

and hence attacks if and only if his expected gross payoff is at least as high as the cost of attack t. As the expected payoff to attacking for a marginal trader receiving a signal x^* must be 0, the "optimal cutoff" condition for x^* is

$$F\left(\frac{\theta^* - x^*}{\sigma}\right) = t. \tag{3.7}$$

Solving for the equilibrium entails solving the pair of equations above. Equation (3.7) gives $\theta^* = x^* + \sigma F^{-1}(t)$; substituting into (3.5) gives $\theta^* = F\left(-F^{-1}(t)\right) = 1 - F\left(F^{-1}(t)\right) = 1 - t$. We obtain the following proposition

Proposition 3.1. *If $\lambda = 0$,*

$$\begin{aligned} x^* &= 1 - t - \sigma F^{-1}(t) \\ \theta^* &= 1 - t \end{aligned}$$

The currency will collapse for any realization of the fundamental θ smaller than $1 - t$, while each individual trader will attack the currency for any realization of his signal below $1 - t - \sigma F^{-1}(t)$.[5] Note that this trigger tends to $1 - t$ as $\sigma \to 0$.

A single large trader

We now consider the opposite extreme case of $\lambda = 1$, in which there is a single large trader. This reduces the game to a single person decision problem, and implies a trivial solution to the coordination problem described above. As this single trader controls the market, there is no need of an equilibrium condition equivalent to the "critical mass condition" (3.5). The

[5] For $t < 1/2$, $F^{-1}(t)$ is a negative number, so that $x^* > \theta^*$. As $\sigma \to 0$, i.e. letting the private signal become arbitrarily precise, the optimal cutoff point will tend to the fundamental threshold, $x^* \to \theta^*$.

only condition that is relevant for a single large risk-neutral trader is the "optimal cutoff": he will attack the currency if and only the expected payoff from a speculative position is non-negative, that is when

$$G\left(\frac{1-y}{\tau}\right) \geq t$$

Thus he attacks if and only if $y \leq y^* = 1 - \tau G^{-1}(t)$. Note that the trigger y is smaller than one, but tends to 1 as $\tau \to 0$.

3.2 Equilibrium with Small and Large Traders

We can now turn to the general case when there are both small and large traders. We will show that there is a unique, dominance solvable equilibrium in this case in which both types of traders follow their respective trigger strategies around the critical points x^* and y^*. The argument will be presented in two steps. We will first confine our attention to solving for an equilibrium in trigger strategies, and then proceed to show that this solution can be obtained by the iterated deletion of strictly interim dominated strategies.

Thus, as the first step let us suppose that the small traders follow the trigger strategy around x^*. Because there is a continuum of small traders, conditional on θ, there is no aggregate uncertainty about the proportion of small traders attacking the currency. Since $F\left(\frac{x^*-\theta}{\sigma}\right)$ is the proportion of small traders observing a signal lower than x^* and therefore attacking at θ, an attack by small traders alone is sufficient to break the peg at θ if $(1-\lambda) F\left(\frac{x^*-\theta}{\sigma}\right) \geq \theta$. From this, we can define a level of fundamentals below which an attack by the small traders alone is sufficient to break the peg. Let $\underline{\theta}$ be defined by:

$$(1-\lambda) F\left(\frac{x^* - \underline{\theta}}{\sigma}\right) = \underline{\theta} \tag{3.8}$$

Whenever θ is below $\underline{\theta}$, the attack is successful irrespective of the action of the large trader. Note that $\underline{\theta}$ lies between 0 and $1-\lambda$. Clearly $\underline{\theta}$ is a function of x^*.

Next, we can consider the additional speculative pressure brought by the large trader. If the small traders follow the trigger strategy around x^*, the incidence of attack at θ attributable to the small traders is $(1-\lambda)\,F\left(\frac{x^*-\theta}{\sigma}\right)$. If the large trader also chooses to attack, then there is an additional λ to this incidence. Hence, if the large trader participates in the attack, the peg is broken whenever $\lambda+(1-\lambda)\,F\left(\frac{x^*-\theta}{\sigma}\right)\geq\theta$. Thus we can define the critical value of the fundamentals at which an attack is successful if and only if the large trader participates in the attack. It is defined by

$$\lambda+(1-\lambda)\,F\left(\frac{x^*-\overline{\theta}}{\sigma}\right)=\overline{\theta} \tag{3.9}$$

$\bar{\theta}$ lies between $\underline{\theta}$ and 1.

Although our notation does not make it explicit, both $\underline{\theta}$ and $\overline{\theta}$ are functions of the switching point x^*. In turn, x^* will depend on the large trader's switching point y^*. Our task is to solve these two switching points simultaneously from the respective optimization problems of the traders. A large trader observing signal y assigns probability $G\left(\frac{\overline{\theta}-y}{\tau}\right)$ to the event that $\theta\leq\overline{\theta}$. Since his expected payoff to attacking conditional on y is $G\left(\frac{\overline{\theta}-y}{\tau}\right)-t$, his optimal strategy is to attack if and only if $y\leq y^*$, where y^* is defined by:

$$G\left(\frac{\overline{\theta}-y^*}{\tau}\right)=t \tag{3.10}$$

Now consider a small trader. Conditional on signal x, the posterior density over θ for this trader is given by

$$\frac{1}{\sigma}f\left(\frac{\theta-x}{\sigma}\right) \tag{3.11}$$

When $\theta \leq \underline{\theta}$, the strategies of the small traders are sufficient for a successful attack. When $\theta \in (\underline{\theta}, \overline{\theta}]$ the peg breaks if and only if the large trader attacks, while if $\theta > \overline{\theta}$, the peg withstands the attacks, irrespective of the actions of the traders. Thus, the expected payoff to attack conditional on signal x can be written as

$$\frac{1}{\sigma}\int_{-\infty}^{\underline{\theta}} f\left(\frac{\theta - x}{\sigma}\right) d\theta + \frac{1}{\sigma}\int_{\underline{\theta}}^{\overline{\theta}} f\left(\frac{\theta - x}{\sigma}\right) G\left(\frac{y^* - \theta}{\tau}\right) d\theta \tag{3.12}$$

The first term is the portion of expected payoff attributable to the region of θ where $\theta \leq \underline{\theta}$. The second term is the portion of expected payoff that is attributable to the interval $(\underline{\theta}, \overline{\theta}]$. Here, one must take into account the fact that the attack is successful if and only if the large trader attacks. The probability that the large trader attacks at θ given his trigger strategy around y^* is given by $G\left(\frac{y^*-\theta}{\tau}\right)$, so that the payoffs are weighted by this value. Beyond $\bar{\theta}$, the attack is never successful, so that the payoff to attack is zero. Since the cost of attack is t, the trigger point x^* for the small trader is defined by the equation:

$$\frac{1}{\sigma}\int_{-\infty}^{\underline{\theta}} f\left(\frac{\theta - x^*}{\sigma}\right) d\theta + \frac{1}{\sigma}\int_{\underline{\theta}}^{\overline{\theta}} f\left(\frac{\theta - x^*}{\sigma}\right) G\left(\frac{y^* - \theta}{\tau}\right) d\theta = t \tag{3.13}$$

There is a unique x^* that solves this equation. To see this, it is helpful to introduce a change of variables in the integrals. Let

$$z \equiv \frac{\theta - x^*}{\sigma} \tag{3.14}$$

and denote

$$\underline{\delta} \equiv \frac{\underline{\theta} - x^*}{\sigma} \text{ and } \overline{\delta} \equiv \frac{\overline{\theta} - x^*}{\sigma}. \tag{3.15}$$

Then, the conditional expected payoff to attacking given signal x^* is

$$\frac{1}{\sigma}\int_{-\infty}^{\underline{\theta}} f\left(\frac{\theta - x^*}{\sigma}\right) d\theta + \frac{1}{\sigma}\int_{\underline{\theta}}^{\overline{\theta}} f\left(\frac{\theta - x^*}{\sigma}\right) G\left(\frac{y^* - \theta}{\tau}\right) d\theta$$

$$= \int_{-\infty}^{\frac{\underline{\theta}-x^*}{\sigma}} f(z)\, dz + \int_{\frac{\underline{\theta}-x^*}{\sigma}}^{\frac{\overline{\theta}-x^*}{\sigma}} f(z)\, G\left(\frac{y^* - \theta}{\tau}\right) dz$$

$$= \int_{-\infty}^{\frac{\underline{\theta}-x^*}{\sigma}} f(z)\, dz + \int_{\frac{\underline{\theta}-x^*}{\sigma}}^{\frac{\overline{\theta}-x^*}{\sigma}} f(z)\, G\left(\frac{\overline{\theta} - x^* - \sigma z}{\tau} - G^{-1}(t)\right) dz$$

$$= \int_{-\infty}^{\underline{\delta}} f(z)\, dz + \int_{\underline{\delta}}^{\overline{\delta}} f(z)\, G\left(\frac{\sigma}{\tau}\left(\overline{\delta} - z\right) - G^{-1}(t)\right) dz \qquad (3.16)$$

where the third line follows from the fact that

$$y^* = \overline{\theta} - \tau G^{-1}(t) \qquad (3.17)$$
$$= x^* + \sigma\overline{\delta} - \tau G^{-1}(t).$$

Hence, (3.13) gives:

$$\int_{-\infty}^{\underline{\delta}} f(z)\, dz + \int_{\underline{\delta}}^{\overline{\delta}} f(z)\, G\left(\frac{\sigma}{\tau}\left(\overline{\delta} - z\right) - G^{-1}(t)\right) dz - t = 0 \qquad (3.18)$$

However, note that both $\underline{\delta}$ and $\overline{\delta}$ are monotonically decreasing in x^*, since

$$\frac{d\underline{\delta}}{dx^*} = -\frac{1}{(1-\lambda) f\left(\underline{\delta}\right) + \sigma} < 0$$
$$\frac{d\overline{\delta}}{dx^*} = -\frac{1}{(1-\lambda) f\left(\overline{\delta}\right) + \sigma} < 0$$

Since the left hand side of (3.18) is strictly increasing in both $\underline{\delta}$ and $\overline{\delta}$, it is strictly decreasing in x^*. For sufficiently small x^*, the left hand side of (3.18) is positive, while for sufficiently large x^*, it is negative. Since the left hand side is continuous in x^*, there is a unique solution to (3.18). Once x^* is determined, the large trader's switching point y^* follows from (3.10).

To this point, we have confined our attention to trigger strategies, and have shown that there is a unique equilibrium within this class of strategies. We can show that confining our attention to trigger strategies is without loss of generality. The trigger equilibrium identified above turns out to be the only set of strategies that survive the iterated elimination of strictly interim dominated strategies. The dominance solvability property is by now well understood for symmetric binary action global games (see Morris and Shin (2000) for sufficient conditions for this property). The contribution here is to show that it also applies in our *asymmetric* global game. The argument is presented separately in appendix A.

3.3 Impact of Large Trader

Having established the uniqueness of equilibrium, we can now address the main question of whether there is any increased fragility of the peg, and how much of this can be attributed to the large trader. There are two natural questions. Do the small traders become more aggressive sellers when the large trader is in the market? Secondly, does the probability of the peg's collapse increase when the large trader is in the market? These questions relate to the following comparative statics questions.

- How does the switching point x^* for the small traders depend on the presence of the large trader?
- How is the incidence of attack by the small traders at a given state (i.e. $\ell(\theta)$) affected

by the large trader's presence?

- How is the probability of the peg's collapse at a given state θ affected by the presence of the large trader?

- How is the ex ante probability of the peg's collapse affected by the large trader?[6]

This final bullet point may seem incongruous when taken at face value, since our model has made use of the assumption that θ has an (improper) uniform prior distribution, so that the ex ante expectations are not well defined. However, there is an interpretation of our model that allows us to comment on this issue for general priors over θ. When the signals received by the traders (small and large) are very precise relative to the information contained in the prior, then a uniform prior over θ serves as a good approximation in generating the conditional beliefs of the traders. Then, the equilibrium obtained under the uniform prior assumption will be a good approximation to the true equilibrium. If we can say something about the critical state θ at which the peg collapses, then we may give an approximate answer for the ex ante probability of collapse by evaluating the prior distribution $H(\cdot)$ at this state. We will comment below on one instance when this type of argument can be made.

A more substantial theme in our comparative statics exercise is to disentangle the effects of the *size* of the large trader (through the size of the trading position that he can amass) from his *precision of information* relative to small traders. If we can interpret the large trader as a coalition of small traders who pool their resources as well as their information,

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[6]We are grateful to a referee for encouraging us to address this question.

then it would be natural to assume that the large trader has very much better information as compared to individual small traders[7]. There may be other reasons to do with resources that a large player can bring to bear on research or access to contacts in policy circles that makes it more reasonable to assume that the large trader is better informed than the typical small trader. However, there is no reason in principle why the large trader must be better informed. In any case, the separation of the effects of size from that of information is a valuable exercise in understanding the impact of each, and so we will be careful in distinguishing these two effects.

For the purpose of the comparative statics exercise, it may help the reader to gather together and restate the key relationships that determine equilibrium. Using the notation $\bar{\delta} = \frac{\bar{\theta}-x^*}{\sigma}$, and $\underline{\delta} = \frac{\underline{\theta}-x^*}{\sigma}$ that we introduced earlier, we restate equations (3.8), (3.9), (3.10) and (3.18) as follows.

$$(1-\lambda)\left(1-F\left(\underline{\delta}\right)\right)=\underline{\theta} \tag{3.19}$$

$$\lambda+(1-\lambda)\left(1-F\left(\bar{\delta}\right)\right)=\bar{\theta} \tag{3.20}$$

$$G\left(\frac{\bar{\theta}-y^*}{\tau}\right)=t \tag{3.21}$$

$$\int_{-\infty}^{\underline{\delta}} f(z)\,dz+\int_{\underline{\delta}}^{\bar{\delta}} f(z)\,G\left(\frac{\sigma}{\tau}\left(\bar{\delta}-z\right)-G^{-1}(t)\right)dz=t \tag{3.22}$$

These four equations jointly determine the switching points x^* and y^* and the critical states $\underline{\theta}$ and $\bar{\theta}$. Obtaining definitive answers to the comparative statics questions can sometimes

[7] We are grateful to a referee for pointing this out to us.

be difficult for general parameter values, although we will examine a number of simulation exercises below that suggest that the equilibrium behaves in intuitive ways. In contrast to the difficulties for general parameter values, the limiting case where both types of traders have very precise information gives us quite tractable expressions that yield relatively clear cut results. Sometimes, even closed form solutions of the equilibrium are possible.

3.3.1 Comparative Statics in the Limiting Case

Let us examine the properities of the equilibrium in the limiting case where

$$\sigma \to 0, \quad \tau \to 0, \quad \text{and} \quad \frac{\sigma}{\tau} \to r$$

In other words, both types of traders have more and more precise information, but the noisiness of the small traders' signals relative to the large trader's signal tends to r. We allow r to be infinite, in which case the large trader's information becomes arbitrarily more precise than the small traders' signals.

One reason for the tractability of the limiting case is that we can identify $\bar{\theta}$ as the critical state at which the peg fails. That is, the peg fails if and only if $\theta < \bar{\theta}$. We can see from equation (3.21) that as $\tau \to 0$, we must have $y^* \to \bar{\theta}$, or else the left hand side of (3.21) will be either 0 or 1 instead of being equal to t. Hence, in the limit, the large trader always attacks in states smaller than $\bar{\theta}$ and refrains from attack at states greater than $\bar{\theta}$. Of course, when the small traders also have very precise information, their switching strategies must be such that they attack precisely when the true state is to the left of $\bar{\theta}$. Thus, in the limit, we must have

$$x^* = y^* = \bar{\theta} \tag{3.23}$$

and the peg fails if and only if $\theta < \bar{\theta}$. This means that the comparative statics questions raised above in the first three bullet points collapse to a single question of whether the true state θ is to the left or right of the critical state $\bar{\theta}$. Also, the fourth bullet point concerning the ex ante probability of the peg's failure can be given an approximate answer. Thus, following the earlier discussion, if $H(\cdot)$ is the prior distribution function for θ, then the ex ante probability of the peg's failure is given approximately by $H\left(\bar{\theta}\right)$. Thus, comparative statics on the prior probability of collapse can be reduced to the behaviour of $\bar{\theta}$. In this sense, the comparative statics questions all hinge on the behaviour of the critical state $\bar{\theta}$.

