Externalities and Efficiency in a Model of Stochastic Job Matching

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1. Introduction

Labor is continuously reallocated across jobs in response to exogenous shocks and the process of learning about new opportunities and possible matches. In this reallocation process, the quality of matches in filled jobs naturally affects the opportunities which are worthwhile for a single worker. This paper explores the possibility of externalities inherent in this description of the labor reallocation process. We analyze equilibrium where all agents are risk neutral and contract to pay compensatory damages[1] in the event of a quit or a firing. We conclude that there is a tendency toward too few double breaches – a simultaneous quit and firing – to permit a particularly good match. The result follows from the fact that the value of externalities generated as a function of the quality of one's match is a convex transform of the value of holding a job with that quality match.

We begin by considering a model with just two types of matches, good and poor, where the probabilities of learning a job are the same whether unemployed, in a poor match, or in a good match. We assume that in equilibrium there are no quits for positions created by a firing, although such a rearrangement would just break even for the parties involved. We should that the present discounted value of aggregate output is raised by carrying out such a rearrangement. We then consider a model with a continuum of match qualities. In steady state equilibrium, we calculate the value function of holding a position with a given quantity match, which equals the present discounted value of expected earnings. We then calculate the value to a person in any given quality match of someone else's filling a vacancy with a particular value match. This externality value is shown to be an increasing convex transform of the value function of holding a match. Thus, whenever a rearrangement holds constant the sum of the value functions of the parties involved and involves a move to a better match, this rearrangement raises the externality value for every person in the economy. This argument is then extended to meeting probabilities that vary with the quality of an individual's position.

2. Basic Model

Consider an economy with equal numbers of jobs and workers. Workers and jobs search for each other. If a pair are well matched, they produce a flow of net output x_2, without any setup costs of delay in learning of the quality of the match. If they are not well matched, output is x_1. When a job and worker are paired, the probability of their matching poorly is p. We denote by $u, e_1,$ and e_2 the proportions of unemployed, poorly matched and well matched. All economic agents are assumed to be risk neutral and to share a common discount rate, r.

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[1] For analysis of compensatory damages, see Mortensen (1978) and Diamond and Maskin (1979).

Search is assumed to be costless, with the unemployed having no advantage over the employed. Each individual finds the matching process to be a Poisson process, with an arrival rate a. For any matching, the probabilities of the job being vacant, poorly filled, or well filled are $u, e_1,$ and e_2.

When an unemployed worker meets a vacancy, they begin production, whether they are well or poorly matched. When an unemployed worker meets a poorly filled match, it is to the combined advantage of the three parties to replace the poor match with a good one if that is the situation. The same conclusion follows when a poorly matched worker learns of a vacancy that makes a good match.

The remaining type of meeting with possible consequences occurs when a poorly matched worker learns of a poorly filled job which would be a good match and would also result in one vacancy and one unemployed worker. We shall restrict analysis to the case where this type of reorganization (one quit and one layoff) is not in the combined interest of the four parties involved, and is not carried out. In the next section we will derive the implication of the unprofitability of this rearrangement.

In addition to the endogenous terminations, it is assumed that any match is subject to exogenous termination, with a Poisson process breakup rate b.

With these assumptions we can write the differential equations for the changing allocation of workers among the three states. Unemployment goes down from matches between unemployed and vacancies and rises from exogenous terminations of existing matches. The numbers in poor matches rise from poor matches of unemployed and vacancies and fall from layoffs and quits because of good matches and from exogenous terminations. Good matches rise from the good matches of unemployed and vacancies and those arising from layoffs and quits for better matches. Good matches fall from exogenous breakup. Thus, we have the three differential equations

$$\begin{aligned} \dot{u} &= -au^2 + b\left(e_1 + e_2\right) \\ \dot{e}_1 &= pau^2 - 2a\left(1-p\right)ue_1 - be_1 \\ \dot{e}_2 &= \left(1-p\right)au^2 + 2a\left(1-p\right)ue_1 - be_2 \end{aligned} \tag{1}$$

