

# 14.452 Economic Growth: Lecture 8, Neoclassical Endogenous Growth

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# First-Generation Models of Endogenous Growth

- Models so far: no sustained long-run growth; relatively little to say about sources of technology differences.
- Models in which technology evolves as a result of firms' and workers' decisions are most attractive in this regard.
- But sustained economic growth is possible in the neoclassical model as well:
  - *AK* model before: relaxed Assumption 2 and prevented diminishing returns to capital.
  - Capital accumulation could act as the engine of sustained economic growth.
- Neoclassical version of the *AK* model:
  - Very tractable and applications in many areas.
  - Shortcoming: capital is essentially the only factor of production, asymptotically share of income accruing to it tends to 1.
- Two-sector endogenous growth models behave very similarly to the baseline *AK* model, but avoid this.

# Demographics, Preferences and Technology I

- Focus on balanced economic growth, i.e. consistent with the Kaldor facts.
- Thus CRRA preferences as in the canonical neoclassical growth model.
- Economy admits an infinitely-lived representative household, household size growing at the exponential rate  $n$ .
- Preferences

$$U = \int_0^{\infty} \exp(-(\rho - n)t) \left[ \frac{c(t)^{1-\theta} - 1}{1-\theta} \right] dt. \quad (1)$$

- Labor is supplied inelastically.
- Flow budget constraint,

$$\dot{a}(t) = (r(t) - n)a(t) + w(t) - c(t), \quad (2)$$

## Demographics, Preferences and Technology II

- No-Ponzi game constraint:

$$\lim_{t \rightarrow \infty} \left\{ a(t) \exp \left[ - \int_0^t [r(s) - n] ds \right] \right\} \geq 0. \quad (3)$$

- Euler equation:

$$\frac{\dot{c}(t)}{c(t)} = \frac{1}{\theta} (r(t) - \rho). \quad (4)$$

- Transversality condition,

$$\lim_{t \rightarrow \infty} \left\{ a(t) \exp \left[ - \int_0^t [r(s) - n] ds \right] \right\} = 0. \quad (5)$$

- Problem is concave, solution to these necessary conditions is in fact an optimal plan.
- Final good sector similar to before, but Assumptions 1 and 2 are *not* satisfied.

## Demographics, Preferences and Technology III

- More specifically,

$$Y(t) = AK(t),$$

with  $A > 0$ .

- Does not depend on labor, thus  $w(t)$  in (2) will be equal to zero.
- Defining  $k(t) \equiv K(t) / L(t)$  as the capital-labor ratio,

$$\begin{aligned} y(t) &\equiv \frac{Y(t)}{L(t)} \\ &= Ak(t). \end{aligned} \tag{6}$$

- Notice output is only a function of capital, and there are no diminishing returns
- But introducing diminishing returns to capital does not affect the main results in this section.

## Demographics, Preferences and Technology IV

- More important assumption is that the Inada conditions embedded in Assumption 2 are no longer satisfied,

$$\lim_{k \rightarrow \infty} f'(k) = A > 0.$$

- Conditions for profit-maximization are similar to before, and require  $R(t) = r(t) + \delta$ .
- From (6) the marginal product of capital is  $A$ , thus  $R(t) = A$  for all  $t$ ,

$$r(t) = r = A - \delta, \text{ for all } t. \quad (7)$$

# Equilibrium I

- A competitive equilibrium of this economy consists of paths  $[c(t), k(t), w(t), R(t)]_{t=0}^{\infty}$ , such that the representative household maximizes (1) subject to (2) and (3) given initial capital-labor ratio  $k(0)$  and  $[w(t), r(t)]_{t=0}^{\infty}$  such that  $w(t) = 0$  for all  $t$ , and  $r(t)$  is given by (7).
- Note that  $a(t) = k(t)$ .
- Using the fact that  $r = A - \delta$  and  $w = 0$ , equations (2), (4), and (5) imply