In solving for the critical state $\bar{\theta}$ in the limiting case, it is important to distinguish two cases. We can distinguish the case when $\bar{\theta} \leq 1 - \lambda$ from the case where $\bar{\theta} > 1 - \lambda$. In the former case, $\underline{\theta} = \bar{\theta}$. However, when $\bar{\theta} > 1 - \lambda$, $\underline{\theta} < \bar{\theta}$. In general, we can characterize the equilibrium value of $\bar{\theta}$ in the limit as follows.

Proposition 3.2. *In the limit as $\sigma \to 0, \tau \to 0$ and $\frac{\sigma}{\tau} \to r$, the critical state $\bar{\theta}$ tends to $\lambda + (1-\lambda)\left(1 - F\left(\bar{\delta}\right)\right)$, where $\bar{\delta}$ falls under two cases. If $\bar{\theta} > 1 - \lambda$, then $\bar{\delta}$ is the unique solution to*

$$\int_{-\infty}^{\bar{\delta}} f(z)\, G\left(r\left(\bar{\delta} - z\right) - G^{-1}(t)\right) dz = t \tag{3.24}$$

If $\bar{\theta} \leq 1 - \lambda$, then $\bar{\delta}$ is the unique solution to

$$\int_{-\infty}^{L} f(z)\, dz + \int_{L}^{\bar{\delta}} f(z)\, G\left(r\left(\bar{\delta} - z\right) - G^{-1}(t)\right) dz = t \tag{3.25}$$

where

$$L = F^{-1}\left(F\left(\bar{\delta}\right) - \frac{\lambda}{1-\lambda}\right).$$

The proof of this result is given in appendix B. We know from (3.20) that $\bar{\theta} = \lambda + (1-\lambda)\left(1 - F\left(\bar{\delta}\right)\right)$, and the main task in the proof is to show that equation (3.22) takes the two cases above when taking into account whether $\bar{\theta}$ is smaller or larger than $1-\lambda$. However, this result allows us to give a definitive answer to the question of how the critical state $\bar{\theta}$ depends on the relative precision of information between the small and large traders. In both equations (3.24) and (3.25), we see that the left hand side is strictly increasing in both r and $\bar{\delta}$. Hence, as r increases, $\bar{\delta}$ must decrease. Since $\bar{\theta}$ tends to $\lambda + (1-\lambda)\left(1 - F\left(\bar{\delta}\right)\right)$, we have:

Proposition 3.3. *In the limit as $\sigma \to 0, \tau \to 0$ and $\frac{\sigma}{\tau} \to r$, the critical state $\bar{\theta}$ is strictly increasing in r.*

In other words, when the small traders' information deteriorates relative to the large trader, the critical state moves up, increasing the incidence of attack and raising the probability of the failure of the peg.

Interestingly, it is not always possible to give a definitive answer to the question of whether $\bar{\theta}$ is increasing in λ - the size of the large trader. When λ is small, so that $\bar{\theta} \leq 1-\lambda$, we are in the range covered by equation (3.25). The left hand side of this equation is decreasing in λ through its effect on L so that $\bar{\delta}$ is increasing in λ. Since $\bar{\theta} = \lambda + (1-\lambda)\left(1 - F\left(\bar{\delta}\right)\right)$, the overall effect of λ is given by

$$\frac{d\bar{\theta}}{d\lambda} = F\left(\bar{\delta}\right) - (1-\lambda) f\left(\bar{\delta}\right) \frac{d\bar{\delta}}{d\lambda}$$

whose sign cannot be tied down definitively. It is only when λ is large (so that $\bar{\theta} > 1-\lambda$) that we have an unambiguous increase in $\bar{\theta}$ as λ increases. This is so, since the left hand side of (3.24) does not depend on λ, so that $\frac{d\bar{\theta}}{d\lambda} = F\left(\bar{\delta}\right) > 0$.

Proposition 3.4. *In the limit as $\sigma \to 0, \tau \to 0$ and $\frac{\sigma}{\tau} \to r$, the critical state $\bar{\theta}$ is strictly increasing in λ provided that $\lambda > 1 - \bar{\theta}$.*

Thus, when we separate the "size effect" of the large trader from the "information effect", we have the following conclusion. Whereas the incidence of attack on the currency is unambiguously increasing in the information precision of the large trader, the local effect of an increase in the size of the large trader may be negligible or even negative when the large trader is small. However, even though the size effect is ambiguous *locally*, we have an argument that shows that it is *always positive globally*. That is, the critical state $\bar{\theta}$ when $\lambda > 0$ cannot be smaller than the critical state when $\lambda = 0$.

We do this by solving for the critical state in two special cases. The first is when $r \to \infty$ (when the large trader is arbitrarily better informed), and the second is when $r = 0$ (when it is the small traders who are arbitrarily better informed). We know from proposition 3.3 that the solution of $\bar{\theta}$ is monotonic in r. Thus, if we can show that the closed form solution to $r = 0$ is non-decreasing in λ, we will have shown that the size effect is non-negative in a global sense. Both closed form solutions can be obtained as a corollary to proposition 3.2.

First, consider the case where $r \to \infty$. Then, both equations (3.24) and (3.25) become

$$\int_{-\infty}^{\bar{\delta}} f(z)\, dz = t$$

so that $F\left(\bar{\delta}\right) = t$. Since $\bar{\theta} = \lambda + (1-\lambda)\left(1 - F\left(\bar{\delta}\right)\right)$, we have

$$\bar{\theta} = \lambda + (1-\lambda)(1-t)$$

Next, consider the case where $r = 0$. Here, we need to keep track of the two cases in proposition 3.2. Equation (3.24) becomes

$$(1-t)\, F\left(\bar{\delta}\right) = t$$

while (3.25) reduces to

$$F\left(\bar{\delta}\right)-t\frac{\lambda}{1-\lambda}=t$$

so that

$$F\left(\bar{\delta}\right)=\begin{cases}\frac{t}{1-t} & \text{if } \bar{\theta}>1-\lambda\\[2ex]\frac{t}{1-\lambda} & \text{if } \bar{\theta}\leq 1-\lambda\end{cases}$$

Since $\bar{\theta}=\lambda+(1-\lambda)\left(1-F\left(\bar{\delta}\right)\right)$, we have $\bar{\theta}>1-\lambda$ if and only if $\lambda>\frac{F(\bar{\delta})}{1+F(\bar{\delta})}$, and

$$F\left(\bar{\delta}\right)=\begin{cases}\frac{t}{1-t} & \text{if } \lambda>t\\[2ex]\frac{t}{1-\lambda} & \text{if } \lambda\leq t\end{cases}$$

Thus, we can obtain the expression for the critical state $\bar{\theta}$ as follows.

$$\bar{\theta}=\begin{cases}\lambda+(1-\lambda)\left(1-\frac{t}{1-t}\right) & \text{if } \lambda>t\\[2ex]1-t & \text{if } \lambda\leq t\end{cases}$$

These closed form solutions are presented in the following table.

Limiting properties of the equilibrium:

Equilibrium value of the critical state $\bar{\theta}$

by size and relative precision of the large trader

	Size:	$\lambda>t$	$t>\lambda>0$	$\lambda=0$
Information precision				
$\frac{\sigma}{\tau}\to\infty$		$1-t+\lambda t$	$1-t+\lambda t$	$1-t$
$\frac{\sigma}{\tau}\to 0$		$1-t+\lambda t-t^2\frac{1-\lambda}{1-t}$	$1-t$	$1-t$

The two closed form solutions define the bounds on the critical state $\bar{\theta}$. We know from proposition 3.3 that the critical state is increasing in r, so that the function that maps λ to the critical state for general values of r must lie in the triangular region bordered by the two closed form solutions. Note, in particular that for any r and any λ, the critical state at λ is no lower than at zero. Thus, the size effect is positive in a global sense, even if it may fail to be positive locally.

As an illustration of the global size effect, we report in figure 3.1 the plot generated by a simulation exercise where $\sigma = 0.01$, $\tau = 0.01$ (so that $r = 1$) and $t = 0.4$. F and G are Standard Normal. The dotted lines are the solutions for $\bar{\theta}$ for the two special cases as

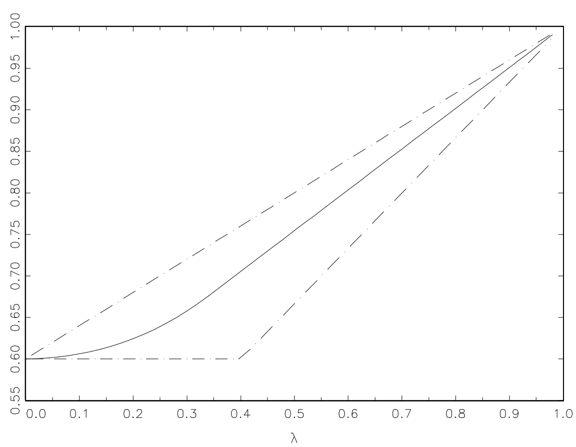

Figure 3.1: $\bar{\theta}$ at $r = 1$ as a function of λ ; $t = 0.4$, $\sigma = \tau = 0.01$

shown already in the table. The solid line is the plot for $\sigma = \tau = 0.01$ as λ varies. We can see that the size effect in this case is positive both locally and globally.

3.3.2 Comparative Statics Away from the Limit

In contrast to the clean comparative statics results in the limiting case, the results away from the limit are not so clear cut. Take, for instance, the question of the probability of the collapse of the peg conditional on some state θ. By the definition of the critical states $\underline{\theta}$ and $\bar{\theta}$, the peg will always fail to the left of $\underline{\theta}$, never fail to the right of $\bar{\theta}$, but in the interval in between $\underline{\theta}$ and $\bar{\theta}$, failure depends on whether the large trader attacks or not. The probability that the large trader attacks at θ is given by $G\left(\frac{y^*-\theta}{\tau}\right)$, but since $y^* = \bar{\theta} - \tau G^{-1}(t)$, the probability of attack is given by $G\left(\frac{\bar{\theta}-\theta}{\tau} - G^{-1}(t)\right)$. Thus, the probability that the peg will fail at state θ is

$$\begin{cases} 1 & \text{if } \theta < \underline{\theta} \\ G\left(\frac{\bar{\theta}-\theta}{\tau} - G^{-1}(t)\right) & \text{if } \underline{\theta} \leq \theta < \bar{\theta} \\ 0 & \text{if } \theta \geq \bar{\theta} \end{cases}$$

In comparison, we know from the analysis of the case where $\lambda = 0$ (i.e. with no large trader) that the fail fails if and only if $\theta < 1 - t$. Thus, the question of whether the peg is more likely to fail with the large trader depends on the relative sizes of $\underline{\theta}$ and $\bar{\theta}$ compared to $1 - t$. To address this and other related questions, it is useful to consider bounds on the critical states $\underline{\theta}$ and $\bar{\theta}$. In particular, we can show that:

Proposition 3.5. *For any σ and τ, and $\lambda > 0$, the critical states $\underline{\theta}$ and $\bar{\theta}$ satisfy*

$$\min\left\{1 - t, \lambda + (1-\lambda)\left(1 - \frac{t}{1+t}\right)\right\} < \bar{\theta} < \lambda + (1-\lambda)(1-t)$$

$$\min\left\{1 - \lambda - t, (1-\lambda)\left(1 - \frac{t}{1+t}\right)\right\} \leq \underline{\theta} \leq \min\{1-\lambda, \lambda + (1-\lambda)(1-t)\}$$

The proof of this result is given in appendix C. For the immediate question at hand concerning the relative sizes of $\underline{\theta}$ and $\bar{\theta}$ compared to $1 - t$, we now that $\bar{\theta} > 1 - t$, but it is possible that $\underline{\theta} < 1 - t$. Thus, in general, we cannot give a definitive answer to whether the presence of the large trader increases the probability of the peg's collapse. For states $\theta \in (\underline{\theta}, 1 - t)$ the probability of collapse decreases, but for states $\theta \in \left(1 - t, \bar{\theta}\right)$, the probability of collapse increases. Beyond this, we cannot say anything further. This lack of a definite answer stands in contrast to the limiting case that we examined above.

In order to examine the effect of the large trader on the small traders' strategies, we examined a number of numerical calculations on the threshold x^* of the small traders' switching strategies. We were particularly interested in putting to the test a conjecture that, in some instances, the presence of the large trader would make the smaller traders *less aggressive* than in the case without the large trader[8]. The reasoning is as follows. The presence of the large trader makes coordination easier and therefore, all else being equal, promotes aggression. However, if the large trader is less well informed than the small traders, his presence may actually make coordination harder because the correlation between his choice and that of the small traders will be low.

This conjecture is intuitively plausible, and we examined a number of numerical solutions for the equilibrium threshold x^* when σ is small but τ is large. However, the simulations have so far proved inconclusive. In those cases where $\sigma \to 0$, we found that the terms of order σ are smaller than the error bounds of the simulations, however accurate. For the various cases we have examined where σ and τ are bounded away from zero, we do not find support for the conjecture - the thresholds x^* are higher with the large trader than without.

[8] This conjecture is due to a referee, and the phrasing of the conjecture is taken verbatim from his/her report. We record our thanks to this referee for suggesting this conjecture.

We regard these simulations as being inconclusive, and so it is an open question whether the conjecture is borne out in a concrete example.

Finally, we report on the numerical calculations of a benchmark case where $F = G = N(0,1)$, and $\sigma = \tau$. We plotted the threshold value x^* for the small traders as a function of the (common) precision of the signals of the two types of traders when $\lambda = 0.5$. Figure 3.2 is for $t = 0.4$, and figure 3.3 is for $t = 0.6$. The dotted line is the threshold without the large trader, while the solid line is the threshold with the large trader. Whether the curves are upward sloping curves or downward sloping depends on whether x^* is positioned the left or right of the critical states $\underline{\theta}$ and $\bar{\theta}$. This, in turn depends on whether t is less than or greater than 0.5. The two figures reveal that the threshold with the large trader is everywhere higher than the threshold without.

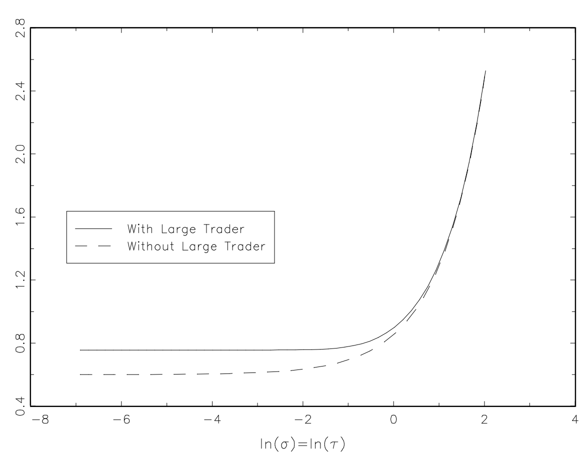

Figure 3.2: x^* with and without the large trader; $t = 0.4$, $\lambda = 0.5$

In gathering together our discussion, the overall conclusion we draw from our analysis is that both the "size effect" and the "information effect" are important determinants in

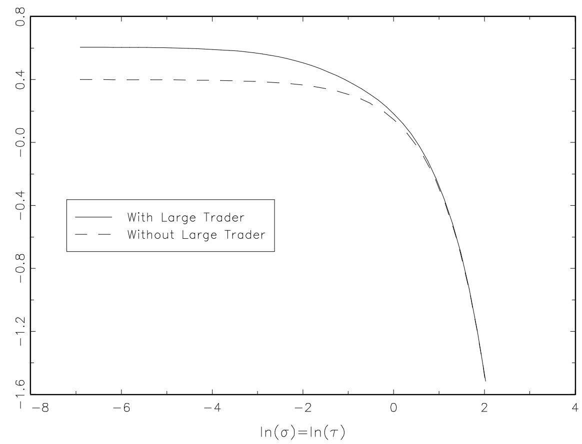

Figure 3.3: x^* with and without the large trader; $t = 0.6$, $\lambda = 0.5$

increasing the probability of collapse. The conclusions are most clear cut in the limiting cases where both σ and τ go to zero, but even away from the limit, numerical calculations reveal that the equilibrium exhibits both types of effects in a consistent way. Having said this, we do not doubt that counterexamples can be obtained for suitably extreme parameter values.

