Interaction Games: A Unified Analysis of Incomplete Information, Local Interaction and Random Matching*

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Abstract

Incomplete information, local interaction and random matching games all share a common structure. A type or player interacts with various subsets of the set of all types/players. A type/player's total payoff is additive in the payoffs from these various interactions. This paper describes a general class of interaction games and shows how each of these three types of games can be understood as special cases. Techniques and results from the incomplete information literature are translated into this more general framework; as a by-product, it is possible to give a complete characterization of equilibria robust to incomplete information (in the sense of Kajii and Morris [1995]) in many player binary action co-ordination games. Only equilibria that are robust in this sense [1] can spread contagiously and [2] are uninvadable under best response dynamics in a local interaction system. A companion paper, Morris [1997], uses these techniques to characterize features of local interaction systems that allow contagion.

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1 Introduction

This paper introduces and analyses a class of *interaction games*. A finite or infinite population interacts strategically. But each player's payoff depends on the population strategy profile in a special way. Each player is involved in a number of *interactions,* consisting of subsets of players. He must choose the same action in each of his interactions. He receives a payoff from each interaction that does not depend on the actions of players not in the interaction. Each interaction has a weight. An equilibrium of an interaction game is a profile of possibly mixed strategies for each player such that each player maximizes the weighted sum of his payoffs from each interaction.

Two restrictions allow an interaction game to have an *incomplete information* interpretation: *N-partite interaction* requires that the players can be partitioned into N groups, such that each interaction consists of exactly one player from each group; *bounded interactions* requires that the weights of the interactions add up to 1. Now consider the N player incomplete information game where each of the N groups represent the set of types of one "big player". Interactions then correspond to type profiles, or states, while the "weight" on an interaction corresponds the probability of the type profile. The definition of equilibrium for general interaction games corresponds to the standard definition of (Bayesian Nash) equilibrium of the incomplete information game. Any incomplete information game can be interpreted as an interaction game in this way.

To interpret an interaction game as a *random matching* game, drop the N-partite interaction assumption but assume bounded interactions and two more restrictions: *player independent payoffs* requires that any player's payoff from an interaction depends only on his action and the actions of others in the group (and not on the identity of the player or interaction); *binary interaction* requires that each positive weight interaction consists of exactly two players. Now interpret an interaction as a match of two players and the weight of an interaction as the probability of that match. Players must choose actions without knowing which match is chosen. Again, the definition of equilibrium for general interaction games corresponds to the standard definition of equilibrium for random matching games and any random matching game can be interpreted as an interaction game in this way. Indeed, only the bounded interactions assumption is necessary to interpret an interaction game as a random matching game. One can easily have many player matches where payoffs depend on the identities of all players in the match.

Finally, to interpret an interaction game as a deterministic *local interaction* game, maintain the player independent payoffs and binary interaction assumptions but replace bounded interactions with the following: *constant weights* requires that each interaction receives a weight of either 0 or 1. Now two players are "neighbours" if the interaction consisting of those two players has weight 1; they are not neighbours if that interaction has weight 0. Again, equilibrium notions coincide. Deterministic local interaction is the broadest interpretation of interaction games since no restriction is necessary for the interpretation: one can have an unbounded number of many player interactions where identities matter and different interactions have different deterministic weights.

These equivalences are more than just a curiosity. By understanding the common structure of interaction games, we understand each of these classes of games better. For example, it transpires that what matters in the analysis of incomplete information games is the additive separability of payoffs across interactions; the fact that types of one player do not interact with other types of the same player is irrelevant for most purposes.

The equivalences are used in this paper to translate and extend earlier results from the incomplete information literature. Kajii and Morris [1995] analysed which equilibria of a complete information game were "robust to incomplete information": a complete information equilibrium is robust if behaviour close to it is played in some equilibrium of every incomplete information game where payoffs are almost always given by the complete information game. Translated into the context of interaction games, a complete information equilibrium is robust if behaviour close to it is an equilibrium of every interaction game where most players in most interactions have payoffs given by that complete information game. Kajii and Morris [1995] provided different sufficient conditions for robustness but none were very powerful in the case of many player games. As a by-product of our analysis of interaction games, we are able to derive a sufficient condition that has significant bite in many player games.

This new result allows a complete characterization of robustness in binary action co-ordination (BC) games. A BC game is a symmetric game where each player must choose one of two actions and the incentive to choose each action is increasing in the number of opponents who choose that action. An action is a "uniform best response" if it is a best response to a conjecture that puts equal probability of each possible number of opponents choosing that action. In a generic BC game, exactly one action is a uniform best response. We show that the unique uniform best response is the unique robust action (in the sense described above). It is also the unique action that might spread contagiously under deterministic or a class of stochastic best response dynamics. Finally, it is the unique action that is itself uninvadable under all such best response dynamics in all local interaction systems.

The results in this paper build heavily on earlier research. Monderer and Samet [1989] introduced techniques (using "belief operators") for analysing higher order beliefs (players' beliefs about other players' beliefs, etc...) in incomplete information games. The basic techniques and results in this paper exploit translations of belief operator techniques, and results proved using them, to general interaction games. In incomplete information games, higher order beliefs are important exactly when players' types are highly *correlated* and belief operators are most useful in such situations. The interaction game viewpoint makes clear that this feature corresponds to highly *local* interaction and highly *non-uniform* random matching. It is thus in these environments that the techniques described are most useful.

Mailath, Samuelson and Shaked [1997] showed that the set of probability distributions over action profiles generated by equilibria of random matching games equals the set of correlated equilibria of the underlying game. This argument (summarized in section 3.3.1) implicitly exploits the equivalence between incomplete information and local interaction / random matching games. The purpose of this paper is to make the equivalence explicit in a more general class of games, but also to develop a unified approach to analysing interaction games. Many results are translations of known results into this general setting, in particular building on Morris, Rob and Shin [1995] and Kajii and Morris [1995]. In a companion piece, Morris [1997], these techniques are applied to give new characterizations of which features of a local interaction system allow behaviour to spread contagiously. That paper also contains a discussion of existing results in the local interaction literature and their (sometimes close) connection to the approach described here.

The paper is organized as follows. In section 2, I describe an "investment example" that has incomplete information, local interaction and random matching interpretations. With the incomplete information interpretation, the example is close to the "electronic mail game" of Rubinstein [1989] that is the canonical example illustrating how higher order beliefs can allow small probability events to have high probability impacts in incomplete information games. With the local interaction interpretation, the example is close to the interaction on a line analysis of Ellison [1993] that is the canonical example of how the behaviour of a small number of players can be bootstrapped to influence the behaviour of all players in a local interaction system. This example goes a long way to providing a feel for the equivalence.

The general class of interaction games is described in section 3, together with the restrictions necessary to support the various interpretations. Four examples in section 3.2 serve two purposes. They all clarify the role of the various restrictions introduced. The latter two examples illustrate the usefulness of a more general perspective on incomplete information results: a standard no trade result and the convention game of Shin and Williamson [1996] are translated from an incomplete information interpretation into a more general interaction game set up.

The unified approach to analysing interaction games is presented in section 4.1; section 4.2 contains equilibrium results; section 4.3 contains results on best response dynamics.