$$\dot{k}(t) = (A - \delta - n)k(t) - c(t) \quad (8)$$

$$\frac{\dot{c}(t)}{c(t)} = \frac{1}{\theta}(A - \delta - \rho), \quad (9)$$

$$\lim_{t \rightarrow \infty} k(t) \exp(-(A - \delta - n)t) = 0. \quad (10)$$

## Equilibrium II

- The important result immediately follows from (9).
  - Since the right-hand side is constant, there must be a constant rate of consumption growth (as long as  $A - \delta - \rho > 0$ ).
  - Growth of consumption is independent of the level of capital stock per person,  $k(t)$ .
  - No transitional dynamics in this model.
- To develop, integrate (9) starting from some  $c(0)$ , to be determined from the lifetime budget constraint,

$$c(t) = c(0) \exp\left(\frac{1}{\theta}(A - \delta - \rho)t\right). \quad (11)$$

- Need to ensure that the transversality condition is satisfied and ensure positive growth ( $A - \delta - \rho > 0$ ). Impose:

$$A > \rho + \delta > (1 - \theta)(A - \delta) + \theta n + \delta. \quad (12)$$



# Equilibrium Characterization

- Substitute for  $c(t)$  from equation (11) into equation (8),

$$\dot{k}(t) = (A - \delta - n)k(t) - c(0) \exp\left(\frac{1}{\theta}(A - \delta - \rho)t\right), \quad (13)$$

- The solution to this differential equation combined with the transversality condition gives (see book for details):

$$\begin{aligned} k(t) &= [(A - \delta)(\theta - 1)\theta^{-1} + \rho\theta^{-1} - n]^{-1} \\ &\quad \times [c(0) \exp(\theta^{-1}(A - \delta - \rho)t)] \\ &= k(0) \exp(\theta^{-1}(A - \delta - \rho)t), \end{aligned} \quad (14)$$

- Second line follows from the fact that the boundary condition has to hold for capital at  $t = 0$ .
- Hence capital and output grow at the same rate as consumption.
- This also pins down the initial level of consumption as

$$c(0) = [(A - \delta)(\theta - 1)\theta^{-1} + \rho\theta^{-1} - n] k(0). \quad (15)$$

## Equilibrium Savings Rate

- No transitional dynamics: growth rates of consumption, capital and output are constant and given in (9).
- Growth is not only sustained, but also endogenous in the sense of being affected by underlying parameters.
  - E.g., an increase in  $\rho$ , will reduce the growth rate.
- Saving rate = total investment (increase in capital plus replacement investment) divided by output:

$$\begin{aligned}
 s &= \frac{\dot{K}(t) + \delta K(t)}{Y(t)} \\
 &= \frac{\dot{k}(t)/k(t) + n + \delta}{A} \\
 &= \frac{A - \rho + \theta n + (\theta - 1)\delta}{\theta A}, \tag{16}
 \end{aligned}$$

- Last equality exploited  $\dot{k}(t)/k(t) = (A - \delta - \rho)/\theta$ .

# Equilibrium Characterization: Summary

- Saving rate, constant and exogenous in the basic Solow model, is again constant.
- But is now a function of parameters, also those that determine the equilibrium growth rate of the economy.

**Proposition** Consider the above-described  $AK$  economy, with a representative household with preferences given by (1), and the production technology given by (6). Suppose that condition (12) holds. Then, there exists a unique equilibrium path in which consumption, capital and output all grow at the same rate  $g^* \equiv (A - \delta - \rho)/\theta > 0$  starting from any initial positive capital stock per worker  $k(0)$ , and the saving rate is endogenously determined by (16).

# Pareto Optimality

- Since all markets are competitive, there is a representative household, and there are no externalities, the competitive equilibrium will be Pareto optimal.
- Can be proved either using First Welfare Theorem type reasoning, or by directly constructing the optimal growth solution.

**Proposition** Consider the above-described  $AK$  economy, with a representative household with preferences given by (1), and the production technology given by (6). Suppose that condition (12) holds. Then, the unique competitive equilibrium is Pareto optimal.