3.4 Sequential Move Game

An important feature of large traders is their visibility in the market - a feature that is only captured to a limited extent by our framework so far. Market participants know the degree of precision of the large trader information, but have no prior information about the exact speculative position of the large trader. In this section, we explore the predictions of our model under a more general assumption regarding observability of actions. Specifically, we let the speculative position taken by any market participant to be observable by the rest of the market. We will see that in equilibrium the large trader will have an incentive to move before the others, so as to maximize his influence.

The analytical framework adopted in this section has essentially the same features of the model presented in Section 2. The main difference is that, instead of analyzing a simultaneous move by all traders, we now allow traders to take a speculative position in either of two periods, 1 and 2, preceding the government decision on the exchange rate. At the beginning of each period, each trader gets a chance to choose an action. However, once he has attacked the currency, he may not do so again and may not reverse his position. So, each trader can choose when, if at all, to attack the currency.

Traders receive their private signal (x_i and y) at the beginning of period 1. In addition, traders are now also able to observe at the beginning of period 2, the action choices of other traders in period 1. Thus, traders can learn from the actions of other market participants, and also use their own actions to signal to other traders. We assume that individual small traders ignore the signalling effect of their actions.[9] Payoffs are the same as in section 2, and are realized at the end of period 2. Payoffs do not depend on the timing of traders' actions, i.e., there are no costs of waiting.[10]

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[9]Levine and Pesendorfer (1995) and others have provided formal limiting justifications for this standard assumption in continuum player games.

[10]Our two period game is best interpreted as a discrete depiction of a continuous time setting, in which the difference between the time periods is very small and represents the time it takes traders to observe and respond to others' actions.

3.4.1 Equilibrium

We begin by making two simple observations about timing incentives in the sequential move game. Small traders will always have an incentive to postpone any action until period 2. Each trader perceives no benefit to signalling, because he believes that he has no power to influence the actions by others by attacking early. On the other hand, he will learn something by waiting to attack: he will find out the large trader's action and he may learn more about the state of the world. There are no costs of waiting, but there is a weak informational benefit to doing so. So it is a dominant strategy for each small trader to wait to period 2 before deciding whether to attack or not. But if small traders wait until period 2, the large trader knows that in equilibrium he can never learn from the actions of the small traders. On the other hand, he knows that if he attacks in period 1, he will send a signal to the small traders, and thereby influence their actions. In particular, since the large trader is concerned with coordinating his actions with those of the continuum of small traders, he benefits from signalling to the small traders. Thus the large trader has a weak incentive to attack in period 1, if he is ever going to attack. Given that small traders wait until period 2, it is a dominant strategy for the larger trader to move early. For these reasons, we assume in the analysis that follows that the large trader moves in period 1 and the small traders moves in period 2.

We first characterize trigger equilibria in this game. Suppose that the large trader, acting first, chooses to attack only if his signal is lower than y^*. If he does not attack in period 1, small traders that receive a low enough signal may nonetheless attack the currency, thinking that they can bring the currency down alone. This will define a threshold $\underline{x}^*$ for the signal of small traders, below which these would attack in period 2 even if the large trader has not attacked in period 1. But if the large trader does attack the currency in period 1, then of course this sends a signal to the small traders that (based upon his information) the large trader believes the economy to be weak enough to risk speculating. When the large trader attacks in period 1, small traders would therefore be inclined to attack for a larger range of signals they might receive. This defines a different threshold $\overline{x}^*$ for their signal, where in equilibrium $\underline{x}^* \leq \overline{x}^*$. We should note here that these thresholds need not be finite. As shown below, there are situations in which the move by the large trader in period 1 will completely determine the behavior of small traders.

Since traders' signals are correlated with fundamentals, corresponding to these triggers are critical mass conditions, i.e. threshold levels for the fundamentals below which there will be always a successful attack. As before, we can derive two conditions, depending on whether the large trader participates in the attack, $(\bar{\theta})$, or not $(\underline{\theta})$.

A trigger equilibrium is then a 5-tuple $\left(y^*, \underline{x}^*, \overline{x}^*, \underline{\theta}, \bar{\theta}\right)$. The equilibrium conditions described above now become:

- y^* solves the equation

$$\Pr\left(\theta \leq \bar{\theta} \mid y = y^*\right) = t \tag{3.26}$$

- $\underline{x}^*$ solves the equation

$$\Pr\left(\theta \leq \underline{\theta} \mid y > y^* \text{ and } x_i = \underline{x}^*\right) = t \tag{3.27}$$

  if a solution exists. If the LHS is strictly larger than the RHS for all x_i, $\underline{x}^* = \infty$. Conversely, if the LHS is strictly smaller than the RHS for all x_i, $\underline{x}^* = -\infty$.

- $\overline{x}^*$ solves the equation

$$\Pr\left(\theta \leq \bar{\theta} \mid y \leq y^* \text{ and } x_i = \bar{x}^*\right) = t \tag{3.28}$$

if a solution exists. If the LHS is strictly larger than the RHS for all x_i, $\overline{x}^* = \infty$. Conversely, if the LHS is strictly smaller than the RHS for all x_i, $\overline{x}^* = -\infty$.

- $\underline{\theta}$ solves the equation

$$(1-\lambda)\Pr(x_i \leq \underline{x}^* \mid \theta = \underline{\theta}) = \underline{\theta} \tag{3.29}$$

- $\bar{\theta}$ solves the equation

$$\lambda + (1-\lambda)\Pr(x_i \leq \overline{x}^* \mid \theta = \bar{\theta}) = \bar{\theta} \tag{3.30}$$

To solve the model, recall that, in our setting, the information system and the definition of the large trader's signal implies

$$\begin{aligned} y &= x_i + \tau\eta - \sigma\varepsilon_i \\ y^* &= \bar{\theta} - \tau G^{-1}(t) \end{aligned}$$

Now, consider a small trader's posterior probability assessment of a successful attack conditional upon observing the large trader attack in period 1 and the signal x_i. Using the above expressions, such probability can be expressed as

$$\begin{aligned} & \Pr\left(\theta \leq \bar{\theta} \mid y \leq y^*\right) \\ = \; & \Pr\left(\varepsilon_i \geq \frac{x_i - \bar{\theta}}{\sigma} \;\middle|\; \tau\eta - \sigma\varepsilon_i \leq \bar{\theta} - x_i - \tau G^{-1}(t)\right) \end{aligned}$$

We can thus derive $\bar{x}^*$ by solving the following equation

$$\frac{\Pr\left(\varepsilon_i \geq \frac{\bar{x}^* - \bar{\theta}}{\sigma}\,,\ \tau\eta - \sigma\varepsilon_i \leq \bar{\theta} - \bar{x}^* - \tau G^{-1}(t)\right)}{\Pr\left(\tau\eta - \sigma\varepsilon_i \leq \bar{\theta} - \bar{x}^* - \tau G^{-1}(t)\right)} = t \tag{3.31}$$

By the same token, $\underline{x}^*$ can be derived by the analogous condition for the case in which the large trader has not attacked the currency in period 1:

$$\frac{\Pr\left(\varepsilon_i \geq \frac{\underline{x}^* - \theta}{\sigma}\ ,\ \tau\eta - \sigma\varepsilon_i > \bar{\theta} - \underline{x}^* - \tau G^{-1}(t)\right)}{\Pr\left(\tau\eta - \sigma\varepsilon_i > \bar{\theta} - \underline{x}^* - \tau G^{-1}(t)\right)} = t \tag{3.32}$$

It is apparent that neither of these equations can be solved in closed form in the general case, without making further parametric assumptions on the distribution functions of the error terms. We therefore resort to two types of analysis. One is to follow the procedure used in section 4, and examine the limiting cases for different relative precisions of the large trader's information.

Before we do so, however, we report the results of some numerical calculations on the critical states $\bar{\theta}$ and $\underline{\theta}$. These critical states take on added significance in the sequential version of our game, since the action of the large trader is observed by the small traders. In figures 3.4 through 3.7 we report the plots for the critical states $\bar{\theta}$ and $\underline{\theta}$ for a variety of parameter combinations. As before, F and G are Standard Normal. In all cases, the numerical plots deliever intuitive answers. As the precision of the large trader's information improves (so that we move left in all the plots), we can see that the upper critical state $\bar{\theta}$ increases, while the lower critical state $\underline{\theta}$ falls. This implies that the pivotal influence of the large trader is greater when his information becomes more precise. Note, in particular, that $\bar{\theta}$ approaches 1 when the large trader's signal becomes more precise. In other words, when τ is small, the large trader's action precipitates the attack whenever the peg can be broken.

This and other properties of the sequential game can be examined by analysing the limiting properties of the equilibrium.

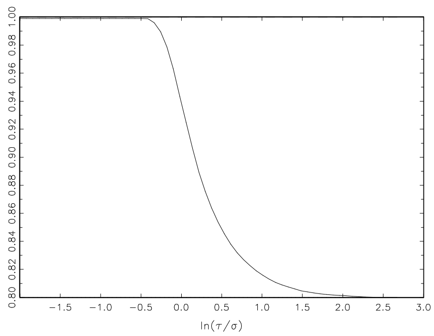

Figure 3.4: $\bar{\theta}$; $t = 0.4$, $\lambda = 0.5$, $\sigma = 1.0$

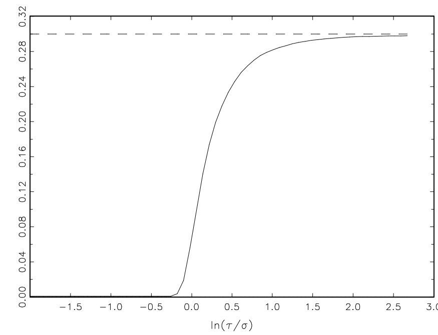

Figure 3.5: $\underline{\theta}$; $t = 0.4$, $\lambda = 0.5$, $\sigma = 1.0$

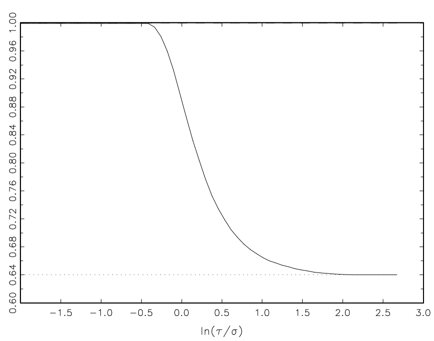

Figure 3.6: $\bar{\theta}$; $t = 0.4$, $\lambda = 0.1$, $\sigma = 1.0$

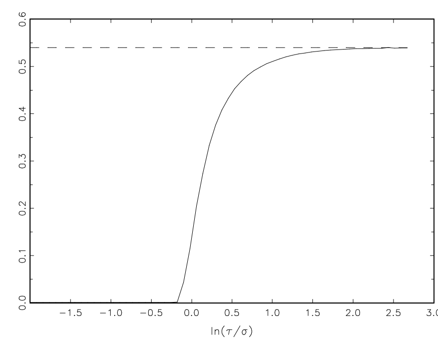

Figure 3.7: $\underline{\theta}$; $t = 0.4$, $\lambda = 0.1$, $\sigma = 1.0$

3.4.2 Comparative Statics in the Limit

We now discuss the limiting properties of the model allowing for differences in the information precision across traders of different size. We consider first the case of a large trader who is arbitrarily better informed than small traders. The following proposition summarizes our result.

Proposition 3.6. *As* $\frac{\sigma}{\tau} \to \infty$*, there is a unique trigger equilibrium in* Γ*, with*

$$\begin{aligned} \frac{1-y^*}{\tau} &\to G^{-1}(t) \\ \bar{x}^* &\to \infty \\ \underline{x}^* &\to -\infty \\ \bar{\theta} &\to 1 \\ \underline{\theta} &\to 0 \end{aligned}$$

Proof: We first rewrite equation (3.31) as

$$\frac{\Pr\left(\varepsilon_i \geq \frac{\bar{x}^*-\bar{\theta}}{\sigma}\ ,\ \frac{\tau}{\sigma}\eta - \varepsilon_i \leq \frac{\bar{\theta}-\bar{x}^*}{\sigma} - \frac{\tau}{\sigma}G^{-1}(t)\right)}{\Pr\left(\frac{\tau}{\sigma}\eta - \varepsilon_i \leq \frac{\bar{\theta}-\bar{x}^*}{\sigma} - \frac{\tau}{\sigma}G^{-1}(t)\right)} = t$$

Taking the limit as $\frac{\tau}{\sigma} \to 0$, the LHS tends to

$$\frac{\Pr\left(\varepsilon_i \geq \frac{\bar{x}^*-\bar{\theta}}{\sigma}\ ,\ -\varepsilon_i \leq \frac{\bar{\theta}-\bar{x}^*}{\sigma}\right)}{\Pr\left(-\varepsilon_i \leq \frac{\bar{\theta}-\bar{x}^*}{\sigma}\right)}$$

which is equal to 1. Thus, in the limit there is no solution to the above equation. Since $t < 1$, we use the definition of $\bar{x}^*$ to set $\bar{x}^* = \infty$. We can then substitute $\bar{x}^*$ into equation (3.30) to derive $\bar{\theta} = 1$. Symmetric arguments establish that $\underline{x}^* = -\infty$ and $\underline{\theta} = 0$. Thus using the definition of y^*, we get $y^* = 1 - \tau G^{-1}(t)$.

In words, this result says that, when the large trader is arbitrarily better informed than the small traders, they follow him blindly, and therefore, he completely internalizes the payoff externality in the currency market. This type of equilibrium corresponds to the strong herding equilibrium in Dasgupta (1999), where all the followers ignore their information completely.

This result implies that, when actions are observable, a relatively well-informed large trader can (but not always will) make small traders either extremely aggressive in selling a currency, or not at all aggressive. His influence in this case is much larger (as should be true, intuitively), in comparison to the case of a simultaneous move game, analyzed in the previous section.

Notably, the size of the large trader never appears in the expressions that define the unique trigger equilibrium. The distinctive feature of a large trader is that he does not ignore the signalling effect of his actions. What emerges from our result is that, when he is significantly better informed than the small traders, his absolute size is irrelevant.

The following proposition states our results corresponding to the case in which the large trader is less precisely informed than the rest of the market.

Proposition 3.7. *As* $\frac{\sigma}{\tau} \to 0$ *there is a unique trigger equilibrium, with*

$$\begin{aligned}
\frac{\lambda + (1-\lambda)(1-t) - y^*}{\tau} &\to G^{-1}(t) \\
\frac{\lambda + (1-\lambda)(1-t) - \overline{x}^*}{\sigma} &\to F^{-1}(t) \\
\frac{(1-\lambda)(1-t) - \underline{x}^*}{\sigma} &\to F^{-1}(t) \\
\bar{\theta} &\to \lambda + (1-\lambda)(1-t) \\
\underline{\theta} &\to (1-\lambda)(1-t)
\end{aligned}$$

Proof: Rewrite equation (3.31) and taking limits as $\frac{\sigma}{\tau} \to 0$, we get

$$\frac{\Pr\left(\varepsilon_i \geq \frac{\bar{x}^* - \bar{\theta}}{\sigma}\ ,\ \eta \leq \frac{\bar{\theta} - \bar{x}^*}{\tau} - G^{-1}(t)\right)}{\Pr\left(\eta \leq \frac{\bar{\theta} - \bar{x}^*}{\tau} - G^{-1}(t)\right)} = t$$

which, given independence of ε_i and τ implies that

$$\bar{x}^* = \bar{\theta} - \sigma F^{-1}(t)$$

Combining with equation (3.30) we get

$$\bar{\theta} \to \lambda + (1 - \lambda)(1 - t)$$

Thus

$$\bar{x}^* \to \lambda + (1 - \lambda)(1 - t) - \sigma F^{-1}(t)$$

The remaining quantities are then uniquely defined.

In words, this proposition means that even a relatively uninformed large trader attempts to influence the market. However, since he does not have any informational signalling ability, his actions affect the equilibrium outcome of the game only inasmuch as his size is relevant. Intuitively, as his signal is quite noisy, he cannot reduce the small traders' uncertainty about the fundamental. By moving first, however, he can eliminate uncertainty about his action. If, in addition, we suppose that $\sigma \to 0$, then

$$\bar{x}^* \to 1 - t + \lambda t$$

Observe that as $\lambda \to 0$, the equilibrium triggers converge exactly to the case in which the large trader does not exist.

3.4.3 A synthesis of our results

We are now in the position to offer a complete overview of our results, and reach some conclusions about the role of a large trader in a currency crisis. As explained in the introduction, there are three main elements in our theory: size, information precision and signalling.