2 Leading Example

Two players (*ROW* and *COL*) must choose action "Invest" (I) or action "Don't Invest" (D). Each player faces a cost 2 of investing. Each player realizes a gross return of 3 from the investment if both (1) the other player invests and (2) investment conditions are *favorable* for that player. Thus if investment conditions are favorable for both players, then payoffs are given by the following symmetric matrix:

Favorable for *ROW* Favorable for *COL*	I	D
I	1,1	-2,0
D	0,-2	0,0

.

This game has two strict Nash equilibria: both players invest and both players don't invest. On the other hand, if conditions are unfavorable for *ROW* (but favorable for player *COL*), payoffs are given by the following matrix:

Favorable for *ROW* Unfavorable for *COL*	I	D
I	-2,1	-2,0
D	0,-2	0,0

.

In this game, *ROW* has a dominant strategy to not invest, and thus the unique Nash equilibrium has both players not investing.

2.1 Incomplete Information

Now allow a small amount of incomplete information about investment conditions. In particular, investment conditions are always favorable for *COL*, but not for *ROW*. *ROW* knows when investment conditions are favorable for him, but *COL* does not.

Specifically, suppose that *ROW* observes a signal $s_R \in \{0, ..., K-1\}$ which is drawn from a uniform distribution. Assume that investment conditions are favorable for *ROW* *unless* $s_R = 0$. *COL* observes a noisy version of *ROW*'s signal, $s_C \in \{0, ..., K-1\}$. In particular, assume that

$$s_C = \begin{cases} s_R, \text{ with probability } 1/2 \\ s_R - 1, \text{ with probability } 1/2 \end{cases},$$

with mod K arithmetic, so that $0-1 = K-1$. Thus if $s_R = 0$, s_C is 0 or $K-1$ with equal likelihood.

The above constitutes a description of an incomplete information game. We can summarize the game in the following diagram:

Type of *COL*

Type of *ROW*

	0	1	2		K-1	
0	×	○	○	·	×	U
1	×	×	○	·	○	F
2	○	×	×	·	○	F
	·	·	·	·	·	·
K-1	○	○	○	·	×	F
	F	F	F	·	F	

.

Types of ROW are represented by rows, types of COL by columns. Boxes with a $\times$ correspond to type profiles which occur with positive probability; given the uniform prior assumption, each occurs with ex ante probability $\frac{1}{2K}$. Boxes with a $\circ$ correspond to type profiles that occur with zero ex ante probability. Payoffs are specified by the letter - F for favorable, U for unfavorable - at the end of the row/column corresponding to the type.

The unique equilibrium of this incomplete information game has each player never investing. To see why, observe first that type 0 of ROW will not invest in any equilibrium. But type 0 of COL attaches probability 1/2 to ROW being of type 0, and therefore not investing. But even if investment conditions are favorable, the best response of a player who believes that his opponent will invest with probability less than or equal to a half is not to invest. Thus type 0 of COL will not invest. But now consider type 1 of ROW. Although investment conditions are favorable, he attaches probability 1/2 to his opponent not investing; so he will not invest. This argument iterates to ensure that no one will invest.

This example is an elaboration of an example of Rubinstein [1989]; this version follows the leading example of Morris, Rob and Shin [1995]. It illustrates the fact that, in order for investment to be an equilibrium outcome, it is not enough that investment conditions are favorable for both players with high probability; nor is it enough that everyone know that everyone know... up to an arbitrary number of levels... that investment conditions are favorable for both players.

2.2 Local Interaction

Now suppose that there are $2K$ players situated on a circle. Player k interacts with his two neighbours, $k-1$ and $k+1$. We use mod $2K$ arithmetic, so that player $2K$'s neighbours are $2K-1$ and 1. Conditions are favorable for all players except the player at location 1. It is common knowledge for whom investment conditions are favorable.

Each player must decide whether to invest or not. His payoff is the sum of his payoff from his two interactions with each of his two neighbours. A strategy profile specifies which players invest, and which do not. A strategy profile is an equilibrium strategy profile if each player's action is a best response given the behaviour of his two neighbours.

This local interaction game can be summarized by the following table:

	2	4	6		2K	
1	×	∘	∘	·	×	U
3	×	×	∘	·	∘	F
5	∘	×	×	·	∘	F
	·	·	·	·	·	·
2K-1	∘	∘	∘	·	×	F
	F	F	F	·	F	

A cross (×) marks a pair of players who interact with each other. Thus, for example, player 3 interacts with players 2 and 4 and no other player.

The unique equilibrium of this game has all players never investing. The argument is as for the incomplete information game. We know that the player at location 1 will never invest. Consider the player at location 2. Since one of his neighbours is not investing, his best response is not to invest. Similarly, the player at location 3 does not invest, and the argument iterates to ensure the result. This iterated deletion of dominated strategies argument is closely related to the best response dynamics on a line argument of Ellison [1993] (the relation is discussed in section 4.3).

The above table is constructed in such a way as to identify an exact relationship between the incomplete information game and the local interaction game. In particular, the odd numbered players in the local interaction game play the role of *ROW*'s types in the incomplete information game, while the even numbered players play the role of *COL*'s types.

2.3 Random Matching

The local interaction game can be easily interpreted as an environment with non-uniform random matching. Suppose in each period, two players are randomly drawn out of a population of $2K$ to play the investment game. The two players are not randomly chosen: players are labelled 1 through $2K$ and only players with consecutive labels may be chosen. Players must decide on an action before knowing who they are matched against. Investment conditions are favorable for all players except player 1.

3 Interaction Games

Fix a finite or countably infinite population of players, $\mathcal{X}$. A standard strategic form game among these players is described by a set of actions for each player, $\{A_x\}_{x\in\mathcal{X}}$, and payoff functions for each player, $\{v_x\}_{x\in\mathcal{X}}$, where each $v_x : \underset{x\in\mathcal{X}}{\times} A_x \to \Re$. Thus the game is described by 3-tuple $\left(\mathcal{X}, \{A_x\}_{x\in\mathcal{X}}, \{v_x\}_{x\in\mathcal{X}}\right)$. A (simple) mixed strategy for player x is a (finite support) probability distribution $\boldsymbol{\alpha}_x \in \Delta(A_x)$. A mixed strategy profile is a vector $\boldsymbol{\alpha} \equiv \{\boldsymbol{\alpha}_x\}_{x\in\mathcal{X}}$. For notational convenience, I want to work with a constant set of actions A (so that $A_x = A$ for all $x \in \mathcal{X}$); we can always re-label actions so that the action set is constant.