# The Role of Policy I

- Suppose there is an effective tax rate of  $\tau$  on the rate of return from capital income, so budget constraint becomes:

$$\dot{a}(t) = ((1 - \tau) r(t) - n)a(t) + w(t) - c(t). \quad (17)$$

- Repeating the analysis above this will adversely affect the growth rate of the economy, now:

$$g = \frac{(1 - \tau)(A - \delta) - \rho}{\theta}. \quad (18)$$

- Moreover, saving rate will now be

$$s = \frac{(1 - \tau)A - \rho + \theta n - (1 - \tau - \theta)\delta}{\theta A}, \quad (19)$$

which is a decreasing function of  $\tau$  if  $A - \delta > 0$ .

## The Role of Policy II

- In contrast to Solow, constant saving rate responds endogenously to policy.
- Since saving rate is constant, differences in policies will lead to permanent differences in the rate of capital accumulation.
  - In the baseline neoclassical growth model even large differences in distortions could only have limited effects on differences in income per capita.
  - Here even small differences in  $\tau$  can have very large effects.
- Consider two economies, with tax rates on capital income  $\tau$  and  $\tau' > \tau$ , and exactly the same otherwise.
- For any  $\tau' > \tau$ ,

$$\lim_{t \rightarrow \infty} \frac{Y(\tau', t)}{Y(\tau, t)} = 0,$$

# The Role of Policy III

- Why then focus on standard neoclassical if *AK* model can generate arbitrarily large differences?
  - ① *AK* model, with no diminishing returns and the share of capital in national income asymptoting to 1, is not a good approximation to reality.
  - ② Relative stability of the world income distribution in the post-war era makes it more attractive to focus on models in which there is a stationary world income distribution.

# The Two-Sector AK Model I

- Model before creates another factor of production that accumulates linearly, so equilibrium is again equivalent to the one-sector *AK* economy.
- Thus, in some deep sense, the economies of both sections are one-sector models.
- Also, potentially blur key underlying characteristic driving growth.
- What is important is not that production technology is *AK*, but that the *accumulation technology* is linear.
- Preference and demographics are the same as in the model of the previous section, (1)-(5) apply as before
- No population growth, i.e.,  $n = 0$ , and  $L$  is supplied inelastically.
- Rather than a single good used for consumption and investment, now two sectors.



# The Two-Sector AK Model II

- Sector 1 produces consumption goods with the following technology

$$C(t) = B(K_C(t))^\alpha L_C(t)^{1-\alpha}, \quad (20)$$

- Cobb-Douglas assumption here is quite important in ensuring that the share of capital in national income is constant
- Capital accumulation equation:

$$\dot{K}(t) = I(t) - \delta K(t),$$

- $I(t)$  denotes investment. Investment goods are produced with a different technology,

$$I(t) = AK_I(t). \quad (21)$$

- Extreme version of an assumption often made in two-sector models: investment-good sector is more capital-intensive than the consumption-good sector.

# The Two-Sector AK Model III

- Market clearing implies:

$$K_C(t) + K_I(t) \leq K(t),$$

$$L_C(t) \leq L,$$

- An equilibrium is defined similarly, but also features an allocation decision of capital between the two sectors.
- Also, there will be a relative price between the two sectors which will adjust endogenously.
- Both market clearing conditions will hold as equalities, so letting  $\kappa(t)$  denote the share of capital used in the investment sector

$$K_C(t) = (1 - \kappa(t)) K(t) \text{ and } K_I(t) = \kappa(t) K(t).$$

- From profit maximization, the rate of return to capital has to be the same when it is employed in the two sectors.

# The Two-Sector AK Model IV

- Let the price of the investment good be denoted by  $p_I(t)$  and that of the consumption good by  $p_C(t)$ , then

$$p_I(t) A = p_C(t) \alpha B \left( \frac{L}{(1 - \kappa(t)) K(t)} \right)^{1-\alpha}. \quad (22)$$

- Define a steady-state (a balanced growth path) as an equilibrium path in which  $\kappa(t)$  is constant and equal to some  $\kappa \in [0, 1]$ .
- Moreover, choose the consumption good as the numeraire, so that  $p_C(t) = 1$  for all  $t$ .
- Then differentiating (22) implies that at the steady state:

$$\frac{\dot{p}_I(t)}{p_I(t)} = - (1 - \alpha) g_K, \quad (23)$$

- $g_K$  is the steady-state (BGP) growth rate of capital.