Focusing on the limiting properties of our equilibria, the following table presents the equilibrium value of the trigger for small traders in the different cases discussed above.

Limiting properties of equilibria

Equilibrium trigger for small traders by relative precision of information

Large trader is:	informed	uninformed
	$(\frac{\tau}{\sigma} \to 0,\ \sigma \to 0)$	$(\frac{\tau}{\sigma} \to \infty,\ \sigma \to 0)$
Actions are:		
unobservable	$x^* = 1 - t + \lambda t$	$x^* = 1 - t - \lambda t - t^2 \frac{1-\lambda}{1-t}$ if $\lambda > t$ $x^* = 1 - t$ if $\lambda \leq t$
observable	$\bar{x}^* = \infty$ $\underline{x}^* = -\infty$	$\bar{x}^* = 1 - t + \lambda t$ $\underline{x}^* = (1 - t)(1 - \lambda)$

In each column of the table, the two thresholds $\bar{x}^*$ and $\underline{x}^*$ in the game where actions are observable are higher and lower, respectively, than the corresponding threshold x^* derived in our game with unobservable action. In other words, regardless of the relative precision of information, a large trader can have a much larger influence in the market if he is able to signal to small traders.

As discussed above, the size of the large trader is irrelevant in the sequential move game when the large trader is relatively well informed – this case corresponds to the bottom left cell of the table. What matter here is not the size per se, but the signalling ability associated with size. Conversely, size matters in all other cases.

Reading the entries on the main diagonal of the table, observe that the critical signal

(x^*) in the unobservable action, information larger trader case is equal to critical signal contingent on the larger trader have attacked ($\bar{x}^*$) in the observable action, uninformed large trader case. This equality provides an interesting link across the two games. When actions are not observable, small traders do not expect a better informed trader to "add noise" to the game. Their problem is to estimate the fundamental as well as possible, given their own signal. When actions are observable, the potential noise added to the game by a relatively uninformed large trader is eliminated by his moving first. So, also in this case, the problem of the smaller traders is the same as above, i.e. to estimate the fundamental as well as possible given their own information.

3.5 Concluding Remarks

Economists and policy makers have long debated whether speculation, especially speculation by large traders, is destabilizing. In our model, a large trader in the market may exacerbate a crisis, and render small traders more aggressive. The small traders' trading strategies as defined by the switching point x^* become more aggressive as the size of the large trader increases. However, the relative precision of the information available to the traders affects this conclusion. If the large trader is less well informed than the small traders, this effect may be quite small. Finally, the influence of the large trader is magnified greatly if the large trader's trading position is publicly revealed to the other traders, although this result also must be qualified by the relative precision of information of the two types of traders.

Crucial to our conclusion is the assumption that the large trader stands to gain in the event of the devaluation. This may not be an assumption that is widely accepted. If the large trader is an investor with a substantial holding of assets denominated in the currency under attack (say, a U.S. pension fund with equity holdings in the target country), he may prefer that an attack not occur, even though, if he thinks the attack is sufficiently likely he will join the attack. In such a case, the presence of a large trader will have the opposite effect, making attacks less likely. This points to the importance of understanding the initial portfolio positions of the traders in such instances.

Our analysis also abstracted from a large trader's incentive to take a position discreetly in order to avoid adverse price movements. If this effect were important, a trader would have an incentive to delay announcing his position until it is fully established. But even once a trader has established his position, he may prefer to avoid public disclosures when he is holding a highly leveraged portfolio in possibly illiquid instruments. One of the motivations for the call for greater public disclosures by banks and hedge funds (see Financial Stability Forum (2000)) is the idea that if leveraged institutions know that their trading positions are to be revealed publicly, they would be wary of taking on large speculative positions. The recent decisions by several well known fund managers (Mr. Soros being one of them) to discontinue their 'macro hedge fund' activities raise deeper questions concerning the trade-off between the sorts of mechanisms outlined in our model against the diseconomies of scale that arise due to the illiquidity of certain markets. It is perhaps not a coincidence that the closure of such macro hedge funds comes at a time when many governments have stopped pursuing currency pegs and other asset price stabilization policies.

Appendix B

In this appendix, we show that the unique equilibrium in switching strategies can be obtained by the iterated deletion of strictly dominated strategies.

Consider the expected payoff to attacking the peg for a small trader conditional on signal x when all other small traders follow the switching strategy around $\hat{x}$ and when the large trader plays his best response against this switching strategy (which is to switch at $y(\hat{x})$, obtained from (3.10)). Denote this expected payoff by $u(x, \hat{x})$. It is given by

$$u(x,\hat{x}) = \frac{1}{\sigma}\int_{-\infty}^{\underline{\theta}(\hat{x})} f\left(\frac{\theta - x}{\sigma}\right) d\theta + \frac{1}{\sigma}\int_{\underline{\theta}(\hat{x})}^{\overline{\theta}(\hat{x})} f\left(\frac{\theta - x}{\sigma}\right) G\left(\frac{y(\hat{x}) - \theta}{\tau}\right) d\theta \qquad (3.33)$$

where $\underline{\theta}(\hat{x})$ indicates the value of $\underline{\theta}$ when small traders follow the $\hat{x}$-switching strategy. $\bar{\theta}(\hat{x})$ is defined analogously. We allow $\hat{x}$ to take the values $-\infty$ and ∞ also, by which we mean that the small traders never and always attack, respectively. Note that $u(\cdot, \cdot)$ is decreasing in its first argument and increasing in its second.

For sufficiently low values of x, attacking the currency is a dominant action for a small trader, irrespective of the actions of the other traders, small or large. Denote by $\underline{x}_0$ the threshold value of x below which it is a dominant action to attack the currency for the small trader. All traders realize this, and rule out any strategy for the small traders which refrain from attacking below $\underline{x}_0$. But then, refraining from attacking cannot be rational for a small trader whenever one's signal is below $\underline{x}_1$ where $\underline{x}_1$ solves

$$u(\underline{x}_1,\ \underline{x}_0) = t \qquad (3.34)$$

This is so, since the switching strategy around $\underline{x}_1$ is the best reply to the switching strategy around $\underline{x}_0$, and even the most cautious small trader (in the sense that he assumes the worst concerning the possibility of a successful attack) believes that the incidence of attack is higher than that implied by the switching strategy around $\underline{x}_0$ and the large trader's best reply $y(\underline{x}_0)$. Since the payoff to attacking is increasing in the incidence of attack by the other traders, any strategy that refrains from attacking for signals lower than $\underline{x}_1$ is dominated. Thus, after *two* rounds of deletion of dominated strategies, any strategy for a small trader that refrains from attack for signals lower than $\underline{x}_1$ is eliminated. Proceeding in this way, one generates the increasing sequence:

$$\underline{x}_0 < \underline{x}_1 < \underline{x}_2 < \cdots < \underline{x}_k < \cdots \tag{3.35}$$

where any strategy that refrains from attacking for signal $x < \underline{x}_k$ does not survive $k + 1$ rounds of deletion of dominated strategies. The sequence is increasing since $u(\cdot, \cdot)$ is decreasing in its first argument, and increasing in its second. The smallest solution $\underline{x}$ to the equation $u(x, x) = t$ is the least upper bound of this sequence, and hence its limit. Any strategy that refrains from attacking for signal lower than $\underline{x}$ does not survive iterated dominance.

Conversely, if x is the largest solution to $u(x, x) = t$, there is an exactly analogous argument from "above", which demonstrates that a strategy that attacks for signals larger than x does not survive iterated dominance. But if there is a *unique* solution to $u(x, x) = t$, then the smallest solution just *is* the largest solution. There is precisely one strategy remaining after eliminating all iteratively dominated strategies. Needless to say, this also implies that this strategy is the only *equilibrium* strategy. This completes the argument.

Appendix B

In this appendix, we give a proof of proposition 3.2. First, suppose that $\lim \underline{\theta} < \lim \bar{\theta}$

(so that $\lim \bar{\theta} \geq 1-\lambda$). Since $x^* \to \bar{\theta}$, we must have

$$\frac{\underline{\theta} - x^*}{\sigma} \to -\infty$$

In other words, $\underline{\delta} \to -\infty$. Thus, in the limit, equation (3.22) can written as (3.24). Now, consider the case where $\lim \underline{\theta} = \lim \bar{\theta}$. This is the case where $\underline{\delta}$ is finite. From (3.19) and (3.20) we have

$$(1-\lambda)(1-F(\underline{\delta})) = \lambda + (1-\lambda)\left(1 - F\left(\bar{\delta}\right)\right)$$

implying that $F\left(\bar{\delta}\right) - F(\underline{\delta}) = \frac{\lambda}{1-\lambda}$, which in turn means that

$$\underline{\delta} = F^{-1}\left(F\left(\bar{\delta}\right) - \frac{\lambda}{1-\lambda}\right) \tag{3.36}$$

Equation (3.22) in the limiting case then becomes (3.25). In both (3.24) and (3.25), the left hand side is strictly increasing in $\bar{\delta}$, and there is a unique value of $\bar{\delta}$ that solves both equations. Then, the proposition follows from (3.20).

Appendix C

In this appendix, we give a proof of proposition 3.5. The bounds can be obtained by manipulating the four equations, (3.19) through (3.22). Let us use the notation: $\overline{\xi} = F\left(\bar{\delta}\right)$ and $\underline{\xi} = F(\underline{\delta})$. From (3.19) and (3.20),

$$\begin{aligned}
\overline{\theta} - \underline{\theta} &= \lambda + (1-\lambda)\left(F\left(\frac{x^* - \overline{\theta}}{\sigma}\right) - F\left(\frac{x^* - \underline{\theta}}{\sigma}\right)\right) \\
&= \lambda + (1-\lambda)\left(1 - F\left(\frac{\overline{\theta} - x^*}{\sigma}\right) - 1 + F\left(\frac{\underline{\theta} - x^*}{\sigma}\right)\right) \\
&= \lambda - (1-\lambda)\left(\overline{\xi} - \underline{\xi}\right).
\end{aligned}$$

This implies that $\overline{\theta} > \underline{\theta}$ and $\overline{\xi} > \underline{\xi}$. Re-arranging, this implies

$$\overline{\xi} - \underline{\xi} < \frac{\lambda}{1-\lambda}. \tag{3.37}$$

By (3.21),

$$y^* = \overline{\theta} - \tau G^{-1}(t)$$

Substituting into (3.22), we obtain

$$\frac{1}{\sigma}\int_{-\infty}^{\underline{\theta}} f\left(\frac{\theta - x^*}{\sigma}\right) d\theta + \frac{1}{\sigma}\int_{\underline{\theta}}^{\overline{\theta}} f\left(\frac{\theta - x^*}{\sigma}\right) G\left(\frac{\overline{\theta} - \theta}{\tau} - G^{-1}(t)\right) d\theta = t \tag{3.38}$$

Observe that for all $\theta \in \left(\underline{\theta}, \overline{\theta}\right)$,

$$1 - t < G\left(\frac{\overline{\theta} - \theta}{\tau} - G^{-1}(t)\right) < 1$$

and that

$$\frac{1}{\sigma}\int_{-\infty}^{\overline{\theta}} f\left(\frac{\theta - x^*}{\sigma}\right) d\theta = F\left(\overline{\xi}\right)$$

so the left hand side of equation (3.38) is strictly less than $\overline{\xi}$ and is strictly more than $(1-t)\,\overline{\xi}$ and $\overline{\xi} - t\left[\overline{\xi} - \underline{\xi}\right]$. By (3.37), this latter expression is strictly more than

$$\overline{\xi} - \frac{\lambda t}{1-t}.$$

Now equation (3.38) and the upper and lower bounds on the left hand side of (3.38) imply

$$t < \overline{\xi} < \min\left\{\frac{t}{1-t}, \frac{t}{1-\lambda}\right\}.$$

The proposition follows from re-arranging the characterizing these characterizing equations.

Chapter 4

A Model of the Origin and Spread of Bank Panics

A commonly held view of financial crises is that they begin locally, in some region, country, or institution, and subsequently "spread" elsewhere.[1] This process of spread is often referred to as *contagion*. What might justify contagion in a rational economy? There are two broad classes of explanations.

The first class of explanations posits that the adverse information that precipitates a crisis in one institution also implies adverse information about the other. This view emphasizes correlations in underlying value across institutions and Bayes learning by rational agents. For example, a currency crisis in Thailand may be driven by adverse information

[1] I am grateful to Ben Polak, Dirk Bergemann, David Pearce, Timothy Guinnane, Debraj Ray, Jonathan Levin, Ady Pauzner, V. V. Chari, Patrick Kehoe, and especially Stephen Morris for helpful discussions. Financial support from the Cowles Foundation and Yale University is gratefully acknowledged. Some of the computations for this project were carried out using Mathematica 3.0.

about underlying asset values in South East Asia, which can then apply to other countries in the region.[2]

A second type of explanation begins with the observation that financial institutions are often linked to each other through direct portfolio or capital connections. For example, entrepreneurs are linked to capitalists through credit relationships; banks are known to hold interbank deposits. While such capital connections may seem to be desirable ex ante, during a crisis the failure of one institution can have direct negative payoff effects upon stakeholders of institutions with which it is linked.[3]

In this paper, we present a model of financial contagion which formalizes this latter view. We focus on a particular (but particularly important) type of financial institution: commercial banks. Throughout history, banks have cross-held deposits (for regulatory and insurance reasons), and thus the failure of some banks had direct consequences on others through capital linkages. The problem of contagious bank failure is particularly complex because it involves an underlying coordination problem amongst depositors of each bank. Even weak banks may not fail if very few depositors withdraw their money early, while strong banks may fail if many depositors withdraw early. This problem of multiple equilibria makes it difficult to examine even individual bank failures, which then compounds the difficulty

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[2]For papers that emphasize this view, see, for example, Kodres and Pritsker 1998, or Chen 1999.

[3]Two leading papers that emphasize this view are Kiyotaki and Moore (2000) and Allen and Gale (2000). An interesting recent paper which highlights the possibility of contagion through financial links between stakeholders in different institutions, rather than the institutions themselves, is Goldstein and Pauzner (2000b).

of isolating contagious effects in many bank settings.[4] Using and extending some recent developments in the theory of equilibrium selection in coordination games (see Morris and Shin 2000) we present a model of an economy with multiple banks where the probability of failure of individual banks, and of systemic crises, is uniquely determined. This then permits us to identify contagion precisely and examine its properties.

While the lessons from the analysis in this paper may have relevance to understanding many financial crises, our model is too stylized to be a precise description of any particular one. The set of financial crises that are, perhaps, best represented by our model are the banking panics of the National Banking System in the United States. For expositional purposes it is useful to provide a brief and stylized description of these panics. We provide one below.

4.0.1 National Banking Era Panics

The description presented here selectively synthesises and summarizes the descriptions provided by Sprague (1910), Wicker (2000), and Calomiris and Gorton (1991). The defining characteristics of the National Banking System were laid out in the National Banking Act of 1864. This act prohibited interstate branching of banks and established a system of reserve pyramiding, under which country banks could hold reserves in designated reserve city banks, which in turn could hold reserves in New York. Thus, throughout this period, the reserve cities including New York directly or indirectly held the deposits of many country banks. As late as 1907, Sprague (1910, p. 223) points out that: "New York still maintained

[4]For the classic multiple equilibrium model of bank runs, see Diamond and Dybvig (1984).

its commanding position as a debtor of national banks."

There were five banking panics of varying intensity in the National Banking Era prior to the Great Depression. They occurred in 1873, 1884, 1890, 1893, and 1907. With the exception of the the crisis of 1893, all of these panics began as localized disturbances in New York and subsequently spread to banks in the interior of the country. What were the sources of these panics? Calomiris and Gorton (1991) argue persuasively that the panics typically began with asset-side shocks. Wicker (2000, p. 1) confirms: "In New York, the banking panics began with an unexpected financial shocks ... the immediate effect being a loss of depositor confidence manifest by bank runs that were bank-specific and sometimes extending to all savings banks." This was typically followed by suspension of payments by New York banks, followed by suspensions in banks at various parts of the country.