This paper is concerned with games with a special form of payoffs. Write $\mathcal{I}$ for the collection of subsets of $\mathcal{X}$ with at least two elements; an element $X \in \mathcal{I}$ will be called an *interaction*. Write $\mathcal{I}(x)$ for the collection of such interactions involving player x, i.e.,

$$\mathcal{I}(x) = \{X \in \mathcal{I} : x \in X\}.$$

Let $P : \mathcal{I} \to \Re_+$, where for all $x \in \mathcal{X}$,

$$0 < \sum_{X \in \mathcal{I}(x)} P(X) < \infty. \tag{1}$$

Write $\mathbf{a}_X = (\mathbf{a}_x)_{x \in X}$ for a typical element of A^X. Now for each $x \in \mathcal{X}$, let $u_x(\mathbf{a}_X, X)$ be the payoff that player x gets from interaction $X \in \mathcal{I}(x)$ if players in X choose according to $\mathbf{a}_X$. Assume that payoffs are bounded, i.e., for each $x \in \mathcal{X}$, there exists M such that $|u_x(\mathbf{a}_X, X)| \leq M$ for all $X \in \mathcal{X}$ and $\mathbf{a}_X \in A^X$. This assumption ensures that total payoffs are well defined:

$$v_x(\mathbf{a}) = \sum_{X \in \mathcal{I}(x)} P(X).u_x(\mathbf{a}_X, X).$$

In this paper, we will be studying *interaction games* of the above form, described by the 4-tuple $\left(\mathcal{X}, P, A, \{u_x\}_{x \in \mathcal{X}}\right)$. Payoff functions can be extended to mixed strategies in the usual way; thus, for any $\boldsymbol{\alpha} \in [\Delta(A)]^{\mathcal{X}}$,

$$\begin{aligned} u_x(\boldsymbol{\alpha}_X, X) &= \sum_{\mathbf{a}_X \in A^X} \left(\prod_{y \in X} \boldsymbol{\alpha}_y(\mathbf{a}_y) \right) u_x(\mathbf{a}_X, X) \\ \text{and } v_x(\boldsymbol{\alpha}) &= \sum_{X \in \mathcal{I}(x)} P(X).u_x(\boldsymbol{\alpha}_X, X). \end{aligned}$$

Definition 1 *Strategy profile $\boldsymbol{\alpha}^* \in [\Delta(A)]^{\mathcal{X}}$ is a (Nash) equilibrium of $\left(\mathcal{X}, P, A, \{u_x\}_{x \in \mathcal{X}}\right)$ if for all $x \in \mathcal{X}$ and all $\alpha \in \Delta(A)$:*

$$v_x\left(\boldsymbol{\alpha}_x^*, \boldsymbol{\alpha}_{-x}^*\right) \geq v_x\left(\alpha, \boldsymbol{\alpha}_{-x}^*\right).$$

The degenerate interaction game with $P(X) = 0$ for all $X \neq \mathcal{X}$ can capture any form of strategic interaction. But this formulation is of interest when $\mathcal{X}$ is large and $P(X) > 0$ only for small X. We will outline a number of alternative interpretations of interaction games below, each of which relies on extra restrictions on the game $\left(\mathcal{X}, P, A, \{u_x\}_{x \in \mathcal{X}}\right)$.

3.1 Interpretations

3.1.1 Incomplete Information

For an incomplete information interpretation, we require first that for some $N \geq 2$, only interactions with N members have positive weight. Writing $\mathcal{I}_N$ for the set of interactions with N elements, we have:

P1 (N-ary Interaction): If $P(X) > 0$, then $X \in \mathcal{I}_N$.

In the special case where $N = 2$, we refer to *binary interaction*. But we will also require the stronger property that the players can be divided into N groups such that each positive weight interaction involves exactly one player from each of the groups.

P1* (N-partite Interaction): There exists a partition of $\mathcal{X}$ into N disjoint subsets $(\mathcal{X}_1, .., \mathcal{X}_N)$ such that if $P(X) > 0$, X consists of exactly one element of each of $\mathcal{X}_1, .., \mathcal{X}_N$.

In the special case where $N = 2$, we refer to *bipartite interaction*. Note that N-partite interaction (**P1***) implies N-ary interaction (**P1**).

Second, the sum of the interaction weights over the whole system is bounded. Without loss of generality, we can assume the sum is equal to one.

P2 (Bounded Interactions): $\sum\limits_{X \in \mathcal{I}} P(X) = 1$.

Now $\left(\mathcal{X}, A, P, \{u_x\}_{x \in \mathcal{X}}\right)$ can be interpreted as an incomplete information game, where there are N "big players", $\{1, ..., N\}$, A is the action set of each player n and $\mathcal{X}_n$ is the set of types of big player n; now writing $\mathcal{I}_N^*$ for the set of interactions consisting of exactly one element of each $\mathcal{X}_n$, each element of $\mathcal{I}_N^*$ corresponds to a type profile in $\mathcal{X}_1 \times .. \times \mathcal{X}_N$, i.e., the cross product of player types, or state space; P is the probability distribution over type profiles. Note that payoffs depend on the type profile (state) X. Now a strategy profile $\boldsymbol{\alpha}$ can be thought of as a collection of mixed strategies for each big player, $\boldsymbol{\alpha} \equiv \{\boldsymbol{\alpha}_n\}_{n=1}^N$, where each $\boldsymbol{\alpha}_n \equiv \{\boldsymbol{\alpha}_x\}_{x \in \mathcal{X}_n}$ describes the behaviour of each type of big player n. The definition of Nash equilibrium for general interaction games given above corresponds to an *interim* definition of Bayesian Nash equilibrium. But this is equivalent to the standard ex ante definition.[1] N-partite interaction (**P1**) and bounded interactions (**P2**) are both necessary for the interpretation of the interaction game as a standard game of incomplete information.[2]

[1]Harsanyi [1967, page 177] described an interim interpretation of incomplete information games (he attributes it to Selten) where each type is treated as a separate player.

[2]Bounded interactions (**P2**) is necessary for a standard *ex ante* interpretation of incomplete information games. But with an interim interpretation, no inconsistency arises if we allow for improper priors (Hartigan [1983]) with infinite mass. Note that equation (1) is a maintained restriction on P that implies that conditional probabilities are always well defined.

3.1.2 Random Matching

If $\mathbf{a} \in A^K$ is a K-vector of actions, write $\widetilde{\pi}[\mathbf{a}] \in \Delta(A)$ for the frequencies of actions in that action profile, i.e.,

$$\widetilde{\pi}[\mathbf{a}](a) = \frac{\#\{k \in \{1,..,K\} : \mathbf{a}_k = a\}}{K}$$

for each $a \in A$. Now suppose that N players are matched together to play a game. Each player cares only about the frequency of actions of his $N-1$ opponents (not who takes which action). Thus if a player chooses action $a \in A$ and his $N-1$ opponents choose action profile $\mathbf{a} \in A^{N-1}$, his payoff is $g(a, \widetilde{\pi}[\mathbf{a}])$. A function $g : A \times \Delta(A) \to \Re$ is a *symmetric payoff function.* For any $N \geq 2$, write $[g, N]$ for the symmetric N-player game where the nth player's payoff from action profile $\{\mathbf{a}_m\}_{m=1}^N$ is $g\left(\mathbf{a}_n, \widetilde{\pi}\left[\{\mathbf{a}_m\}_{m\neq n}\right]\right)$.

Write X/x for the group consisting of all members of X except x. Requiring that each player's payoff from each interaction is given by some symmetric payoff function gives us:

P3 (Symmetric Payoffs): For each $x \in \mathcal{X}$ and $X \in \mathcal{I}(x)$, there is a symmetric payoff function g such that $u_x(\mathbf{a}_X, X) = g\left(\mathbf{a}_x, \widetilde{\pi}\left[\mathbf{a}_{X\setminus x}\right]\right)$ for all $\mathbf{a}_X \in A^X$.