# The Two-Sector AK Model V

- Euler equation (4) still holds, but interest rate has to be for *consumption-denominated loans*,  $r_C(t)$ .
- I.e., the interest rate that measures how many units of consumption good an individual will receive tomorrow by giving up one unit of consumption today.
- Relative price of consumption goods and investment goods is changing over time, thus:
  - By giving up one dollar, the individual will buy  $1/p_I(t)$  units of capital goods.
  - This will have an instantaneous return of  $r_I(t)$ .
  - Individual will get back the one unit of capital, which has experienced a change in its price of  $\dot{p}_I(t)/p_I(t)$ .
  - Finally, he will have to buy consumption goods, whose prices changed by  $\dot{p}_C(t)/p_C(t)$ .

# The Two-Sector AK Model VI

- Therefore,

$$r_C(t) = \frac{r_I(t)}{p_I(t)} + \frac{\dot{p}_I(t)}{p_I(t)} - \frac{\dot{p}_C(t)}{p_C(t)}. \quad (24)$$

- Given our choice of numeraire, we have  $\dot{p}_C(t) / p_C(t) = 0$ .
- Moreover,  $\dot{p}_I(t) / p_I(t)$  is given by (23).
- Finally,

$$\frac{r_I(t)}{p_I(t)} = A - \delta$$

given the linear technology in (21).

- Therefore, we have

$$r_C(t) = A - \delta + \frac{\dot{p}_I(t)}{p_I(t)}.$$

# The Two-Sector AK Model VII

- In steady state, from (23):

$$r_C = A - \delta - (1 - \alpha) g_K.$$

- From (4), this implies a consumption growth rate of

$$g_C \equiv \frac{\dot{C}(t)}{C(t)} = \frac{1}{\theta} (A - \delta - (1 - \alpha) g_K - \rho). \quad (25)$$

- Finally, differentiate (20) and use the fact that labor is always constant to obtain

$$\frac{\dot{C}(t)}{C(t)} = \alpha \frac{\dot{K}_C(t)}{K_C(t)},$$

- From the constancy of  $\kappa(t)$  in steady state, implies the following steady-state relationship:

$$g_C = \alpha g_K.$$

# The Two-Sector AK Model VIII

- Substituting this into (25), we have

$$g_K^* = \frac{A - \delta - \rho}{1 - \alpha(1 - \theta)} \quad (26)$$

and

$$g_C^* = \alpha \frac{A - \delta - \rho}{1 - \alpha(1 - \theta)}. \quad (27)$$

- Because labor is being used in the consumption good sector, there will be positive wages.
- Since labor markets are competitive,

$$w(t) = (1 - \alpha) p_C(t) B \left( \frac{(1 - \kappa(t)) K(t)}{L} \right)^\alpha.$$

# The Two-Sector AK Model IX

- Therefore, in the balanced growth path,

$$\begin{aligned}\frac{\dot{w}(t)}{w(t)} &= \frac{\dot{p}_C(t)}{p_C(t)} + \alpha \frac{\dot{K}(t)}{K(t)} \\ &= \alpha g_K^*,\end{aligned}$$

- Thus wages also grow at the same rate as consumption.

**Proposition** In the above-described two-sector neoclassical economy, starting from any  $K(0) > 0$ , consumption and labor income grow at the constant rate given by (27), while the capital stock grows at the constant rate (26).

- Can do policy analysis as before



# The Two-Sector AK Model X

- Different from the neoclassical growth model, there is continuous *capital deepening*.
- Capital grows at a faster rate than consumption and output. Whether this is a realistic feature is debatable:
  - Kaldor facts include constant capital-output ratio as one of the requirements of balanced growth.
  - For much of the 20th century, capital-output ratio has been constant, but it has been increasing steadily over the past 30 years.
  - Part of the increase is because of relative price adjustments that have only been performed in the recent past.