A particularly good example of such contagious panics was the panic of 1907.[5] In 1907, the panic began due to an unsuccessful attempt to corner the Copper market by a group of speculators who were associated with several Trust Companies and National Banks in New York. When news of this speculatory failure became public in October there were runs on Knickerbocker Trust Company. This was followed by runs on the National Bank of North America and on other institutions thought to be linked to the Copper speculators. While some attempts were made to ease the crisis by private bankers led by J. P. Morgan, an unfortunate delay in reaction by the large New York Clearing House led to a widespread panic, followed by several suspensions and bank closures. Sprague (1910, p. 259) points

[5]Sprague (1910) points out that the panic of 1907 was preceded by no systemic shocks that might conflate our analysis (p. 216): "For our purposes, therefore, we are fortunate in being provided with a crisis which was preceded by no legislation or monetary conditions unfavorable to sound banking."

out: "Everywhere the banks suddenly found themselves confronted with demands for money by frightened depositors ... Country banks drew money from city banks and all banks throughout the country demanded the return of funds deposited or on loan in New York." Finally, the panic that began with a localized asset shock in New York led to suspensions (or effective suspensions) through much of the country. In the panic of 1907, therefore, we find a clear example of how a financial panic may spread along the channels of direct capital connections between institutions.

To summarize, some of the stylized features of the National Banking System panics are as follows:

- Panics originated due to asset-side shocks. They were inherently dynamic, starting in New York and spreading to the interior of the country.
- While other factors may also play a role,[6] panics appeared to diffuse nationally through the correspondent network, from debtor New York banks to creditor banks in the interior.

In what follows, we develop a dynamic Bayesian game theoretic model in which many of these stylized features emerge in equilibrium.

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[6] Wicker (2000) identifies four channels of transmission from New York to the interior, including diffusion via the correspondent network. However, not all of these factors are independent, and at least two of these may formally reduce to consequences of existing correspondent relationships.

4.0.2 Summary of Model and Results

We consider a 3-period economy with two non-overlapping regions, each with a representative competitive bank. Regional banks have access to a storage technology (cash) and a region-specific risky technology that pays a higher expected return (than cash) if held to maturity, but pays less than par value if liquidated early. The return on the risky asset is revealed in period three, and is an increasing function of uniformly distributed region-specific underlying fundamentals.

There are two continuums of risk-averse consumers, one in each region, each of whom lives three periods. The consumers receive private liquidity shocks: with some probability they may need to consume in period one. They begin life with their endowments deposited in the representative bank of their region. The aggregate level of liquidity demand in the economy is fixed, but the two regions may experience negatively correlated regionally aggregate shocks. The two representative banks insure against such regional liquidity shocks by holding interbank deposits. Consumer deposits, as well as interbank deposits take the form of standard demand deposit contracts, but interbank deposits are assumed to have seniority to (settle earlier than) individual demand deposits.

Within period one, regional liquidity shocks are realized first and become public knowledge. Then, nature selects the depositors of one of the banks to receive private signals about their bank's fundamentals and make their choices. The depositors of the other bank observe the net proportion of the depositors of the first bank who withdraw their money. Shortly thereafter, the depositors of the other bank receive private information about their bank's fundamentals, and decide whether to remain or withdraw.[7] The structure of the model and the seniority of interbank deposits implies that when the depositors of either bank are called upon to make their choices, there is an ex-interim asymmetry between the two banks: one bank (in the region with high initial liquidity demand) is a net debtor to the other bank.

Under weak assumptions on the distribution of fundamentals, we prove that there is a unique threshold in asset returns below which each bank will fail (Propositions 4.1, 4.2, and 4.3). These results imply that banks fail upon the release of adverse information about them. The probability of failure is determined endogenously. In our central result, we show that contagion exists: there are regions of fundamentals in which one bank may fail if and only if the other bank fails (Proposition 4.4). Conditional on the failure of the debtor bank, the creditor bank fails for a wider range of its own fundamentals than if the debtor bank survived. However, the failure of the creditor bank does not affect the probability of failure of the debtor bank. Contagion flows from debtors to creditors, and thus spreads along the channels of interbank deposits, but only in a specific direction. Hence, contagion can be localized and not all institutions become potential targets.

[7]There are two natural ways to interpret this non-simultaneity in the model One can think of this model as a discrete approximation of a continuous time model with generically staggered release of information. One can also think of this as a genuine discrete time setting in which (for reasons we do not model) the depositors of one bank suddenly receive information and choose to act. The depositors of the other bank (in another region) read about the events in the first bank in their morning newspaper the following day, and then actively seek out information about their own bank immediately thereafter, and choose whether to withdraw or remain. We prefer this latter interpretation.

Interbank deposits enable banks to hedge regional liquidity shocks, but expose them to the risk of contagion. We illustrate the conditions under which banks would want to hold significant levels of interbank deposits. In particular, our simulations suggest that when banks runs are rare, financial institutions will insure fully against regional liquidity demand shocks. However, when bank runs are frequent, only partial insurance will be optimal.

Finally, under the assumption of full liquidity insurance, we present comparative statics results to demonstrate that contagion is increasing in the size of regional liquidity demand shocks (Proposition 4.5). This is a testable implication of the model.

4.0.3 Related Literature

Our paper is connected with a diverse literature. We apply the equilibrium selection techniques summarized in Morris and Shin (2000). Goldstein and Pauzner (2000a) were the first to apply these techniques to the analysis of bank runs. Our model shares features with theirs.[8] They investigate the probability of bank runs in a single-bank setting, while we are interested in the problem of contagion with multiple banks. Rochet and Vives (2000) also analyze bank runs using similar techniques, but do not concern themselves with the problem of contagion. Goldstein and Pauzner (2000b), like us, examine contagion of self-fulfilling crises, but their mechanism for contagion (through common lenders) is different from ours. Kiyotaki and Moore (2000) explore the method by which contagion flows through credit chains amongst lenders and entrepreneurs. Their model shares with ours the feature that capital connections are the channels for contagion, but does not concern itself with coordi-

[8] Importantly, in both models, payoffs fail to satisfy global strategic complementarities.

nation problems. Rochet and Tirole (1996) examine correlated bank failures via monitoring: the failure of one bank is assumed to mean that other banks have not been monitored, and thus triggers multiple collapses. Other papers with similar themes that have less clear connections to ours are Allen and Gale (1998), Chari and Kehoe (1997), and Cole and Kehoe (2000).

The paper that comes closest to us in theme is Allen and Gale (2000). Their purpose is to model contagion as an equilibrium phenomenon in a many-bank setting. While our model contains features of Allen and Gale's framework, there are important differences between our models. Allen and Gale work with perfect information and bank panics occur due to aggregate (random) liquidity shocks on the part of the depositors. Such aggregate liquidity shocks are necessary and sufficient for contagion. Our model features incomplete information. Bank runs occur due to adverse information about asset returns. Regional liquidity shocks are necessary *but not sufficient* for contagion. This is in keeping with the data on historical bank panics in the US. Allen and Gale assume that bank panics are zero probability phenomena, while our set-up allows us to derive the probability of bank panics in equilibrium. The existence of contagion in Allen and Gale's model is intricately connected with an ad hoc incompleteness in the interbank deposit market. Without such incompleteness, contagion vanishes, except for extremely large global liquidity shocks. This may suggest that complete interbank deposits may essentially eliminate contagion, thus reducing the need for a lender of last resort. We show that contagion occurs with positive probability even with complete interbank deposits. Our results, therefore, also have bearing on the findings of Rochet and Vives (2000), who consider the necessity of a lender of last resort within a banking system. Finally, in Allen and Gale (2000), contagion is a static phenomenon, while we model it as a dynamic one. This allows us to explicate an endogenous direction of flow for contagious effects.

The rest of the paper is organized as follows. In the next section we present the model. In section 4.2 we prove the existence and uniqueness of threshold equilibria. Section 4.3 contains our central result. The optimal level of interbank deposit holdings is illustrated numerically in Section 4.4. Section 4.5 provides closed forms and comparative statics. Section 4.6 discusses and concludes.

4.1 The Model

4.1.1 Regional Liquidity Shocks

We consider an economy with two non-overlapping "regions," A and B. There are three time periods $t = 0, 1, 2$. The regions are populated by distinct continuums of weakly risk averse agents with utility functions $u(\cdot)$ $[u'(\cdot) > 0,\ u''(\cdot) \leq 0]$ who each live for three periods. Each agent has an endowment of 1 unit. Agents face private (uninsurable) liquidity shocks: they need either to consume in period 1 (impatient) or in period 2 (patient). In the aggregate, there is no uncertainty about liquidity in the economy: there is exactly a proportion $w \in (0, 1)$ of agents who require early liquidity. However, individual regions experience (regionally) aggregate liquidity shocks of size $x > 0$. In particular, there are two states of the world: $\lambda = A$ or $\lambda = B$, corresponding to the cases where region A and region B have high early liquidity demands respectively. Observe that since aggregate liquidity is constant, regional liquidity shocks are negatively correlated. The state λ is realized and

	A	B
$\lambda = A$	$w + x$	$w - x$
$\lambda = B$	$w - x$	$w + x$

Table 4.1: Regional Liquidity Shocks

publicly known immediately at the beginning of period 1. States A and B occur with equal probability.

4.1.2 Banks, Demand Deposits, and Interbank Insurance

We consider two symmetric (representative, competitive) banks which lie in two regions of the economy. Agents begin their lives with their endowments deposited in the bank of their region. There are two classes of assets available to banks: a safe and liquid storage technology with a low (unit) gross rate of return, and a risky, illiquid asset with high expected return but with costs to premature liquidation. The storage technology is common to both banks. One unit stored at time t produces one unit at time $t+1$. In addition, region i's residents also have access to risky illiquid technology R_i, with returns given by:

$$R_i(t) \quad = \quad \begin{cases} 0 < r < 1 & \text{when } t = 1, \\ R(\theta_i) & \text{when } t = 2\text{, where } \theta_i \text{ is distributed Uniform on } [L, U] \end{cases}$$

where t is the time of liquidation, $R(\cdot)$ is any increasing function. θ_i indexes some underlying "fundamentals" related to the bank's assets, which determine the level of the bank's asset returns. These fundamentals θ_i are independent and identically distributed for $i = A, B$. We assume that $E_{\theta_i}[u(R(\theta_i))] \geq u(1)$ i.e., the risky asset pays a higher expected return if held till period 2.

Banks are constrained to offer depositors *demand deposit contracts.*[9] Demand deposit contracts offer conversion of deposits into cash at par on demand in period 1 conditional on sufficient cash being available. If, however, sufficient cash is not available, then the contract specifies that the bank will divide up evenly what cash it can generate by liquidating its portfolio amongst the depositors who demand early withdrawal. At this point of time, the bank goes out of business. For those depositors who choose to remain in the bank till period 2, the bank promises to pay a stochastic amount, which is contingent upon the returns on the bank's assets, the proportion of early withdrawals, and payouts to any senior liabilities.

The two banks face aggregate demand shocks in period 1, in keeping with the regionally aggregate liquidity shocks outlined above. However, since these aggregate regional liquidity shocks are negatively correlated, banks insure against these by holding interbank deposits.[10] In particular, we assume that banks hold cash reserves equal to w, the *average* level of liquidity demand in the economy, and insure against regional liquidity shocks by holding

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[9]Hence we are not solving here for the optimal contractual form. Demand deposit contracts are a standard feature of banking systems and we take them as given.

[10]We are implicitly assuming that banks have access to only interbank deposits as a tool to insure, and hence can insure only against shocks to their liabilities. While this assumption is not central to our analysis, we assume it for algebraic simplicity as well as to eliminate correlation between the bank's asset returns. Such correlation would introduce a second channel of contagion, through learning. Our purpose in this paper to explore the extent to which capital connections, *by themselves* can contribute to contagion.

interbank deposits of size $D \in [0, x]$ with the other bank.[11] Thus, in this symmetric scheme, banks exchange deposits of size D, and distribute their net wealth of 1, putting w in cash, and $1 - w$ in long term investment projects. We note that D is a *choice variable* for banks. In equilibrium, the selected size of D shall depend on the parameters of the model.

These interbank deposits have the feature that in each period they "clear before" claims to individual depositors are paid. In other words, as soon as λ is realized at the beginning of period 1, the bank in the high liquidity demand region receives a payment of x from the bank in the other region, before individual depositors can claim money from the bank. Similarly, in period 2, the banks use any proceeds of liquidated investments to first pay their fellow bankers, and then pay their patient depositors out of the remaining proceeds. It is helpful to consider an example.

For clarity, assume a truthtelling equilibrium (i.e. an equilibrium in which only impatient agents withdraw money in period 1) and maximal interbank deposit levels, $D = x$. Suppose state A is realized, so that region A has a higher immediate liquidity shock. Upon the realization of $\lambda = A$, immediately bank A receives from bank B its redeemed deposit of x, so that bank A now has $w + x$ in cash, which matches (in the truthtelling equilibrium) the amount of withdrawals it faces. Similarly, bank B now has $w - x$ in cash, which is precisely the demand it faces in period 1. Bank A now owes bank B $xR(\theta_A)$, and owes its own customers $(w - x)R(\theta_A)$. But it has exactly $(1 - w)$ invested in the illiquid asset $R(\theta_A)$,

[11] Given the cash holdings of the banks, and given the timing of the model to be explicated below, interbank deposits of size larger than x will not be desirable to banks. Such deposits would leave banks unable to pay their own early consumers in *any* equilibrium without prematurely liquidating some of their long term assets. In practice, there were often regulatory restrictions on the size of D.

so its proceeds in period 2 are $(1-w)R(\theta_A)$, which is exactly the sum of its liabilities. Similarly, promises and earnings clear for bank B.

4.1.3 Information and Timing

In period 1 nature selects at random (and with equal probability) one of the sets of depositors to receive information about their bank and to act. Information is received in the form of private signals about the underlying fundamentals of their bank. Suppose region i is selected first. Depositor j of region i receives signal $\theta_{j,i} = \theta_i + \epsilon_{j,i}$, where $\epsilon_{j,i}$ are independent and identically distributed uniform on $[-\epsilon, \epsilon]$. Shortly thereafter, the depositors of the other bank (in region $-i$) receive information about their own bank, and get to act themselves. The information is symmetric. Depositor j of region $-i$ receives signal $\theta_{j,-i} = \theta_{-i} + \epsilon_{j,-i}$, where $\epsilon_{j,-i}$ are independent and identically distributed uniform on $[-\epsilon, \epsilon]$. Importantly, before choosing, the depositors who move second learn what happened in the first bank. Thus, the timing of this game can be described shown below in itemized form:

- Period 0
  - Interbank deposits are initiated.
- Period 1
  - λ is realized.
  - Period 1 interbank claims settle.
  - Depositors in bank i receive information and choose actions.
  - Depositors of bank i who demand early withdrawal are paid.

- Depositors in bank $-i$ receive information and choose actions.
- Depositors in bank $-i$ who demand early withdrawal are paid.

- Period 2
  - Period 2 interbank claims settle.
  - Residual depositor claims on the two banks settle.

We can now write down the payoffs of this game.

4.1.4 Depositor Payoffs and Interbank Payments

We are now ready to write down the payoffs to depositors in this game. In period 1, depositors choose whether to demand conversion of their deposits into cash at par (withdraw) or to retain their deposits with the bank (remain). Impatient depositors can only consume in period 1. They will therefore always withdraw. However, the patient depositors face a non-trivial decision problem. We explicate their payoffs below.[12]

Recall that in period 1 one bank will be a debtor and one bank will be a creditor. Thus, without loss of generality, we can label the payoff matrices for the patient depositors of the two banks as those of the debtor bank and the creditor bank respectively.