Note that the symmetric payoffs assumption is empty for those X with $\#X = 2$. Requiring in addition that each player's payoff function does not depend on which interaction he is involved in gives us:

P3*(Interaction Ind't Payoffs): For each $x \in \mathcal{X}$, there is a symmetric payoff function g such that $u_x(\mathbf{a}_X, X) = g\left(\mathbf{a}_x, \widetilde{\pi}\left[\mathbf{a}_{X\setminus x}\right]\right)$ for all $X \in \mathcal{I}(x)$ and $\mathbf{a}_X \in A^X$.

Finally, requiring also that payoff functions do not depend on the identity of the player gives us:

P3 (Player Ind't Payoffs):** There is a symmetric payoff function g such that $u_x(\mathbf{a}_X, X) = g\left(\mathbf{a}_x, \widetilde{\pi}\left[\mathbf{a}_{X\setminus x}\right]\right)$ for all $x \in \mathcal{X}$, $X \subseteq \mathcal{I}(x)$ and $\mathbf{a}_X \in A^X$.

The most standard one population model of random matching assumes binary interaction (**P1**, with $N = 2$), bounded interactions (**P2**) and player ind't payoffs (**P3****). Now $\mathcal{X}$ is a collection of players. Each (positive probability) match consists of two players. Thus $\mathcal{I}$ is the set of possible matches and P is a probability distributions over matches. Payoffs are independent of all features of the match. An equilibrium has the following interpretation. Each player picks a possibly mixed strategy. He does not know with whom he will interact. His mixed strategy is a best response to the expected distribution over actions.

Only the bounded interactions assumption (**P2**) is *necessary* for this interpretation. Matches may consist of more than two players. Payoffs may be different for each player and may depend on who they interact with.

3.1.3 Local Interaction

A standard model of local interaction considers a *graph* $(\mathcal{X}, \sim)$, where $\mathcal{X}$ is the set of players (or "locations") and $\sim$ is an irreflexive symmetric relation; player x is a "neighbour" of player y if $x \sim y$. Players must choose the same action against each neighbour, all players have the same payoff function from all interactions, and their total payoff is the sum of their payoffs from each neighbour.

This model corresponds in this framework to assuming binary interaction (**P1**, with $N = 2$), player ind't payoffs (**P3****) and

P4 (Constant Weights): $P(X) \in \{0, c\}$ for all $X \in \mathcal{I}$ for some $c > 0$.

Now x and y are neighbours exactly if $P(\{x, y\}) = c$. An equilibrium has the following interpretation. Each player picks a possibly mixed strategy. His mixed strategy maximizes the sum of his payoffs from all interactions, given the strategies of others.

The local interaction interpretation is the most general, in the sense that no restriction is necessary for the interpretation. We can allow an unbounded quantity of interactions involving many players with varying payoffs that depend on the interactions and the opponents' identities. We can drop the constant weights assumption. If $P(X) > 0$, we would say that the group X interacts and $P(X)$ measures the importance of that interaction.

3.2 Examples

Four examples will illustrate the general structure of interaction games. The investment example (section 3.2.1) and co-ordination on a lattice example (section 3.2.2) illustrate the various properties that we have introduced in the alternative interpretations. No trade (section 3.2.3) and convention (section 3.2.4) examples illustrate how results that hold for incomplete information games generalize to interaction games.

3.2.1 Investment Game

The following is a formal description of the example of section 2. Let $\mathcal{X} = \{1, .., 2K\}$; $A = \{I, D\}$;

$$P(X) = \begin{cases} \frac{1}{2K}, \text{ if } X = \{x, y\} \text{ and either } |x - y| = 1 \text{ or } \{x, y\} = \{1, 2K\} \\ 0, \text{ otherwise} \end{cases};$$

$$u_1(\mathbf{a}_X, X) = \begin{cases} -2, \text{ if } \mathbf{a}_1 = I \\ 0, \text{ if } \mathbf{a}_1 = D \end{cases}$$

and if $x \neq 1$, then

$$u_x(\mathbf{a}_X, X) = \begin{cases} 1, \text{ if } \mathbf{a}_y = I \text{ for all } y \in X \\ -2, \text{ if } \mathbf{a}_x = I \text{ and } \mathbf{a}_y = D \text{ for some } y \in X \\ 0, \text{ if } \mathbf{a}_x = D \end{cases}.$$

- This game satisfies bipartite interaction (**P1***, with $N = 2$), bounded interactions (**P2**), interaction ind't payoffs (**P3***), constant weights (**P4**), but not player ind't payoffs (**P3****). To check for bipartite interaction, let $\mathcal{X}_1 = \{x : x \text{ is odd}\}$ and $\mathcal{X}_2 = \{x : x \text{ is even}\}$.

- The argument given in section 2 showed unique equilibrium $\boldsymbol{\alpha}^*$ has $\boldsymbol{\alpha}_x^*(D) = 1$ for all $x \in \mathcal{X}$. This is also the unique strategy profile satisfying iterated deletion of strictly dominated strategies (we provide a formal definition for this in the next section).

3.2.2 Co-ordination on a Lattice

Versions of this example have been studied in the local interaction literature (Blume [1995], Ellison [1994], Anderlini and Ianni [1996]). Suppose that the set of players consists of all points on a two dimensional lattice, each player interacts with his nearest neighbours and each player's payoffs from each interaction are given by the symmetric matrix

	I	D
I	$1,1$	$0,0$
D	$0,0$	$2,2$

.

This game may be formally represented at follows. Writing $\mathcal{Z}$ for the set of integers, $\mathcal{X} = \mathcal{Z}^2$; $A = \{I, D\}$;

$$P(X) = \begin{cases} 1, \text{ if } X = \{x, y\} \text{ and } |x_1 - y_1| + |x_2 - y_2| = 1 \\ 0, \text{ otherwise} \end{cases};$$

$$\text{and } u_x(\mathbf{a}_X, X) = \begin{cases} 1, \text{ if } \mathbf{a}_y = I \text{ for all } y \in X \\ 2, \text{ if } \mathbf{a}_y = D \text{ for all } y \in X \\ 0, \text{ otherwise} \end{cases}.$$

- This game satisfies bipartite interaction (**P1***, with $N = 2$), player ind't payoffs (**P3****), constant weights (**P4**), but not bounded interactions (**P2**). To check for bipartite interaction, let $\mathcal{X}_1 = \{x : x_1 + x_2 \text{ is odd}\}$ and $\mathcal{X}_2 = \{x : x_1 + x_2 \text{ is even}\}$.

- There are many equilibria (see Blume [1995] for a characterization). For example, $\boldsymbol{\alpha}^*$ is an equilibrium where $\boldsymbol{\alpha}_x^*(I) = 1$ if $x_1 \geq k$ and $\boldsymbol{\alpha}_x^*(D) = 1$ if $x_1 < k$, for some integer k.

3.2.3 No Trade Theorem

The standard no trade theorem for incomplete information games states that if there are no ex ante gains from trade, no trade will take place in any trading game where players always have the option of not trading. As a number of researchers have noted, this result remains true if there are no *interim* gains from trade (a weaker assumption, and thus a stronger result). One special case where there are no interim gains from trade is when (i) there are no *ex post* gains from trade; (ii) players are risk neutral; and (iii) players share a common prior. This result has a natural analogue in all interaction games.