In a steady state equilibrium we have the values

$$\begin{aligned} u &= \frac{\left(b^2 + 4ab\right)^{1/2} - b}{2a} \\ e_1 &= \frac{apu^2}{2a\left(1-p\right)u + b} \\ e_2 &= \left(1-p\right)ab^{-1}u\left(u + 2e_1\right) \end{aligned} \tag{2}$$

3. Individual Values

Individual shares of output are determined by two assumptions. In the event of a layoff or quit there is compensation which leaves the laid-off worker or quit job unaffected. That is, we assume compensatory damages. Parties starting production are assumed to share equally in the surplus from their finding each other by appropriately sharing the damage payment and then dividing output equally. With these assumptions we can calculate the expected present discounted value of earnings of workers in all three positions. We use the asset value equilibrium condition that the rate of interest times the value equal dividends plus expected capital gains. Thus, we have

$$\begin{aligned} rV_u &= 0 + au\left(pV_1 + (1-p)V_2 - V_u\right) + ae_1(1-p)(V_2 - V_1) \\ rV_1 &= \frac{1}{2}x_1 + au(1-p)(V_2 - V_1) - b(V_1 - V_u) \\ rV_2 &= \frac{1}{2}x_2 - b(V_2 - V_u) \end{aligned} \tag{3}$$

where the gain to an unemployed worker getting a good match after a firing is $V_2 - V_1$, equal to one half the increased value of positions of worker and firm, $V_2 - V_1 - V_u$, less compensatory damages, $V_1 - V_u$. A layoff and quit to generate a good match are not worthwhile if the combined value of positions decreases:

$$2V_1 \geq V_2 + V_u \tag{4}$$

Solving from the steady state value equations this inequality is equivalent to

$$\left(r + b + au + a(1-p)e_1\right)x_2 \leq \left(2r + 2b + a(2-p)u + a(1-p)e_1\right)x_1 \tag{5}$$

4. Aggregate Output

The present discounted value of aggregate output satisfies

$$W = \int_0^\infty e^{-rt}\left(x_1e_1(t) + x_2e_2(t)\right)dt \tag{6}$$

Aggregate value also satisfied the asset value equation. Let us write the aggregate value of output from initial position (u, e_1, e_2) and satisfying the differential equations (1) as $W(u, e_1, e_2)$. Then we have

$$rW = x_1e_1 + x_2e_2 + W_u\dot{u} + W_1\dot{e}_1 + W_2\dot{e}_2 \tag{7}$$

Since this equation holds for all values, we can differentiate it with respect to all three arguments.[2] Evaluating the derivatives at the steady state values where $u, e_1,$ and e_2 are all zero we have a considerable simplification:

$$\begin{aligned} rW_u &= W_u \frac{\partial \dot{u}}{\partial u} + W_1 \frac{\partial \dot{e}_1}{\partial u} + W_2 \frac{\partial \dot{e}_2}{\partial u} \\ &= -2auW_2 + \left(2aupu - 2a\left(1-p\right)e_1\right)W_1 \\ &\qquad + \left(a\left(1-p\right)u + 2a\left(1-p\right)e_1\right)W_2 \\ rW_1 &= x_1 + bW_u - \left(2a\left(1-p\right)u + b\right)W_1 + 2a\left(1-p\right)uW_2 \\ rW_2 &= x_2 = bW_u - bW_2 \end{aligned} \tag{8}$$

Starting at the steady state equilibrium we shall evaluate the effect on aggregate output of a simultaneous layoff and quit to accomplish a good match. This rearrangement adds an unemployed worker and a well matched worker and subtracts two poorly matched workers. The aggregate value of this rearrangement is $W_u + W_2 - 2W_1$. Solving from (8) we have

$$\begin{aligned} &\left(r+b+2au\right)\left(W_u + W_2 - 2W_1\right) = x_2 - 2x_1 + \left(W_2 - W_1\right)\left(2apu + 2a\left(1-p\right)e_1\right) \\ &= x_2 - 2x_1 + \left(x_2 - x_1\right)\frac{2apu + 2a\left(1-p\right)e_1}{r+b+2a\left(1-p\right)u} \end{aligned} \tag{9}$$

To see the possible social value of this rearrangement when it is privately unprofitable, we evaluate (8), where (5) holds with equality. Substituting for x_1, one can check that expression (8) is indeed positive.