Begin by considering the debtor bank, i.e. the bank that experienced a high liquidity shock in period 1. There is a mass $1-(w+x)$ of patient agents in the debtor region. Let n_d represent the proportion of the patient depositors who choose to withdraw in period 1. If n_d proportion of patient depositors withdraw, then, since impatient agents (of measure $w+x$)

[12] The payoffs presented below are motivated by the those in Goldstein and Pauzner (2000) and Allen and Gale (2000). They contain features from both papers.

always withdraw in period 1, total demand for cash in period 1 is $(w+x)+n_d(1-(w+x))$. The bank had w in cash and received D in cash from the creditor bank at the beginning of period 1 (and hence became a debtor to the creditor bank). Thus, its total cash holdings are $w+D$. If demand for cash exceeds $w+D$, the bank can obtain more cash by liquidating its long assets. It has $1-w$ invested in the long asset, from which it can generate $(1-w)r$ in cash in period 1. Thus, observe that if $[w+x]+(1-[w+x])n_d \geq [w+D]+(1-w)r$, i.e., if

$$n_d \geq \frac{(1-w)r+D-x}{1-(w+x)} \tag{4.1}$$

then the debtor bank becomes insolvent and goes out of business in period 1, and in the process divides up the proceeds of its liquidated asset portfolio equally amongst its claimants in period 1. However, if the bank remains solvent in period 1, then it must first settle its debt of $DR(\theta_i)$ to the creditor bank (because interbank deposits have seniority, within each period, to regular demand deposits). In order to pay the proportion n_d early demands by patient agents in period 1, the debtor bank had to liquidate $\frac{(1-w-x)n_d+(x-D)}{r}$ of the illiquid asset in period 1. Its original investment in the long asset was $1-w$. The remaining proceeds are $(1-w-\frac{(1-(w+x))n_d+(x-D)}{r})R(\theta_i)$. As long as $(1-w-\frac{(1-(w+x))n_d+(x-D)}{r})R(\theta_i) > DR(\theta_i)$, (i.e., $n_d < \frac{(1-w)r+(D-x)-rD}{1-w-x}$) the debtor bank pays $DR(\theta_i)$ to the creditor bank in period 2, and divides up the remainder equally amongst its residual depositors who chose to remain in the bank. This means that each patient depositor who chooses to remain receives $\frac{1-w-\frac{(1-(w+x))n_d+(x-D)}{r}-D}{(1-w-x)(1-n_d)}R(\theta_i)$ However, if $n_d \geq \frac{(1-w)r+(D-x)-rD}{1-w-x}$, residual depositors receive nothing, and the creditor bank receives (due to seniority) $(1-w-\frac{(1-(w+x))n_d+(x-D)}{r})R(\theta_i)$. Thus, the period 2 payments on the interbank deposits from the debtor to the creditor bank can be written as:

$$g(\theta_i, n_d) \quad = \quad \begin{cases} DR(\theta_i) & \text{if } n_d < \frac{(1-w)r+(D-x)-rD}{1-w-x} \\ (1-w-\frac{(1-(w+x))n_d+(x-D)}{r})R(\theta_i) & \text{if } \frac{(1-w)r+(D-x)-rD}{1-w-x} \leq n_d < \frac{(1-w)r+(D-x)}{1-(w+x)} \\ 0 & \text{if } n_d \geq \frac{(1-w)r+(D-x)}{1-(w+x)} \end{cases}$$

Correspondingly, the payoffs to the patient depositors are given by:

$$u_W(\theta_i, n_d) \quad = \quad \begin{cases} u[1] & \text{if } n_d < \frac{(1-w)r+(D-x)}{1-(w+x)} \\ u[\frac{(w+D)+(1-w)r}{(w+x)+(1-(w+x))n_d}] & \text{if } n_d \geq \frac{(1-w)r+(D-x)}{1-(w+x)} \end{cases} \tag{4.2}$$

$$u_R(\theta_i, n_d) \quad = \quad \begin{cases} u[\frac{1-w-\frac{(1-(w+x))n_d+(x-D)}{r}-D}{(1-w-x)(1-n_d)}R(\theta_i)] & \text{if } n_d < \frac{(1-w)r+(D-x)-rD}{1-w-x} \\ u[0] & \text{if } n_d \geq \frac{(1-w)r+(D-x)-rD}{1-w-x} \end{cases} \tag{4.3}$$

Now consider the payoffs to the depositors of the creditor bank. Observe that the creditor bank's payoffs are complicated by the fact that they depend on the condition of the debtor bank. If the debtor bank were to become insolvent in period 1 (i.e. condition (4.1) holds), then the creditor bank receives no money from the debtor bank in period 2, and has to divide up a smaller pool of proceeds amongst its residual claimants. However, regardless of the condition of the debtor bank, the creditor bank may itself be run out of business. Let n_c denote the proportion of the patient depositors of the creditor bank who choose to withdraw in period 1. Observe that if

$$n_c \geq \frac{(1-w)r + (x-D)}{1-(w-x)} \tag{4.4}$$

the creditor bank shall become insolvent. It is thus possible that the creditor bank shall become insolvent while the debtor bank remains solvent. In the simplest possible interpretation of bankruptcy laws, we then assume that the proceeds from the debtor bank will be divided equally amongst all the depositors of the creditor bank.

Conditional on the failure of the debtor bank, i.e., if condition (4.1) holds, the payoffs are:

$$u_W(\theta_i, n_c) = \begin{cases} u[1] & \text{if } n_c < \frac{(1-w)r+(x-D)}{1-(w-x)} \\ u[\frac{(w-D)+(1-w)r}{(w-x)+(1-(w-x))n_c}] & \text{if } n_c \geq \frac{(1-w)r+(x-D)}{1-(w-x)} \end{cases} \tag{4.5}$$

$$u_R(\theta_i, n_c) = \begin{cases} u[\frac{(x-D)-(1-w+x)n_c+(1-w)R(\theta_i)}{(1-n_c)(1-w+x)}] & \text{if } n_c < \frac{x-D}{1-w+x} \\ u[\frac{1-w-\frac{D-x+n_c(1-(w-x))}{r}}{(1-n_c)(1-(w-x))}R(\theta_i)] & \text{if } \frac{x-D}{1-w+x} \leq n_c < \frac{(1-w)r+(x-D)}{1-(w-x)} \\ u[0] & \text{if } n_c \geq \frac{(1-w)r+(x-D)}{1-(w-x)} \end{cases} \tag{4.6}$$

Conditional on the survival of the debtor bank the payoffs are:

$$u_W(\theta_i, n_c) = \begin{cases} u[1] & \text{if } n_c < \frac{(1-w)r+(x-D)}{1-(w-x)} \\ u[\frac{(w-D)+(1-w)r}{(w-x)+(1-(w-x))n_c} + g(\theta_{-i}, n_d)] & \text{if } n_c \geq \frac{(1-w)r+(x-D)}{1-(w-x)} \end{cases}$$

$$u_R(\theta_i, n_c) = \begin{cases} u[\frac{(x-D)-(1-w+x)n_c+(1-w)R(\theta_i)+g(\theta_{-i},n_d)}{(1-n_c)(1-w+x)}] & \text{if } n_c < \frac{x-D}{1-w+x} \\ u[\frac{(1-w-\frac{D-x+n_c(1-(w-x))}{r})R(\theta_i)+g(\theta_{-i},n_d)}{(1-n_c)(1-(w-x))}] & \text{if } \frac{x-D}{1-w+x} \leq n_c < \frac{(1-w)r+(x-D)}{1-(w-x)} \\ u[g(\theta_{-i}, n_d)] & \text{if } n_c \geq \frac{(1-w)r+(x-D)}{1-(w-x)} \end{cases}$$

4.1.5 Notation

We label the entire game Γ. We label the realization of Γ in which the depositors of the debtor bank are chosen to act first as Γ_d, and the complementary game Γ_c. Finally, within Γ_i, for $i = c, d$, we denote the stage static coordination games by $\Gamma_{i,1}$ and $\Gamma_{i,2}$ respectively.

The structure of the game is common knowledge amongst participants. In what follows, we first outline our assumptions, and then look for Bayesian Nash equilbria of this game.

4.1.6 Parameter Restrictions

Γ is characterized by the triple of parameters: (r, w, x). In what follows, we lay down conditions on these parameters that prevent the problem from becoming degenerate.

Recall that the proportion of impatient agents in the high liquidity demand state is $w + x$. Clearly this proportion is bounded above by 1. In particular, we assume that there is always at least *some* positive proportion of patient agents. Otherwise the game Γ would be vacuous. Hence, we assume that $w + x < 1$, or, in other words

$$x < 1 - w \tag{4.7}$$

Similarly, in the low liquidity demand state, we insist that there are at least some impatient agents, and hence:

$$x < w \tag{4.8}$$

Finally, observe that in order to keep the stage coordination games non-trivial, we must insist that both the debtor bank and the creditor bank can become insolvent for some ranges of withdrawals by patient agents. In other words, we insist that $\frac{(1-w)r+(D-x)}{1-w-x} < 1$ for all D. In particular, since the LHS is maximized at $D = x$, we require that $\frac{(1-w)r}{1-w-x} < 1$, i.e,

$$x < (1 - r)(1 - w) \tag{4.9}$$

Since we wish all the features implied by the above restrictions to hold in our model, we consider parameters in the region implied by their intersection, i.e., $(r \in (0, 1), w > 0, x > 0)$ are chosen such that

$$x < \min[w, (1 - r)(1 - w)] \tag{4.10}$$

We are now ready to find equilibria of this model.

4.2 Equilibrium

In each of the static coordination games $\Gamma_{i,1}$ and $\Gamma_{i,2}$ for $i = c, d$, with common knowledge of θ_i, there are (at least) two equilibria. In one of these equilibria, all patient agents remain in the bank (tell the truth) because they expect other patient agents to do so. In the other, they all withdraw money from the bank, because they expect other patient agents to do so. With common knowledge of θ_i, patient agents have nothing to condition their beliefs upon, and hence *any* feasible belief about the actions and beliefs of others is permissible. However, with private signals, the situation changes.

In the game with incomplete information, agents are able to condition their beliefs on their private signals, which are known to be correlated with the private signals of others. Since $R(\theta_i)$ is increasing in θ_i, and private signals are positively correlated with fundamentals, high signals convey "good news" to patient agents, and, ceteris paribus, make remaining in the bank more attractive. A natural class of strategies that emerges are those in which agents beliefs, and therefore their actions, are increasing in their information: for good information, agents are more optimistic and more easily persuaded to remain in the bank; the opposite is true for bad information. Since this is a binary action game, such strategies take a particularly simple form: Agents choose to remain in the bank if their private information $\theta_{i,j}$ is above some threshold $\theta^*_{i,j}$, and choose to withdraw otherwise. We shall call such strategies *monotone strategies*, and equilibria in such strategies *monotone equilibria*. Monotone equilibria are characterized by threshold levels, and hence we shall sometimes refer to these as equilibrium thresholds. In what follows, we restrict attention to such monotone equilibria, as the natural class of equilibria to arise in applications, and demonstrate their existence and uniqueness. However, first, we need to make two weak assumptions, which we shall refer to as the lower and upper dominance assumptions. We explicate these below.

Assumption 4.1 (Lower Dominance). *For each depositor of each bank, in each stage static coordination game* $\Gamma_{i,j}$ *for* $i \in \{c, d\}$, $j \in \{1, 2\}$, *if* $\theta_{i,j} = L$, *it is* strictly *dominant to withdraw.*

In other words, if depositors knew that the bank's returns were going to be at its lowest possible level, it is strictly dominant to withdraw. This is an extremely weak assumption, and emerges essentially endogenously from the payoffs of the game. Observe that since at $\theta = L$ the dominance is strict, there is some (possibly vanishing measure) region in the neighborhood of L over which dominance holds. Call this region $[L, \underline{\theta}]$, the *lower dominance region.*

Assumption 4.2 (Upper Dominance). *For each depositor of each bank, in each stage static coordination game* $\Gamma_{i,j}$ *for* $i \in \{c, d\}$, $j \in \{1, 2\}$, *if* $\theta_{i,j} = U$, *it is* strictly *dominant to remain.*

In other words, if depositors knew that the bank's returns were going to be at its highest possible level, it is strictly dominant to remain. This is an extremely weak assumption, and we do not model it explicitly in our payoffs. It can be supported by a number of assumptions. For example, we could assume that for very high θ, the risky asset in each region pays a premium over cash even in period 1. The strict dominance inherent in this assumption, implies, as before, that there exists some region $[\bar{\theta}, U]$, in the interior of which the dominance result holds. We call this the *upper dominance region.* Note that this region can be vanishingly small.

Given these two assumptions, we are able to uniquely characterize the equilibrium outcome of each of these static coordination games.

Proposition 4.1 (Static Existence). *For each stage static coordination game $\Gamma_{i,j}$ for $i \in \{c, d\}$, $j \in \{1, 2\}$ there exists a threshold level in the fundamentals $\theta^*(\Gamma_{i,j})$ such that agents who receive signals above $\theta^*(\Gamma_{i,j})$ will remain in the bank, while those that receive signals below it will withdraw.*

Proof: We demonstrate this proof for only one of the static coordination games: the coordination game of the debtor bank's patient depositors. The proofs for the other games are simpler than but otherwise identical to the proof for this one.

For the purposes of this proof, denote by θ, the underlying fundamentals of the bank concerned, and by θ_i the signal received by agent i. Upon receiving signal θ_i, the agent has to decide whether to remain or withdraw. The quantity she is interested in is the expected payoff difference between withdrawing and remaining. Suppose all other agents were following threshold strategies with threshold θ^*. Conditional upon receiving signal θ_i, the agent knows that fundamentals lie between $\theta_i - \epsilon$ and $\theta_i + \epsilon$, and has uniform beliefs over this interval. For any θ, therefore, the agent believes that a proportion

$$n(\theta, \theta^*) = \begin{cases} 1 & \text{if } \theta \leq \theta^* - \epsilon \\ \frac{1}{2} + \frac{\theta^* - \theta}{2\epsilon} & \text{if } \theta^* - \epsilon < \theta < \theta^* + \epsilon \\ 0 & \text{if } \theta \geq \theta^* + \epsilon \end{cases} \tag{4.11}$$

of agents will withdraw from the bank. For a particular (θ, θ^*), the payoff premium to remaining is given by:

$$\pi(\theta,n) = \begin{cases} u[0]-u[\frac{w+D+(1-w)r}{w+x+(1-w-x)n(\theta,\theta^*)}] & \text{if } \frac{(1-w)r+(D-x)}{1-w-x} \le n \le 1 \\ u[0]-u[1] & \text{if } \frac{(1-w)r+(D-x)-rD}{1-w-x} \le n \le \frac{(1-w)r+(D-x)}{1-w-x} \\ u[\frac{1-w-\frac{(1-(w+x))n_d+(x-D)}{r}-D}{(1-w-x)(1-n_d)}R(\theta_i)]-u[1] & \text{if } 0 \le n \le \frac{(1-w)r+(D-x)-rD}{1-w-x} \end{cases}$$

Thus, the quantity of interest to the agent is

$$\Pi(\theta_i) = \int_{\theta_i-\epsilon}^{\theta_i+\epsilon} \pi(\theta, n(\theta,\theta^*))d\theta$$

θ^* is a monotone equilibrium if the following hold:

1. $\Pi(\theta^*) = 0$
2. $\Pi(\theta_i) > 0$ if $\theta_i > \theta^*$
3. $\Pi(\theta_i) < 0$ if $\theta_i < \theta^*$

Observe that the existence of the upper and lower dominance regions implies that $\Pi(\theta^*)$ is negative for sufficiently low θ^* and positive for sufficiently high θ^*. Thus, it must cross the θ^* axis somewhere. This establishes (1) above.