Let $\mathcal{X}$ be finite and $A = \{I, D\}$. For each $X \in \mathcal{I}$, let $f_X : X \to \Re$ satisfy $\sum\limits_{x \in X} f_X(x) \leq 0$. Let

$$u_x(\mathbf{a}_X, X) = \begin{cases} f_X(x) - \varepsilon, \text{ if } \mathbf{a}_y = I \text{ for } \textit{all } y \in X \\ 0, \text{ if } \mathbf{a}_y = D \text{ for } \textit{some } y \in X \end{cases}$$

where $\varepsilon > 0$. The interpretation is that player x must decide whether to participate (I) or not (D). If he participates, he pays a transaction cost ε. Each interaction in which he participates is zero sum.

- This game satisfies bounded interactions (**P2**) and symmetric payoffs (**P3**) but for non-trivial functions f_X will fail interaction ind't payoffs (**P3***). It may or may not satisfy N-ary interaction (**P1**) or constant weights (**P4**).

Let $\boldsymbol{\alpha}^*$ be any equilibrium and let $\beta^*(X)$ be the corresponding probability that all players participate in interaction X, i.e., $\beta^*(X) = \prod\limits_{x \in X} \boldsymbol{\alpha}_x^*(I)$. Now player x's payoff is $u_x^* = \sum\limits_{X \in \mathcal{I}(x)} P(X)\beta^*(X)(f_X(x) - \varepsilon) \geq 0$ (since he can guarantee himself 0 by choosing D). So

$$\begin{aligned} 0 &\leq \sum_{x \in \mathcal{X}} u_x^* \\ &= \sum_{x \in \mathcal{X}} \sum_{X \in \mathcal{I}(x)} P(X)\beta^*(X)(f_X(x) - \varepsilon) \\ &= \sum_{X \in \mathcal{I}} \sum_{x \in X} P(X)\beta^*(X)(f_X(x) - \varepsilon) \\ &\leq -\varepsilon \sum_{X \in \mathcal{I}} \#X.P(X).\beta^*(X). \end{aligned}$$

Thus $P(X) > 0 \Rightarrow \beta^*(X) = 0 \Rightarrow \boldsymbol{\alpha}_x^*(D) = 1$ for some $x \in X$. In other words:

- In every positive probability interaction, at least one player chooses D.

In the incomplete information interpretation, the common prior assumption plays a crucial role in this result (it ensures that the ex post zero sum property implies no interim gains from trade). The analogous property in interaction games (built into this formulation) is that each player uses the same interaction weights.

3.2.4 Conventions

Shin and Williamson [1996] described and analysed (a more general version of) the following game (with an incomplete information interpretation). Let $\mathcal{X}$ be finite, $A = [0,1]$ and

$$u_x(\mathbf{a}_X, X) = \widetilde{u}_x\left(\mathbf{a}_{X\setminus x}, X\right) - \varepsilon\left(\mathbf{a}_x - \frac{1}{\#X-1}\sum_{y\in X\setminus x}\mathbf{a}_y\right)^2$$

for some $\varepsilon > 0$. Thus player x's payoff from interaction X is additively separable in two components. The first component, $\widetilde{u}_x\left(\mathbf{a}_{X\setminus x}, X\right)$, does not depend on player x's action. The second component is a quadratic loss function proportional to the squared distance between player x's action and the weighted average of the actions of others in the interaction.

- This game satisfies bounded interactions (**P2**); it may or may not satisfy N-ary interaction (**P1**), symmetric payoffs (**P3**) and constant weights (**P4**).

Each player's best response is always to choose an action that is a weighted average of actions chosen by the other players in the interactions he is a member of. Thus this is a convention game where each player wants to mimic those he interacts with. Thus for any $\psi \in [0,1]$, there is an equilibrium where $\boldsymbol{\alpha}_x(\psi) = 1$ for all $x \in \mathcal{X}$. More surprisingly, if every player is linked, directly or indirectly, to every other player, *all* equilibria take this form. More precisely, this is true if the following property is satisfied.

P5 (Connectedness): For all $x, y \in \mathcal{X}$, there exists a sequence of interactions $X_1, ..., X_K$ such that $x \in X_1$; $y \in X_K$; $P(X_k) > 0$ for all $k = 1,..,K$; and $X_k \cap X_{k+1} \neq \emptyset$ for all $k = 1,..,K-1$.

The argument is straightforward. Let $\overline{\psi}$ be largest action played with positive probability by any player (say it is player x). Since each player's action is a strictly convex combination of the actions played by all players he interacts with, we have $\boldsymbol{\alpha}_y\left(\overline{\psi}\right) = 1$ for all $y \in \bigcup_{\{X\in\mathcal{I}\{x\}:P(X)>0\}} X$. Iterating this argument, connectedness ensures that $\boldsymbol{\alpha}_y\left(\overline{\psi}\right) = 1$ for all $y \in \mathcal{X}$.

It might be highly inefficient to have all players choose a constant action, i.e., if ε is very small and $\widetilde{u}_x$ depends non-trivially on $\mathbf{a}_{X\setminus x}$.[3]

[3] Morris [1997] contains positive results on the co-existence of conventions with *discrete* actions. See also Sugden [1995] and Young [1996].

3.3 Related Literature and Further Solution Concepts

3.3.1 Role Dependent Payoffs, Player Independent Payoffs and Correlated Equilibria

N-partite interaction ($\mathbf{P1}^*$) was the defining characteristic of an incomplete information game. But it also has a natural interpretation in a local interaction / random matching setting: each player has a role and each interaction consists of exactly one player in each of N roles. Under this interpretation, it is natural to consider settings where a player's payoff depends on his role, but nothing else. This restriction can be described formally as follows. Again write $\mathcal{I}_N^*$ for the set of interactions with exactly one player from each of the N roles, and $\mathcal{I}_N^*(x) \equiv \mathcal{I}(x) \cap \mathcal{I}_N^*$. Any $X \in \mathcal{I}_N^*$ can be written as $X = \{\nu(n, X)\}_{n=1}^N$, where $\nu(n, X)$ is the unique (by N-partite interaction) element of $X \cap \mathcal{X}_n$. An N-player game (not necessarily symmetric) is parameterized by payoff functions $\{g_n\}_{n=1}^N$, with each $g_n : A^N \to \Re$.

P3a (Role Dependent Payoffs): There is an N-player game $\{g_n\}_{n=1}^N$, such that $u_x(\mathbf{a}_X, X) = g_n\left(\left(\mathbf{a}_{\nu(m,X)}\right)_{m=1}^N\right)$ for all $x \in \mathcal{X}_n$, $X \in \mathcal{I}_N^*(x)$ and $\mathbf{a}_X \in A^X$.[4]

Mailath, Samuelson and Shaked [1997] studied interaction games (with a random matching interpretation) satisfying N-partite interaction (**P1**), bounded interactions (**P2**) and role dependent payoffs (**P3a**). They showed the following. Let $\mu \in \Delta(A^N)$ be the probability distribution over action profiles generated by some equilibrium $\boldsymbol{\alpha}$ of an interaction game, i.e.,

$$\mu(\mathbf{a}) = \sum_{X \in \mathcal{I}} P(X) \left(\prod_{n=1}^{N} \alpha_{\nu(n,X)}\left(\mathbf{a}_{\nu(n,X)}\right) \right).$$

This probability distribution μ is a correlated equilibrium of the N-player game parameterized by $\{g_n\}_{n=1}^N$. Under the incomplete information interpretation, the role dependent payoffs assumption (**P3a**) is equivalent to assuming that each big player's payoffs are independent of his type. Thus the above result is equivalent to Aumann's [1987] classic characterization of correlated equilibrium.[5]

A related result holds in the case where N-partite interaction ($\mathbf{P1}^*$) is weakened to N-ary interaction (**P1**), although in this case it is necessary to have player ind't payoffs ($\mathbf{P3}^{**}$) (see Mailath, Samuelson and Shaked [1997] and

[4] Player ind't payoffs ($\mathbf{P3}^{**}$) implies role dependent payoffs (**P3a**), but role dependent payoffs need not imply even symmetric payoffs (**P3**).