Just because a single rearrangement of this type is worthwhile, it does not follow that it is always worthwhile to follow the path undertaking all such rearrangements that arise. That is, even with (5) satisfied with equality, $\left(x_1e_1 + x_2e_2\right)/r$, can exceed

$$\begin{aligned} &\int_0^\infty e^{-rt}\left(x_1e_1\left(t\right) + x_2e_2\left(t\right)\right)dt \\ &\dot{u} = -au^2 + b\left(e_1 + e_2\right) + a\left(1-p\right)e_1^2 \\ &\dot{e}_1 = pau^2 - 2a\left(1-p\right)ue_1 - be_1 - 2a\left(1-p\right)e_1^2 \\ &\dot{e}_2 = \left(1-p\right)au^2 = 2a\left(1-p\right)ue_1 - be_2 + a\left(1-p\right)e_1^2 \\ &u\left(0\right), e_1\left(0\right), \text{and } e_2\left(0\right) \text{ satisfy (2)} \end{aligned} \tag{10}$$

[2] This analysis follows that presented more fully in Diamond (1980).

For example, consider the parameters

$$a = 10^4, r = .1, b = 1, p = .97, x_2 / x_1 = 1.02286$$

Then, the initial conditions satisfy

$$e_1 = .0484834 \quad e_2 = .9483593$$

Remaining at equilibrium yields 1.018530 x_1 units of output, while following the alternative path yields 10.18461 x_1 units. It is easy to find examples where it is worthwhile to carry out all double breaches.

5. Continuum of Match Qualities

The fact that a layoff and quit for a good match which is just worthwhile privately generates a positive externality that carries over to more general models. Here we consider one change in the model above, that the distribution of match qualities in a continuum with distribution $F(x)$. (It is straightforward to add a uniform setup cost for production to this model.)

Let u be the proportion of unemployed and $G(y)$ the equilibrium distribution of existing matches. Let V_u and $V(x)$ be the private values of being unemployed or in a match producing output x. As before, an unemployed worker meeting a vacancy will always commence production. An employed worker finding a better match with a vacancy will always take it. An employed worked in a match producing x will quit an fill a position, formerly producing y and created by a layoff, if the new output level is at least as large as $z(x, y)$ defined implicitly by

$$V(z) + V_u - V(x) - V(y) = 0 \tag{11}$$

The rate of interest times the value of a position equals the dividend plus the expected capital gain.

$$\begin{aligned} rV_u &= au\int_0^\infty \left(V(z)-V_u\right)dF(z)+a\int_0^\infty \int_y^\infty \left(V(z)-V(y)\right)dF(z)\,dG(y) \\ rV(x) &= \frac{1}{2}x+au \int_x^\infty \left(V(z)-V(x)\right)dF(z) \\ &\quad +a\int_0^\infty \int_{z(x,y)}^\infty \left(V(z)+V_u-V(x)-V(y)\right)dF(z)\,dG(y) \\ &\quad -b\left(V(x)-V_u\right) \end{aligned} \tag{12}$$

We calculate the first two derivatives of $V(x)$

$$\begin{aligned} rV'(x) &= \frac{1}{2}-auV'(x)\left(1-F(x)\right)-aV'(x) \int_0^\infty \left(1-F\left(z(x,y)\right)\right)dG(y) \\ &\quad -bV'(x) \end{aligned}$$

or

$$\left[r+b+au\left(1-F(x)\right)+a\int_0^\infty \left(1-F\left(z(x,y)\right)\right)dG(y)\right]V'(x)=\frac{1}{2} \tag{13}$$

Noting that $\partial z/\partial x$ equals $V'(x)/V'(z)$ we have

$$V''(x)=-2\left(V'(x)\right)^2\left[-auF'(x)-\int_0^\infty F'\left(z(x,y)\right)V'(x)/V'\left(z(x,y)\right)dG(y)\right]>0 \tag{14}$$

Thus V is a convex function of x. If the only terminations were exogenous V would be linear in x.

$$(r+b)V(x)=\frac{1}{2}x+bV_u \tag{15}$$

The gains from endogenous terminations (i.e., from quits) are greater the poorer the match and so the greater the opportunity for improvement. Thus V asymptotes to a straight line from above.