To prove (2) and (3) observe that changing θ_i, holding θ^* constant only changes the bounds of integration in $\Pi(\cdot)$. In particular, note that since $\pi(\theta,n) < 0$ for $\theta \le \theta^* - \epsilon$ and $\pi(\theta,n) > 0$ for $\theta \ge \theta^* + \epsilon$, , there exists $\hat{\theta} \in (\theta^* - \epsilon, \theta^* + \epsilon)$ such that $\pi(\hat{\theta}) = 0$. Since $\Pi(\theta^*) = 0$, the positive and negative parts of the integral exactly offset each other. Increasing θ_i above θ^* increases the positive part of the integral and reduces the negative part, and thus makes $\Pi(\cdot)$ strictly positive. By the same token, reducing θ_i below θ^* makes $\Pi(\cdot)$ strictly negative. Thus, we have established (2) and (3). ■

Having thus shown existence of monotone equilibria, we now demonstrate that they are unique:

Proposition 4.2 (Static Uniqueness). *For each stage static coordination game* $\Gamma_{i,j}$ *for* $i \in \{c, d\}$, $j \in \{1, 2\}$ *there is only one threshold level in the fundamentals* $\theta^*(\Gamma_{i,j})$ *such that agents who receive signals above* $\theta^*(\Gamma_{i,j})$ *will remain in the bank, while those that receive signals below it will withdraw.*

Proof: Again, we prove this only for the coordination game of the patient depositors, and extend by symmetry to all other games. Write $n^d = \frac{(1-w)r+(D-x)}{1-w-x}$. Note that if $n(\theta, \theta^*) < n^d$, then $\theta > \theta^* + \epsilon(1 - 2n^d)$. Thus, we can express $\Pi(\cdot)$, as a sum of integrals over θ, with limits of integration given by functions of θ^*, following the piecewise definition of $\pi(\theta, n)$ above. Since the limits of integration are always linear with slope 1 in θ^*, integrating over constant terms gives us final products that are independent of θ^*. Thus, we can rewrite $\Pi(\cdot)$ as

$$\int_{\theta^*+\epsilon(1-2r)}^{\theta^*+\epsilon} u[\frac{1 - w - \frac{(1-(w+x))n_d+(x-D)}{r} - D}{(1 - w - x)(1 - n_d)} R(\theta_i)]d\theta$$
$$- \int_{\theta^*-\epsilon}^{\theta^*+\epsilon(1-2n^d)} u[\frac{w + D + (1 - w)r}{w + x + (1 - w - x)n(\theta, \theta^*)}]d\theta + K$$

where K proxies for the terms that do not involve θ^*. Taking the other parameters as given, we write:

$$f(\theta, \theta^*) = u[\frac{1 - w - \frac{(1-(w+x))n_d+(x-D)}{r} - D}{(1 - w - x)(1 - n_d)} R(\theta_i)]$$

and

$$g(\theta, \theta^*) = u[\frac{w + D + (1 - w)r}{w + x + (1 - w - x)n(\theta, \theta^*)}]$$

and differentiate with respect to θ^*:

$$\frac{d}{d\theta^*}\Pi(\theta,\theta^*) = \frac{d}{d\theta^*}\int_{\theta^*+\epsilon(1-2r)}^{\theta^*+\epsilon} f(\theta,\theta^*)d\theta - \frac{d}{d\theta^*}\int_{\theta^*-\epsilon}^{\theta^*+\epsilon(1-2n^d)} g(\theta,\theta^*)d\theta$$

Since the limits of integration, in each case are linear in θ^*, their derivatives are simply unity, and thus differentiating under the integral:

$$\frac{d}{d\theta^*}\int_{\theta^*+\epsilon(1-2r)}^{\theta^*+\epsilon} f(\theta,\theta^*)d\theta = f(\theta^*+\epsilon,\theta^*) - f(\theta^*+\epsilon(1-2r),\theta^*) + \int_{\theta^*+\epsilon(1-2r)}^{\theta^*+\epsilon} \frac{d}{d\theta^*}f(\theta,\theta^*)d\theta$$

We can rewrite this to be:

$$\frac{d}{d\theta^*}\int_{\theta^*+\epsilon(1-2r)}^{\theta^*+\epsilon} f(\theta,\theta^*)d\theta = \int_{\theta^*+\epsilon(1-2r)}^{\theta^*+\epsilon} \frac{d}{d\theta}f(\theta,\theta^*)d\theta + \int_{\theta^*+\epsilon(1-2r)}^{\theta^*+\epsilon} \frac{d}{d\theta^*}f(\theta,\theta^*)d\theta$$

Similarly,

$$\frac{d}{d\theta^*}\int_{\theta^*-\epsilon}^{\theta^*+\epsilon(1-2n^d)} g(\theta,\theta^*)d\theta = \int_{\theta^*-\epsilon}^{\theta^*+\epsilon(1-2n^d)} \frac{d}{d\theta}g(\theta,\theta^*)d\theta + \int_{\theta^*-\epsilon}^{\theta^*+\epsilon(1-2n^d)} \frac{d}{d\theta^*}g(\theta,\theta^*)d\theta$$

Now, we make the following set of observations:

1. $f(\theta,\theta^*)$ decreases in $n(\theta,\theta^*)$.

2. $g(\theta,\theta^*)$ decreases in $n(\theta,\theta^*)$.

3. $n(\theta,\theta^*)$ increases in θ^*.

4. $n(\theta,\theta^*)$ decreases in θ.

5. $|\frac{dn(\theta,\theta^*)}{d\theta}| = |\frac{dn(\theta,\theta^*)}{d\theta^*}|$, since θ and θ^* enter $n(\theta,\theta^*)$ symmetrically.

6. $R(\theta)$ increases in θ, but is unaffected by θ^*.

Now, (1) and (3) imply that $f(\theta, \theta^*)$ decreases in θ^*. (1) and (4) imply that $f(\theta, \theta^*)$ increases in θ. (1), (3), (4), (5), and (6) imply that $|\frac{df(\theta,\theta^*)}{d\theta}| > |\frac{df(\theta,\theta^*)}{d\theta^*}|$. Thus,

$$\frac{d}{d\theta^*}\int_{\theta^*+\epsilon(1-2r)}^{\theta^*+\epsilon} f(\theta, \theta^*)d\theta > 0$$

Similarly, (2) and (3) imply that $g(\theta, \theta^*)$ decreases in θ^*. (2) and (4) imply that $g(\theta, \theta^*)$ increases in θ. (2), (3), and (5) imply that $|\frac{dg(\theta,\theta^*)}{d\theta}| = |\frac{dg(\theta,\theta^*)}{d\theta^*}|$. Thus,

$$\frac{d}{d\theta^*}\int_{\theta^*-\epsilon}^{\theta^*+\epsilon(1-2n^d)} g(\theta, \theta^*)d\theta = 0$$

In the net, we have just shown that $\Pi(\cdot)$ is strictly increasing in θ^*. Thus, there is only one value of θ^* that solves $\Pi(\theta, \theta^*) = 0$. ■

This theorem implies unique monotone equilibria in the component stage games of Γ. Does this mean that there is a unique monotone equilibrium in the general game Γ? In order to investigate this, we introduce some additional notation. There are four possible static coordination games. The creditor bank may go first. Denote the threshold of this game by $\theta^*_{c,1}$. But the creditor bank may also go second. In particular, it may do so after observing that the debtor bank has failed or survived. Denote the threshold of the creditor bank's depositors conditional upon the survival of the debtor bank by $\theta^*_{c,S}$. Conditional upon the failure of the debtor bank, call this threshold $\theta^*_{c,F}$. Correspondingly denote by $\theta^*_{d,1}$, $\theta^*_{d,S}$, and $\theta^*_{d,F}$ the respective thresholds for the depositors of the debtor bank.

In general, several of these thresholds are interlinked by intricate functional relationships. Thus, uniqueness in the component static games does not necessarily imply uniqueness in the general dynamic game. However, note that a solvent debtor bank is assumed to always pay its debt to the residual claimants of the creditor bank, *regardless of whether the creditor bank remains solvent or not.*[13] Thus, the thresholds of the debtor bank's depositors are independent of the outcome in the creditor bank, and, therefore, of the thresholds of the creditor bank's depositors. Thus, $\theta^*_{d,1}$, $\theta^*_{d,S}$, and $\theta^*_{d,F}$ are all uniquely defined in the general dynamic game (in particular, they are equal to each other). Thus, $\theta^*_{c,1}(\theta^*_{d,S}, \theta^*_{d,F})$ is uniquely defined, as is $\theta^*_{c,S}(\theta^*_{d,1})$. $\theta^*_{c,F}$ is uniquely defined, independent of $\theta^*_{d,1}$, $\theta^*_{d,S}$, and $\theta^*_{d,F}$. Thus, we have just argued that the following global uniqueness result holds:

Proposition 4.3 (Dynamic Uniqueness). *There is a unique monotone equilibrium in* Γ. *In* Γ_c *it is characterized by the triple* $(\theta^*_{c,1}, \theta^*_{d,S}, \theta^*_{d,F})$. *In* Γ_d*, it is characterized by the triple* $(\theta^*_{d,1}, \theta^*_{c,F}, \theta^*_{c,S})$.

Observe that one straightforward interpretation of propositions 4.1, 4.2, and 4.3 taken together is that banks are run and fail upon the release of adverse news about them, and therefore for lower levels of asset returns (when noise goes to zero, signals and asset returns are perfectly correlated in the limit). This matches the commonly made observation that bank failures are positively correlated with the release of bank-specific or region-specific adverse information.

Having established uniqueness of monotone equilibria, it is now of interest to us to explore some of their properties.

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[13]This assumption is not one of convenience (though it is convenient) but is simply a stylized interpretation of common bankruptcy law. The failure of the creditor bank leaves a set of residual claimants to its assets. These assets naturally include expected payments from the debtor bank.

4.3 Contagion

Contagion emerges as a natural property of the unique equilibrium thresholds of this game. In order to examine the phenomenon of contagion, it is necessary to provide a precise theoretical definition of what we mean by the term. In the context of bank runs, the most natural concept of contagion that emerges is as follows: Consider any two banks within a banking system, i and j. Both banks i and j have some probability of failure/insolvency independent of what happens in the other banks. Thus, even if bank i does not fail, bank j may fail for some realized level of adverse information about it. However, if bank i fails, this may create an adverse effect on bank j, and bank j may fail for a *larger* range of realized adverse information. Thus, we say that the failure of bank i *has a contagious effect* on bank j, if, conditional on the failure of bank i, bank j fails with higher probability than it would have had bank i not failed. Formally, we can define this as follows:

Definition 4.1 (Contagion). *Consider a pair of banks, each with asset returns indexed by θ_k of $k = 1, 2$. Let $\theta^*_{k,F}$ and $\theta^*_{k,S}$ denote the failure threshold of bank k conditional on the failure and success of bank $-k$ respectively. We say that the failure of bank $-k$ contagiously affects bank k if there exists some positive measure region of fundamentals $[\theta^*_{k,S}, \theta^*_{k,F}]$ within which bank k fails if and only if bank $-k$ also fails.*

Having thus defined contagion, we are ready to state a central result of this paper.

Proposition 4.4 (Contagion). *In Γ_d, the failure of the debtor contagiously affects the creditor, i.e., there exists a region of fundamentals $[\theta^*_{c,S}, \theta^*_{c,F}]$ in which the creditor bank fails if and only if the debtor bank fails. But in Γ_c, $\theta^*_{d,F} = \theta^*_{d,S}$. Thus, the failure of the creditor does not contagiously affect the debtor.*

Proof: To prove this result, we begin by writing down the threshold equation for the coordination game amongst depositors at the creditor bank conditional on the failure of the debtor bank. First, we write $n_1^c = \frac{x-D}{1-w+x}$, and $n_2^c = \frac{(1-w)r+x-D}{1-w+x}$. Let $l_1 = 1 - 2n_1^c$, and $l_2 = 1 - 2n_2^c$. Finally, for brevity, we let $m = 1 - w + x$, and suppress the arguments of $n(\theta, \theta^*)$. Then, the threshold equation for patient depositors of the creditor bank conditional upon the failure of the debtor bank can be written as $L_f(\theta^*) = R_f(\theta^*)$ where,

$$L_f(\theta^*) = \int_{\theta^*+\epsilon l_1}^{\theta^*+\epsilon l_2} u[\frac{1 - w - \frac{x-D+nm}{r}}{(1-n)m} R(\theta)]d\theta + \int_{\theta^*+\epsilon l_2}^{\theta^*+\epsilon} u[\frac{(x-D) - mn + (1-w)R(\theta_i)}{(1-n)m}]d\theta$$

$$R_f(\theta^*) = \int_{\theta^*-\epsilon}^{\theta^*+\epsilon l_1} (u[\frac{w - D + (1-w)r}{w - x + mn}] - u[0])d\theta + K_1$$

where $K_1 = \int_{\theta^*+\epsilon l_1}^{\theta^*+\epsilon} u[1]d\theta$. We know by our previous results that there is a unique $\theta_{c,F}^*$ that solves this equation. Now, we write down the corresponding threshold equation for the depositors of the creditor bank conditional upon the survival of the debtor bank as $L_s(\theta^*) = R_s(\theta^*)$, where

$$L_s(\theta^*) = \int_{\theta^*+\epsilon l_1}^{\theta^*+\epsilon l_2} u[\frac{(1 - w - \frac{x-D+nm}{r})R(\theta) + g}{(1-n)m}]d\theta + \int_{\theta^*+\epsilon l_2}^{\theta^*+\epsilon} u[\frac{(x-D) - mn + (1-w)R(\theta_i) + g}{(1-n)m}]d\theta$$

$$R_s(\theta^*) = \int_{\theta^*-\epsilon}^{\theta^*+\epsilon l_1} (u[\frac{w - D + (1-w)r}{w - x + mn} + g] - u[g])d\theta + K_1$$

where K_1 is as before. Observe that since $g > 0$, $L_s(\theta^*) > L_f(\theta^*)$ for all θ^*. Since $u(\cdot)$ is a concave function, $u(x+y) - u(y) \leq u(x) - u(0)$, for all $x, y > 0$. Thus, $R_s(\theta^*) \leq R_f(\theta^*)$ for all θ^*. In particular, this means that

$$L_s(\theta_{c,F}^*) > R_s(\theta_{c,F}^*)$$

i.e., $\theta_{c,F}^* \neq \theta_{c,S}^*$. Now, observe that by analogy to the proof of proposition 4.2 we know that $L_s(\theta^*)$ is increasing in θ^*, while $R_s(\theta^*)$ is invariant with θ^*. Thus, in order to make the indifference equations hold, we need to reduce θ^* below $\theta^*_{c,F}$, and thus, we have just shown that $\theta^*_{c,S} < \theta^*_{c,F}$. ■

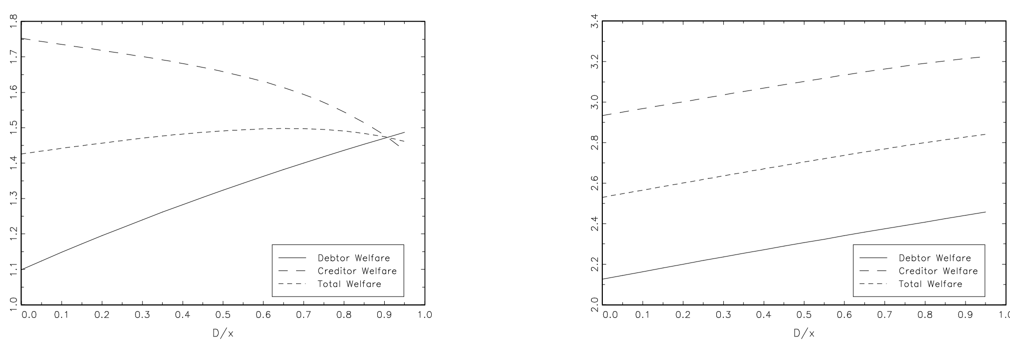
In other words, contagious effects flow in a specific direction, from debtors to creditors. The failure of debtors, naturally adversely affects the failure of creditors, and thus, ceteris paribus, makes it likelier that the creditor shall fail. On the other hand, the failure of a creditor makes the debtor no better or worse off, since we have assumed that the debtor has to always pay the residual claimants of the failed creditor the amount originally owed.[14] Focussing purely on capital linkages as a channel for contagion, this result cleanly characterizes why individual bank runs may not necessarily become widespread bank panics. Conditional upon the failure of a bank, this theorem also characterizes *which* other banks, ceteris paribus, are likelier to face runs. Thus, this result, while in no way a complete description of contagion, gives one rationale for why panics may be "local" in some specific but unobvious sense.

4.4 Should banks hold interbank deposits?

In our discussion to date, we have not commented on the size of interbank deposit holdings. We have shown above that when banks cross-hold deposits to hedge against regional liquidity shocks, the failure of one bank may contagiously affect the other. Thus, in deciding the amount of interbank deposit holdings, banks trade off the benefit of insuring liquidity shocks

[14]Any other assumption would imply that the debtor actually *benefits* from the failure of the creditor, thus naturally ruling out contagion, leaving our results unchanged.

against the cost of exposing themselves to the risk of contagion. We have demonstrated that for a given set of parameters of our stylized banking system (w, x, r, U, L), for each choice of $D \in [0, x]$, there is a unique monotone equilibrium with an associated level of social welfare. This makes it possible to determine the optimal interbank deposit amount by mazimizing ex ante social welfare. While the model is too complex to derive an analytical formula for such optimal interbank deposit levels, it is possible to characterize the solution numerically.