# Growth with Externalities I

- Romer (1986): model the process of “knowledge accumulation”.
- Difficult in the context of a competitive economy.
- Solution: knowledge accumulation as a *byproduct* of capital accumulation.
- Technological spillovers: arguably crude, but captures that knowledge is a largely *non-rival* good.
- Non-rivalry does not imply knowledge is also non-excludable.
- But some of the important characteristics of “knowledge” and its role in the production process can be captured in a reduced-form way by introducing technological spillovers.

# Preferences and Technology I

- No population growth (we will see why this is important).
- Production function with labor-augmenting knowledge (technology) that satisfies Assumptions 1 and 2.
- Instead of working with the aggregate production function, assume that the production side of the economy consists of a set  $[0, 1]$  of firms.
- The production function facing each firm  $i \in [0, 1]$  is

$$Y_i(t) = F(K_i(t), A(t)L_i(t)), \quad (28)$$

- $K_i(t)$  and  $L_i(t)$  are capital and labor rented by a firm  $i$ .
- $A(t)$  is not indexed by  $i$ , since it is technology common to all firms.

## Preferences and Technology II

- Normalize the measure of final good producers to 1, so market clearing conditions:

$$\int_0^1 K_i(t) di = K(t)$$

and

$$\int_0^1 L_i(t) di = L,$$

- $L$  is the constant level of labor (supplied inelastically) in this economy.
- Firms are competitive in all markets, thus all hire the same capital to effective labor ratio, and

$$w(t) = \frac{\partial F(K(t), A(t)L)}{\partial L}$$
$$R(t) = \frac{\partial F(K(t), A(t)L)}{\partial K(t)}.$$

## Preferences and Technology III

- Key assumption: firms take  $A(t)$  as given, but this stock of technology (knowledge) advances endogenously for the economy as a whole.
- Lucas (1988) develops a similar model, but spillovers work through human capital.
- Extreme assumption of sufficiently strong externalities such that  $A(t)$  can grow continuously at the economy level. In particular,

$$A(t) = BK(t), \quad (29)$$

- Motivated by “learning-by-doing.” Alternatively, could be a function of the cumulative output that the economy has produced up to now.
- Substituting for (29) into (28) and using the fact that all firms are functioning at the same capital-effective labor ratio, production function of the representative firm:

$$Y(t) = F(K(t), BK(t)L).$$

## Preferences and Technology IV

- Using the fact that  $F(\cdot, \cdot)$  is homogeneous of degree 1, we have

$$\begin{aligned}\frac{Y(t)}{K(t)} &= F(1, BL) \\ &= \tilde{f}(L).\end{aligned}$$

- Output per capita can therefore be written as:

$$\begin{aligned}y(t) &\equiv \frac{Y(t)}{L} \\ &= \frac{Y(t)}{K(t)} \frac{K(t)}{L} \\ &= k(t) \tilde{f}(L),\end{aligned}$$

- Again  $k(t) \equiv K(t) / L$  is the capital-labor ratio in the economy.

# Preferences and Technology V

- Normalized production function, now  $\tilde{f}(L)$ .
- We have

$$w(t) = K(t) \tilde{f}'(L) \quad (30)$$

and

$$R(t) = R = \tilde{f}(L) - L\tilde{f}'(L), \quad (31)$$

which is constant.

# Equilibrium I

- An equilibrium is defined as a path  $[C(t), K(t)]_{t=0}^{\infty}$  that maximize the utility of the representative household and  $[w(t), R(t)]_{t=0}^{\infty}$  that clear markets.
- Important feature is that because the knowledge spillovers are external to the firm, factor prices are given by (30) and (31).
- I.e., they do not price the role of the capital stock in increasing future productivity.
- Since the market rate of return is  $r(t) = R(t) - \delta$ , it is also constant.
- Usual consumer Euler equation (e.g., (4) above) then implies that consumption must grow at the constant rate,

$$g_C^* = \frac{1}{\theta} (\tilde{f}(L) - L\tilde{f}'(L) - \delta - \rho). \quad (32)$$



## Equilibrium II

- Capital grows exactly at the same rate as consumption, so the rate of capital, output and consumption growth are all  $g_C^*$ .
- Assume that

$$\tilde{f}(L) - L\tilde{f}'(L) - \delta - \rho > 0, \quad (33)$$

so that there is positive growth.