[5] The common prior assumption was necessary for Aumann's [1987] characterization. Dropping the common prior assumption, his assumptions imply only that each player x (with $x \in \mathcal{X}_n$) chooses an action that survives iterated deletion of strictly dominated strategies (for player n in the N-player game $\{g_n\}_{n=1}^N$). The same conclusion would follow if we relaxed the assumption (in interaction games) that players use the same weights in calculating payoffs.

Ianni [1996]). Since player ind't payoffs is satisfied, assume that payoffs of all players are given by symmetric payoff function g. For some equilibrium $\boldsymbol{\alpha}$ of an interaction game, we can calculate the probability distribution over *unordered* profiles of actions. We can then construct a probability distribution $\mu \in \Delta\left(A^N\right)$ over *ordered* action profiles by assuming that any ordering is equally likely. This probability distribution is a symmetric correlated equilibrium of the symmetric N-player game $[g, N]$. The formal construction is

$$\mu(\mathbf{a}) = \frac{1}{\#\left\{\mathbf{a}' \in A^N : \widetilde{\pi}(\mathbf{a}') = \widetilde{\pi}(\mathbf{a})\right\}} \sum_{X \in \mathcal{I}} \sum_{\left\{\mathbf{a}' \in A^N : \widetilde{\pi}(\mathbf{a}') = \widetilde{\pi}(\mathbf{a})\right\}} P(X) \left(\prod_{x \in X} \boldsymbol{\alpha}_x\left(\mathbf{a}'_x\right)\right)$$

(note that $\widetilde{\pi}(\mathbf{a}') = \widetilde{\pi}(\mathbf{a})$ exactly if $\mathbf{a}'$ and $\mathbf{a}$ represent the same collection of actions - possibly in a different order).

3.3.2 Iterated Deletion of Strictly Dominated Strategies

The natural definitions of equilibrium in incomplete information games, random matching games and local interaction games all correspond to the natural definition of equilibrium in the general interaction games. However, other solution concepts do not translate quite as straightforwardly. Consider the following definition of iterated deletion of strictly dominated strategies for interaction games.

Definition 2 *Define $\left\{\mathcal{U}_x^k\right\}_{x \in \mathcal{X}}$, each $\mathcal{U}_x^k \subseteq A$, iteratively as follows: $\mathcal{U}_x^0 = A$;*

$$\mathcal{A}^k = \left\{\boldsymbol{\alpha} \in [\Delta(A)]^{\mathcal{X}} : \boldsymbol{\alpha}_x(a) = 0 \text{ if } a \notin \mathcal{U}_x^k\right\};$$

$$\mathcal{U}_x^{k+1} = \left\{a \in \mathcal{U}_x^k : v_x\left(a, \boldsymbol{\alpha}_{-x}\right) \geq v_x\left(a', \boldsymbol{\alpha}_{-x}\right) \text{ for all } a' \in A, \text{ for some } \boldsymbol{\alpha} \in \mathcal{A}^k\right\}.$$

Action a survives iterated deletion of strictly dominated strategies for player x if $a \in \mathcal{U}_x^\infty \equiv \bigcap_{k \geq 1} \mathcal{U}_x^k$.

This definition corresponds to iterated deletion of strictly *interim* dominated strategies in an incomplete information game [Fudenberg and Tirole 1991, p. 226].

4 A Unified Analysis of Interaction Games

Some tools for analyzing interaction systems $(\mathcal{X}, P)$ are introduced in section 4.1; these tools are applied to characterizing equilibrium behaviour and best response dynamics in interaction games in sections 4.2 and 4.3 respectively.

4.1 The Structure of Interaction

Throughout section 4.1, we assume N-ary interaction (**P1**).

4.1.1 Neighbourhood Operators and Cohesion

Let $Y \subseteq \mathcal{X}$ be a group of players. We are interested in the set of players who interact mostly with players within group Y. In particular, write $\pi_k(Y|x)$ for the proportion of player x's interactions that involve exactly k neighbours within Y (and thus $N-k-1$ neighbours outside Y); i.e.,

$$\pi_k(Y|x) \geq \frac{\sum_{\{X \in \mathcal{I}_N(x): \# X \cap Y = k+1\}} P(X)}{\sum_{X \in \mathcal{I}_N(x)} P(X)}.$$

Intuitively, x interacts more with group Y if $\pi_k(Y|x)$ is large for large k and small for small k. We want a one dimensional measure of how much x interacts with group Y, so we will aggregate the π_k using different weights: let Γ_N be the set of $\gamma \equiv (\gamma_0, ..., \gamma_{N-1}) \in \Re^N$ with

$$\gamma_0 \leq \gamma_1 \leq .. \leq \gamma_{N-1}. \tag{2}$$

Now for any $\gamma \in \Gamma_N$, let $B^\gamma(Y)$ be the set of players within Y for whom the γ-weighted proportion of interactions involving players in Y is non-negative, i.e.,

$$B^\gamma(Y) \equiv \left\{ x \in Y : \sum_{k=0}^{N-1} \gamma_k \pi_k(Y|x) \geq 0 \right\}.$$

The operators B^γ are referred to as neighbourhood operators. For different values of γ, such operators have quite different interpretations. Consider a few special cases.

- Let $\gamma = (-p, ... - p, 1-p)$, i.e., $\gamma_k = \begin{cases} -p, \text{ if } k < N-1 \\ 1-p, \text{ if } k = N-1 \end{cases}$. In this case, $B^\gamma(Y)$ is the set of players in Y for whom at least proportion p of interactions involve *exclusively* players in Y. This case will be important later, and we will write $\overline{\gamma}^p \equiv (-p, ... - p, 1-p)$.

- More generally, let $\gamma = \left(\overbrace{-p, .., -p}^{m \text{ times}}, \overbrace{1-p, .., 1-p}^{N-m \text{ times}} \right)$, i.e., $\gamma_k = \begin{cases} -p, \text{ if } k < m \\ 1-p, \text{ if } k \geq m \end{cases}$ for some $m \in \{1, ..., N-1\}$. In this case, $B^\gamma(Y)$ is the set of players in Y for whom at least proportion p of interactions involve at least m players in Y.

- Let $\gamma = (-p(N-1), 1-p(N-1), 2-p(N-1), .., N-1-p(N-1))$, i.e., $\gamma_k = k - p(N-1)$. In this case, $B^{\gamma}(Y)$ is the set of players in Y for whom the *average* proportion of interacting players from Y is at least p.