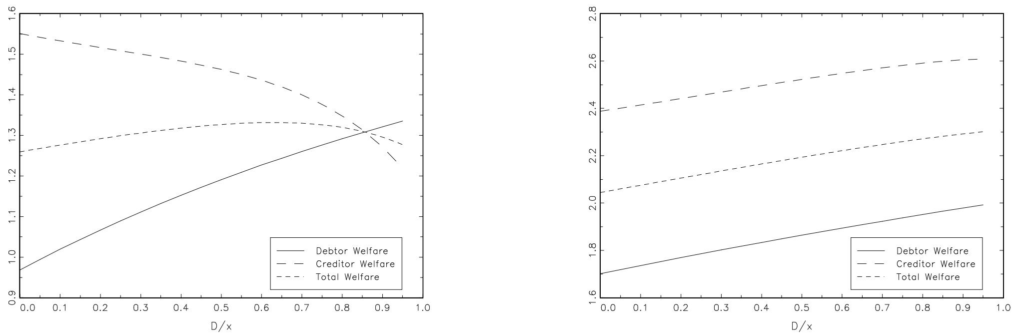


bank runs likely　　　　bank runs rare

Figure 4.1: Welfare Comparisons: $w = 0.3$, $x = \frac{w}{2}$, $r = 0.7$

Figures 4.1 and 4.2 illustrate the optimal holding of interbank deposits in a risk neutral economy.[15] In figure 4.1, we set the constant aggregate liquidity demand (w) to be 30%, and the level of regional variation at $\frac{w}{2}$. We fix the lower bound of fundamentals (θ) to

[15]The features of the examples shown in figures 4.1 and 4.2 are representative of a large number of simulations that we have completed but do not present here for brevity.



bank runs likely   bank runs rare

Figure 4.2: Welfare Comparisons: $w = 0.5$, $x = \frac{w}{4}$, $r = 0.7$

be $L = 0$, and vary the upper bound (U) to change the equilibrium probability of bank runs. The left panel in each image corresponds to the case in which bank runs are relatively frequent ($U = 10$), and the right panel to the case in which runs are rare ($U = 30$). The early liquidation return r is set to 0.7, and $R(\theta) = \sqrt{\theta}$. Figure 4.2 portrays a similar economy in which average liquidity demand is greater ($w = 0.5$), but the proportional level of regional variation is smaller ($x = \frac{w}{4}$). In all cases, the analysis if carried out "close" to the full information limit. The bound on idiosyncratic noise (ϵ) is set to $\frac{1}{1000}$.

The x-axis in each image shows $\frac{D}{x}$, the proportionate size of interbank deposit holdings. The central locus in each figure represents the ex ante social welfare at to the corresponding level of D. From the figures it is apparent that when bank runs are frequent, social welfare is maximized for *intermediate* values of D, whereas when runs are rare, welfare is maximized for *maximal* values of D.

Intuition for this property can be derived upon inspection of the other two locii in each of the images. These locii represent the ex ante welfare of banks under the hypothetical assumption that they know whether they are going to be interim debtors or creditors (i.e., receive high or low idiosyncratic regional shocks in period 1). Since the two regions receive idiosyncratic liquidity shocks with equal probability in the model, the central welfare locus is simply the arithmetic average of these two locii. It follows from the model that interbank deposit holdings are *always* beneficial for debtors. They are not affected by contagion. Thus interbank holdings insure them against regional liquidity shocks without exposing them to any additional risk. This is evidenced in the figures: the locii representing debtor welfare are always strictly increasing in $\frac{D}{x}$. For the bank that is the interim creditor, there is a tangible risk of contagion. However, contagion is a *conditional* event. A necessary condition for contagion is that the debtor bank must actually experience a run. Thus, the attractiveness of interbank deposits to creditor banks depends crucially on their assessment of the probability of runs on the debtor bank. When this probability is high, a higher level of interbank deposit holdings exposes the creditor to greater contagion risk, thus lowering its ex ante welfare. This can be seen in the left panels of the figures. When the probability that the debtor bank will fail is low, it becomes much more attractive for the creditor bank to hold interbank deposits, which enable it to eliminate idle reserves and increase expected payments to its depositors. Under these circumstances, holding higher interbank deposits increases the welfare of the interim debtor bank. This can be seen in the right panels of the figures.

Ex ante social welfare is given by the average of debtor and creditor welfares. When bank runs are rare, and both debtor and creditor welfare increasing in $\frac{D}{x}$, social welfare is maximized at $D = x$. Thus, banks should hold a maximal level of interbank deposits. However, when bank runs are more likely, creditor welfare decreases in $\frac{D}{x}$ while debtor welfare increases. Thus, the social welfare locus is hump-shaped, and there is a strictly interior level of optimal interbank deposit holdings.

4.5 Closed Forms, Comparative Statics

Historically, bank runs have been rare. The simulations presented above suggest that when bank runs are rare, banks will find it optimal to insure against idiosyncratic liquidity shocks by holding a maximal level of interbank deposits. For the remainder of the paper, we shall, therefore, assume that banks fully insure against regional liquidity shocks by setting $D = x$.

This assumption enables us to demonstrate further properties of contagion. Under the assumption of complete interbank deposit holdings, we can solve the linear realization of the model (with risk neutral consumers and $R(\theta) = \theta$) in closed form.

Below, we present the closed forms for the equilibrium thresholds for this simplified model. This requires a small intermediate step. The thresholds of depositors of the creditor bank are functions of the interbank payments from the debtor bank. These interbank payments, in turn, are functions of the proportion of premature withdrawals from the creditor bank (which may or may not be observable to the depositors of the creditor bank, depending on whether Γ_c or Γ_d is realized) and the underlying fundamentals of the debtor bank (which are never observable to the depositors of the creditor bank). The depositors of the creditor bank, therefore, have to compute the *expected* interbank payments from the debtor bank. In preparation for writing down the closed forms for equilibrium thresholds, we explicate below the functional form of the expected interbank payments for the game.

4.5.1 Expected Interbank Payments

There are two situations in which payments from the debtor bank must be anticipated by depositors in the creditor bank. In game Γ_c the depositors of the creditor bank must, upon acting first, decide what probability with which the debtor bank will remain solvent, and, if solvent, the expected payment that they shall make. Observe that the depositors of the creditor bank can calculate θ_d^*, which is uniquely defined in terms of the parameters. Given θ_d^*, they can assign probabilities to ranges of $n_d(\theta_d, \theta_d^*)$, and thus compute $\hat{g} = E(g|\theta_d^*)$.

In the game Γ_d when the debtor bank survives, the depositors of the creditor bank can observe n_d. When $0 < n_d < 1$, there is a one-to-one relationship between n_d and θ_d, and thus for strictly interior n_d, depositors at the creditor bank can compute $g(n_d, \theta_d)$ exactly. However, when $n_d = 0$, the depositors of the creditor bank know simply that $\theta_d > \theta_d^* + \epsilon$, or, in the limit as noise vanishes, $\theta_d > \theta_d^*$. Thus, their expected value of θ_d is $\frac{\theta_d^*+U}{2}$. Hence, for this game, we can write $\hat{g}$ as follows:

$$\hat{g}(\theta_d|\theta_d^*, n_d) = \begin{cases} xR(\frac{\theta_d^*+U}{2}) & \text{if } n_d = 0 \\ g(\theta_d, n_d) & \text{if } 1 > n_d > 0 \end{cases} \tag{4.12}$$

We are now ready to write down the equilibrium thresholds.

4.5.2 Limiting Thresholds

The equilibrium thresholds are particularly tractable in the limit as $\epsilon \to 0$. The limit threshold points of interest are as follows:

- For the coordination game amongst depositors at the debtor bank:

$$\theta_d^* = r\frac{r(1-w) + (r(1-w)+w+x)\ln\left[\frac{1}{r(1-w)+w+x}\right]}{(1-w-x)(r+(1-r)\ln[1-r])} \tag{4.13}$$

- For the coordination game amongst depositors at the creditor bank conditional upon the survival of the debtor bank (in the game Γ_d):

$$\theta_{c,S}^* = r\frac{r(1-w) + \hat{g}(n_d,\theta_d^*)\ln\left[1-\frac{1-w}{1-w+x}r\right] + (r(1-w)+w-x)\ln\left[\frac{1}{r(1-w)+w-x}\right]}{r(1-w)+(1-r(1-w)-w+x)\ln\left[1-\frac{1-w}{1-w+x}r\right]} \tag{4.14}$$

where $\hat{g}(\cdot)$ is defined as in (4.12).

- For the coordination game amongst depositors at the creditor bank conditional upon the failure of the debtor bank (in the game Γ_d):

$$\theta_{c,F}^* = r\frac{r(1-w) + (r(1-w)+w-x)\ln\left[\frac{1}{r(1-w)+w-x}\right]}{r(1-w)+(1-r(1-w)-w+x)\ln\left[1-\frac{1-w}{1-w+x}r\right]} \tag{4.15}$$

Now, it is of interest to us to explore some of the comparative statics of our model by using these closed form expressions for the thresholds. In particular, we examine below how the magnitude of the contagious effect changes as the volatility of the financial system, or, equivalently,the depth of inter-institutional financial linkages increases.[16] It turns out that the degree of the contagious effect increases in the strength of financial linkages between banks. In order to prove this, we need to first prove the following two preliminary results.

Lemma 4.1. $r(1-w)+(1-r(1-w)-w+x)\ln\left[1-\frac{1-w}{1-w+x}r\right]$ *is decreasing in* x *and positive over the permissible range of* x.

[16]Note that in a fully rational set-up, regional liquidity shocks are always anticipated ex ante. Since these shocks are negatively correlated, larger shocks imply larger interbank insurance levels, under the maintained assumption of complete interbank deposit insurance.

Proof: Let

$$E(r, w, x) = r(1-w) + (1 - r(1-w) - w + x) \ln \left[1 - \frac{1-w}{1-w+x} r\right]$$

Write $a = (1-w)(1-r)$ and $b = (1-w)$. Observe that $a = b(1-r)$. Now,

$$E(r, a, b, x) = (a+x) \ln \left[1 - \frac{br}{b+x}\right]$$

which simplifies to

$$E(a, b, x) = (a+x) \ln \left[\frac{a+x}{b+x}\right]$$

We differentiate with respect to x to obtain

$$E'[a, b, x] = \frac{b-a}{b+x} + \ln \left[\frac{a+x}{b+x}\right]$$

Is this expression always negative over the permissible range of x? To investigate, we differentiate again with respect to x, and obtain

$$E''(a, b, x) = \frac{b-a}{b+x} \left[\frac{1}{a+x} - \frac{1}{b+x}\right] > 0$$

Thus, if $E' < 0$ for the maximal permissible x, we shall be done. But note that $x < \min[w, (1-w)(1-r)] = \min[1-b, a]$. We check that for $x = 1-b$, E' is negative.

$$E'(a, b, 1-b) = (b-a) + \ln(1-(b-a))$$

Let $y = b - a$. This is then equivalent to

$$y > 1 - \exp(-y)$$

which is always true for $y \geq 0$. Thus, we have just shown that $r(1-w) + (1 - r(1-w) - w + x) \ln \left[1 - \frac{1-w}{1-w+x} r\right]$ is decreasing in x. To show that it is always positive, we simply need to check that it is positive at the highest global value of x, which is $\frac{1}{2}$. Note that $x = \frac{1}{2}$ implies that $w = \frac{1}{2}$ to keep everything well defined. Thus, our target expression reduces to

$$E(r, \frac{1}{2}, \frac{1}{2}) = \frac{r}{2} + (1 - \frac{r}{2}) \ln \left[1 - \frac{r}{2}\right]$$

which is clearly positive for all $r \in (0, 1)$. ■

Lemma 4.2. *θ_d^* increases in x.*

Proof: Observe that

$$\theta_d^* = \left[\frac{r}{r + (1-r)\ln(1-r)}\right] \left[\frac{(1-w) + (r(1-w) + w_x)\ln\left[\frac{1}{r(1-w)+w+x}\right]}{(1-w-x)}\right]$$

Observe that the left term in the product is always positive and is not affected by x. Thus, to analyze the dependence of θ_d^* on x, it is sufficient to examine the right term in the product. We differentiate this with respect to x, to obtain

$$\frac{[x - (1-r)(1-w)] - [1 + r(1-w)]\ln[r(1-w) + w + x]}{(1-w-x)^2}$$

Since the denominator is always positive, if we could show that the numerator is positive, then we shall be done. Observe that by differentiating the numerator with respect to x, we get

$$1 - \frac{1 + r(1-w)}{r(1-w) + w + x}$$

Since $x < (1-r)(1-w)$, the numerator is decreasing in x. Thus, if the numerator is positive for the maximal x, then it is positive for all x, and we are done. We check that the numerator is 0 for $x = (1-r)(1-w)$, and so we are done. ■

Given these results, we can now state:

Proposition 4.5. *The size of contagion increases with the size of the regional liquidity shocks, i.e.,* $ct(r, w, x) = \theta^*_{c,F} - \theta^*_{c,S}$ *is increasing in* x.

Proof: In the limit, the survivor of the debtor bank implies that $n_d = 0$, so that $\hat{g}(\cdot) = x(\frac{\theta^*_d + U}{2})$. Now, we can use the expressions for the equilibrium thresholds shown above and write:

$$ct(r, w, x) = \frac{[-x(\frac{\theta^*_d + U}{2})]r \ln\left[1 - \frac{1-w}{1-w+x}r\right]}{r(1-w) + (1 - r(1-w) - w + x)\ln\left[1 - \frac{1-w}{1-w+x}r\right]}$$

The numerator of $ct(x)$ can be rewritten as $[(\frac{\theta^*_d + U}{2})]r(-x \ln\left[1 - \frac{1-w}{1-w+x}r\right])$. $\frac{\theta^*_d + U}{2}$ is increasing in θ^*_d, which in turn is increasing in x by Lemma 4.2. $-x\ln\left[1 - \frac{1-w}{1-w+x}r\right]$ is increasing in x. By Lemma 4.1 we know that the denominator decreases in x. Thus, $ct(x)$ increases in x. ■

This proposition has a natural interpretation. Contagion flows from debtors to creditors through the channels of interbank deposits. The larger the interbank deposits, the larger the "pipe" through which the contagious effect can flow. In a setting of complete interbank deposits (or, indeed, in any setting in which interbank deposits are increasing in the size of the negatively correlated regional liquidity shocks), the larger the anticipated regional liquidity shocks, the larger the dollar value of interbank deposits, and thus the larger the effect of contagion when it occurs. In another, somewhat looser, interpretation, note that the size of the regional liquidity shocks can be seen to be a measure of the level of intra-economy financial volatility. On this interpretation, the proposition above says that as financial volatility in the system increases, the damage caused by the failure of a financial institution also increases. This too is intuitive.

4.6 Discussion

We conclude with a few thoughts on the robustness of these results, and on potential extensions.

4.6.1 More regions? Aggregate liquidity shocks?

Our model extends naturally to more than two regions, and none of the results change in this extension. With more than two regions, holding aggregate liquidity constant, there would be some level of negative correlation across regional liquidity demands. This would create, as before, the incentive to insure against regional liquidity demand shocks using interbank deposits. The only significant change would be one of algebraic complexity in computing the level of interbank deposits, since imperfect negative correlation across regional liquidity shocks would lead to multi-party cross holdings of interbank deposits. The existence of interbank deposits, however complex, along with the seniority of institutional claims, shall create ex interim asymmetries amongst banks (some shall be debtors, others creditors) exactly as in the simpler two-region model discussed above. Thus, the contagious effect shall re-emerge in equilibrium exactly as before.

Adding aggregate liquidity shocks to our model creates a second source of bank failure in our model without changing the internal structure of interbank deposits and contagion. With large aggregate liquidity shocks, banks may fail simply because there is just not enough money in the system to meet all claims in period 1 even without expectations-based runs. While we do not deny that fully aggregate liquidity shocks may, indeed, emerge in an economy, we argue that the strategic nature of the game remains the same even in their presence. Therefore, we limit our attention to constant aggregate liquidity economies and show that contagion occurs *even in such economies.* Naturally, adding more sources for contagion will increase its occurence.

4.6.2 Learning?

Perhaps the most significant drawback of our model is that we abstract from social learning. We assume that fundamentals in the two regions are independent, and thus eliminate any conclusions that agents in one region can draw about their own bank from the observed failure or survival of a bank in a different region. While this assumption seems fitting when discussing banks in New York and Oregon, for banks in neighboring regions, it seems less natural. Incorporating correlations amongst assets across the regions of our economy would introduce Bayes learning into our model and would complicate our arguments significantly. However, in recent work we have laid out techniques to analyze dynamic coordination games with social learning (Dasgupta 2001). Incorporating learning into a model similar to ours is a promising direction for future research. We conjecture that incorporating learning into our model would increase the occurence of contagion, without modifying the main qualitative features outlined above.

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