- But also that growth is not fast enough to violate the transversality condition,

$$(1 - \theta) (\tilde{f}(L) - L\tilde{f}'(L) - \delta) < \rho. \quad (34)$$

**Proposition** Consider the above-described Romer model with physical capital externalities. Suppose that conditions (33) and (34) are satisfied. Then, there exists a unique equilibrium path where starting with any level of capital stock  $K(0) > 0$ , capital, output and consumption grow at the constant rate (32).

## Equilibrium III

- Population must be constant in this model because of the *scale effect*.
- Since  $\tilde{f}(L) - L\tilde{f}'(L)$  is always increasing in  $L$  (by Assumption 1), a higher population (labor force)  $L$  leads to a higher growth rate.
- The scale effect refers to this relationship between population and the equilibrium rate of economic growth.
- If population is growing, the economy will not admit a steady state and the growth rate of the economy will increase over time (output reaching infinity in finite time and violating the transversality condition).

# Pareto Optimal Allocations I

- Given externalities, not surprising that the decentralized equilibrium is not Pareto optimal.
- The per capita accumulation equation for this economy can be written as

$$\dot{k}(t) = \tilde{f}(L) k(t) - c(t) - \delta k(t).$$

- The current-value Hamiltonian to maximize utility of the representative household is

$$\hat{H}(k, c, \mu) = \frac{c(t)^{1-\theta} - 1}{1-\theta} + \mu [\tilde{f}(L) k(t) - c(t) - \delta k(t)],$$

# Pareto Optimal Allocations II

- Conditions for a candidate solution

$$\hat{H}_c(k, c, \mu) = c(t)^{-\theta} - \mu(t) = 0$$

$$\hat{H}_k(k, c, \mu) = \mu(t) [\tilde{f}(L) - \delta] = -\dot{\mu}(t) + \rho\mu(t),$$

$$\lim_{t \rightarrow \infty} [\exp(-\rho t) \mu(t) k(t)] = 0.$$

- $\hat{H}$  strictly concave, thus these conditions characterize unique solution.

## Pareto Optimal Allocations III

- Social planner's allocation will also have a constant growth rate for consumption (and output) given by

$$g_C^S = \frac{1}{\theta} (\tilde{f}(L) - \delta - \rho),$$

which is always greater than  $g_C^*$  as given by (32)—since  $\tilde{f}(L) > \tilde{f}(L) - L\tilde{f}'(L)$ .

- Social planner takes into account that by accumulating more capital, she is improving productivity in the future.

**Proposition** In the above-described Romer model with physical capital externalities, the decentralized equilibrium is Pareto suboptimal and grows at a slower rate than the allocation that would maximize the utility of the representative household.

# Conclusions I

- Linearity of the models (most clearly visible in the *AK* model):
  - Removes transitional dynamics and leads to a more tractable mathematical structure.
  - Essential feature of any model that will exhibit sustained economic growth.
  - With strong concavity, especially consistent with the Inada, sustained growth will not be possible.
- But most models studied in this chapter do not feature technological progress:
  - Debate about whether the observed total factor productivity growth is partly a result of mismeasurement of inputs.
  - Could be that much of what we measure as technological progress is in fact capital deepening, as in *AK* model and its variants.

# Conclusions II

- Important tension:
  - neoclassical growth model (or Solow growth model) have difficulty in generating very large income differences
  - models here suffer from the opposite problem.
  - Both a blessing and a curse: also predict an ever expanding world distribution.
- Issues to understand:
  - 1 Era of divergence is not the past 60 years, but the 19th century: important to confront these models with historical data.
  - 2 “Each country as an island” approach is unlikely to be a good approximation, much less so when we endogenize technology.