By definition, $B^{\gamma}(Y) \subseteq Y$. Group Y is γ-*cohesive* if each member of Y has non-negative γ-weighted proportion of his interactions within Y, i.e., $Y \subseteq B^{\gamma}(Y)$. Iterating the operator, we have:

$$C^{\gamma}(Y) = \underset{k\geq 1}{\cap} [B^{\gamma}]^k (Y).$$

It is straightforward to show that if $\gamma \in \Gamma_N$, B^{γ} satisfies the following two properties:

$$\text{if } Y \subseteq Y', \text{ then } B^{\gamma}(Y) \subseteq B^{\gamma}(Y'); \qquad \text{(monotonicity)}$$

$$\text{if } Y_{k+1} \subseteq Y_k \text{ for all } k, \text{ then } \underset{k\geq 1}{\cap} B^{\gamma}(Y_k) \subseteq B^{\gamma}\left(\underset{k\geq 1}{\cap} Y_k\right). \qquad \text{(continuity)}$$

The following result is a consequence of these two properties.

Proposition 1 *For all groups Y: (1) $C^{\gamma}(Y)$ is γ-cohesive; (2) If Y' is γ-cohesive and $Y' \subseteq Y$, then $Y' \subseteq C^{\gamma}(Y)$; (3) $x \in C^{\gamma}(Y)$ if and only if there exists a γ-cohesive group Y' such that (i) $x \in Y'$ and (ii) $Y' \subseteq Y$.*

Proof. $C^{\gamma}(Y) \equiv \underset{k\geq 1}{\cap} [B^{\gamma}]^k (Y) \subseteq \underset{k\geq 2}{\cap} [B^{\gamma}]^k (Y) \subseteq B^{\gamma}\left(\underset{k\geq 1}{\cap} [B^{\gamma}]^k (Y)\right) = B^{\gamma}(C^{\gamma}(Y))$, by continuity, so (1) $C^{\gamma}(Y)$ is γ-cohesive. Now for all $Y' \subseteq Y$, $[B^{\gamma}]^k (Y') \subseteq [B^{\gamma}]^k (Y)$ for all $k \geq 1$, by iterated application of monotonicity; thus $C^{\gamma}(Y') \subseteq C^{\gamma}(Y)$. If in addition Y' in γ-cohesive, then $Y' = C^{\gamma}(Y') \subseteq C^{\gamma}(Y)$, proving (2). For the "only if" part of (3), set $Y' = C^{\gamma}(Y)$. For the "if" part of (2), we have $x \in Y'$ by (i), $Y' = C^{\gamma}(Y')$ by assumption that Y' is γ-cohesive and $C^{\gamma}(Y') \subseteq C^{\gamma}(Y)$ by (ii) and part (2). So $x \in Y' = C^{\gamma}(Y') \subseteq C^{\gamma}(Y)$. ■

4.1.2 Interpretation of Neighbourhood Operators and Cohesion

Incomplete Information Assume N-partite interaction (**P1***) and bounded interactions (**P2**). Thus under the incomplete information interpretation, $\mathcal{X}_n$ is the set of types of big player n. For any $X \in \mathcal{I}_N^*$, write $\nu(n, X)$ for the unique element of $X \cap \mathcal{X}_n$: $\nu(n, X)$ is the type of big player n if the state (i.e., the interaction) is X; and $\mathcal{I}_N^*(x) = \mathcal{I}_N^* \cap \mathcal{I}(x)$. For arbitrary *events* $E \subseteq \mathcal{I}$, define $\widetilde{B}_n^p(E)$ to be the set of states where player n believes event E with probability at least p. Thus $\widetilde{B}_n^p : 2^{\mathcal{I}} \to 2^{\mathcal{I}}$ is defined by

$$\widetilde{B}_n^p(E) \equiv \left\{ X \in \mathcal{I} : \frac{\sum_{\left\{X' \in \mathcal{I}_N^*(\nu(n,X)) : X' \in E\right\}} P(X')}{\sum_{X' \in \mathcal{I}_N^*(\nu(n,X))} P(X')} \right\}.$$

Monderer and Samet [1989] introduced the operator "everyone believes event E with probability at least p". Thus let $\widetilde{B}_*^p(E)$ be the set of states where *all* big players believe event E with probability at least p, i.e., $\widetilde{B}_*^p(E) \equiv \bigcap_{n=1}^{N} \widetilde{B}_n^p(E)$. This belief operator $\widetilde{B}_*^p$ is closely related to the neighbourhood operator B^γ in the special case where $\gamma = (-p, .., -p, 1-p) = \overline{\gamma}^p$. Under the incomplete information interpretation, a group Y is a collection of types. We can associate with each collection of types an event $\widetilde{E}(Y) \equiv \{X \in \mathcal{I}_N^* : X \subseteq Y\}$. Now it is true by definition that

$$\begin{aligned}
\widetilde{E}\left(B^{\overline{\gamma}^p}(Y)\right) &= \left\{X \in \mathcal{I}_N^* : X \subseteq B^{\overline{\gamma}^p}(Y)\right\} \\
&= \left\{X \in \mathcal{I}_N^* : \frac{\sum\limits_{\left\{X' \in \mathcal{I}_N^*(x) : X' \subseteq Y\right\}} P(X')}{\sum\limits_{X' \in \mathcal{I}_N^*(x)} P(X')} \geq p \text{ for all } x \in X\right\} \\
&= \bigcap_{n=1}^{N} \left\{X \in \mathcal{I}_N^* : \frac{\sum\limits_{\left\{X' \in \mathcal{I}_N^*(\nu(n,X)) : X' \subseteq Y\right\}} P(X')}{\sum\limits_{X' \in \mathcal{I}_N^*(\nu(n,X))} P(X')} \geq p\right\} \\
&= \bigcap_{n=1}^{N} \left\{X \in \mathcal{I}_N^* : \frac{\sum\limits_{\left\{X' \in \mathcal{I}_N^*(\nu(n,X)) : X' \in \widetilde{E}(Y)\right\}} P(X')}{\sum\limits_{X' \in \mathcal{I}_N^*(\nu(n,X))} P(X')} \geq p\right\} \\
&= \bigcap_{n=1}^{N} \widetilde{B}_n^p\left(\widetilde{E}(Y)\right) \\
&= \widetilde{B}_*^p\left(\widetilde{E}(Y)\right).
\end{aligned}$$

Thus neighbourhood operators $B^{\overline{\gamma}^p}$ can be thought of as belief operators restricted to *simple* events that have the form $\widetilde{E}(Y)$ (for the game theory applications that we will discuss in the next section, these are *exactly* the events we are interested in). Proposition 1 is thus a simple corollary of Proposition 3 of Monderer and Samet [1989].[6] In the language of Monderer and Samet [1989], an event $\widetilde{E}(Y)$ is "evident p-belief" if and only if the group Y is $\overline{\gamma}^p$-cohesive; and $\widetilde{E}\left(C^{\overline{\gamma}^p}(Y)\right)$ is the set of states where the event $\widetilde{E}(Y)$ is "common p-belief."[7]

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[6] In fact, the restriction to simple events simplifies the argument: $\widetilde{B}_*^p$ is monotonic when restricted to simple events, but not otherwise.