The second step in this analysis is to consider the external diseconomy generated by filling a job with a particular quality match rather than remaining vacant. Let $E(x,y)$ be the externality value to a worker in a match of quality x from the filling of a vacant job with a match of quality y. The external diseconomy is the probability of meeting, a, times the difference in surplus from this meeting rather than one with a vacancy.

$$E(x,y) = -a'\int_{z(x,y)}^{\infty}\left(V(z)+V_u - V(x) - V(y)\right)dF(z) + a\int_{x}^{\infty}\left(V(z)-V(x)\right)dF(z) \qquad (16)$$

Differentiating twice with respect to y we have

$$\begin{aligned} E_y(x,y) &= a'V'(y)\left(1-F\left(z(x,y)\right)\right) \\ E_{yy}(x,y) &= a'V''(y)\left(1-F\left(z(x,y)\right)\right) - aV'(y)F'\left(z(x,y)\right)\frac{V'(y)}{V'\left(z(x,y)\right)} \end{aligned} \qquad (17)$$

Thus for each x we have

$$\frac{E_{yy}(x,y)}{E_y(x,y)} = \frac{V''(y)}{V'(y)} - \frac{F'\left(z(x,y)\right)V'(y)}{\left(1-F\left(z(x,y)\right)\right)V'\left(z(x,y)\right)} \qquad (18)$$

That is, for each x, $E(x,y)$ is an increasing convex transform of $V(y)$. Recognizing that V_u is the same as $V(0)$ and $E(x,0)$ is zero, we have

$$E\left(x,z(s,t)\right) - E(x,s) - E(x,t) < 0 \qquad (19)$$

That is, a rearrangement through a quit and a layoff which is just worthwhile privately generates a positive externality to every other person in the economy.

6. Varying Meeting Probabilities

In the models above, the flow probability of any meeting is a. This misses the fact that the unemployed may be easier to meet. More generally, assume that the flow probability of a meeting between a worker in a job match of quality x and a position in a job match of quality y is a symmetric function $a(x,y)$ with $a_y \le 0,\ a_{yy} \ge 0$. Then, the argument of the previous section goes through. To see this we note that (16) becomes

$$
\begin{aligned}
E(x,y) = & -a(x,y)\int_{z(x,y)}^{\infty}\left(V(z)+V_u-V(x)-V(y)\right)dF(z) \\
& +a(x,0)\int_{x}^{\infty}\left(V(z)-V(x)\right)dF(z)
\end{aligned}
\tag{20}
$$

Differentiating twice we have

$$
E_y = a(x,y)V'(y)\left(1-F\left(z(x,y)\right)\right)-a_y(x,y)\int_{z(x,y)}^{\infty}\left(V(z)+V_u-V(x)-V(y)\right)dF(z) \tag{21}
$$

$$
\begin{aligned}
E_{yy} = & \, a(x,y)V''(y)\left(1-F\left(z(x,y)\right)\right) \\
& +2a_y(x,y)V'(y)\left(1-F\left(z(x,y)\right)\right)-a_{yy}\int_{z(x,y)}^{\infty}\left(V(z)+V_u-V(x)-V(y)\right)dF \\
& -a(x,y)V'(y)F'\left(z(x,y)\right)\frac{V'(y)}{V'\left(z(x,y)\right)}
\end{aligned}
\tag{22}
$$

Thus it remains true that E_{yy}/E_y is less than V''/V' since

$$
\begin{aligned}
& E_y > aV'(1-F) \\
& E_{yy} < aV''(1-F).
\end{aligned}
\tag{23}
$$

References

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