[7] When γ is not in the simple form $\overline{\gamma}^p$, $B^\gamma(Y)$ will still have a natural interpretation. For example, if $\gamma_k = k - p(N-1)$, then Y is γ-cohesive only if at each state in $\widetilde{E}(Y)$ each player's expected proportion of players who think $\widetilde{E}(Y)$ possible is at least p.

Local Interaction With binary local interaction, the neighbourhood operator B^{γ} with $\gamma = \overline{\gamma}^p = (-p, 1-p)$ is especially relevant to the existing literature. Under the local interaction interpretation, a group Y is $\overline{\gamma}^p$-cohesive if at least proportion p of the interactions of each member involve only members of that group. The local interaction interpretation of cohesion is discussed extensively in a companion piece, Morris [1997]. That paper explores a simple form of local interaction described by a graph $(\mathcal{X}, \sim)$, where $\sim$ is a symmetric and irreflexive relation. Two players x and y are said to be neighbours if $x \sim y$. This corresponds (in the language of this paper) to the case of binary interaction (**P1**, with $N = 2$) and constant weights (**P4**), i.e., $P(X) = \begin{cases} 1, \text{ if } X = \{x, y\} \text{ and } x \sim y \\ 0, \text{ otherwise} \end{cases}$. In this simple setting, it was natural to consider an operator defined by:

$$\Pi^p(Y) = \left\{ x \in \mathcal{X} : \frac{\#\{y \in Y : y \sim x\}}{\#\{y \in \mathcal{X} : y \sim x\}} \geq p \right\}.$$

This operator is related to the proportion operator of this paper as follows:

$$\begin{aligned} B^{\overline{\gamma}^p}(Y) &= \left\{ x \in \mathcal{X} : \frac{\sum\limits_{\{\{x,y\} \in \mathcal{I}_2(x) : \{x,y\} \subseteq Y\}} P(\{x,y\})}{\sum\limits_{\{x,y\} \in \mathcal{I}_2(x)} P(\{x,y\})} \geq p \right\} \\ &= \begin{cases} \left\{ x \in \mathcal{X} : \frac{\#\{y \in Y : y \sim x\}}{\#\{y \in \mathcal{X} : y \sim x\}} \geq p \right\}, \text{ if } x \in Y \\ \emptyset, \text{ if } x \notin Y \end{cases} \\ &= Y \cap \Pi^p(Y). \end{aligned}$$

Random Matching Under the random matching interpretation, group Y is $\overline{\gamma}^p$-cohesive if each member of Y attaches probability at least p to any interaction he is in involving only members of group Y.

4.1.3 The Size of Cohesive Groups

It will be useful to know something about the relation between the size of group Y and the size of the group $C^{\gamma}(Y)$. Some more notation is required before proving a result on this subject. Write $\overline{Y}$ for the complement of Y in $\mathcal{X}$. Say that group Y is *finite* if

$$\sum_{\{X \in \mathcal{I} : X \cap Y \neq \emptyset\}} P(X) < \infty.$$

A sufficient condition for Y to be finite is that Y contains a finite number of players. But an infinite number of players may constitute a finite group if the sum of the weights of the interactions involving that infinite set of players is a convergent sequence. In particular, if bounded interactions (**P2**) is satisfied, all groups are finite. Now if Y is finite, write $P^*[Y]$ for the total weight of interactions involving some players in Y, i.e.,

$$P^*[Y] = \sum_{\{X \in \mathcal{I}: X \cap Y \neq \emptyset\}} P(X).$$

Finally, let $\Gamma_N^+ = \left\{\gamma \in \Gamma_N : \sum_{k=1}^{N-1} \gamma_k > 0\right\}$.

Proposition 2 *If Y is finite and $\gamma \in \Gamma_N^+$, then*

$$P^*\left[\overline{Y} \cap \overline{C^\gamma\left(\overline{Y}\right)}\right] \leq \left(\frac{\sum_{k=0}^{N-1} |\gamma_k|}{\sum_{k=0}^{N-1} \gamma_k}\right) N^2 P^*[Y].$$

Proof. Step 1. We introduce some notation. Let Y be finite, $Z = \overline{Y} \cap \overline{C^\gamma\left(\overline{Y}\right)}$ and $Z_j = \overline{[B^\gamma]^j\left(\overline{Y}\right)} \cap [B^\gamma]^{j-1}\left(\overline{Y}\right)$. Thus the collection of groups $\{Z_j\}_{j=1}^\infty$ partitions Z. So for each $x \in Z$, let $j(x)$ be the unique j such that $x \in Z_j$; let $\succ$ be any complete ordering on Z with $j(x) < j(y) \Rightarrow x \prec y$ and a minimal $x_0 \in Z$ with $x_0 \prec x$ for all $x \in Z \backslash \{x_0\}$. For each $x \in Z$, write $Z^+(x) = C^\gamma\left(\overline{Y}\right) \cup \{z \in Z : z \succ x\}$ and $Z^-(x) = Y \cup \{z \in Z : z \prec x\}$. Note that $\left\{[B^\gamma]^j\left(\overline{Y}\right)\right\}_{j=0}^\infty$ is a decreasing sequence of events and that $x \succ z$ implies that x survives longer in that sequence. Let $\zeta_k(x) = \pi_k(Z^-(x))$. Intuitively, $\zeta_k(x)$ is the proportion of x's interactions where k of x's partners in the interaction survived a smaller number of iterations. Finally, let $c = \frac{1}{N} \sum_{k=0}^{N-1} \gamma_k > 0$ and $\gamma_k' = \gamma_k - c$. By construction, $\sum_{k=0}^{N-1} \gamma_k' = 0$.

Step 2. We show the following fact; for all $x \in Z$,

$$\sum_{k=0}^{N-1} \gamma_{N-1-k}' \zeta_k(x) < -c. \tag{3}$$

For each $x \in Z_j$, $x \notin [B^\gamma]^j\left(\overline{Y}\right)$. Since $Z^+(x) \subseteq [B^\gamma]^{j-1}\left(\overline{Y}\right)$, we also have $x \notin B^\gamma(Z^+(x))$, i.e., $\sum_{k=0}^{N-1} \gamma_k \pi_k(Z^+(x)) < 0$. But $\pi_k(Z^+(x)) = \pi_{N-1-k}(Z^-(x)) = \zeta_{N-1-k}(x)$, so $\sum_{k=0}^{N-1} \gamma_k \zeta_{N-1-k}(x) < 0$. Now

$$\begin{aligned} \sum_{k=0}^{N-1} \gamma_{N-1-k}' \zeta_k(x) &= \sum_{k=0}^{N-1} \gamma_{N-1-k} \zeta_k(x) - c \\ &< -c. \end{aligned}$$

Step 3. We introduce more notation. Let

$$f_k(x) = \sum_{\{X \in \mathcal{I}_N : \# X \cap Z^-(x) = k\}} P(X);$$

$f_k(x)$ is the weight of interactions involving exactly k players lower ranked under $\prec$. Also write

$$\Pi(x) = \sum_{X \in \mathcal{I}(x)} P(X).$$

Observe that by construction

$$\begin{aligned}
f_0(x) &= f_0(x_0) - \sum_{\{z \in Z : z \prec x\}} \Pi(z) \zeta_0(z); \\
f_k(x) &= f_k(x_0) + \sum_{\{z \in Z : z \prec x\}} \Pi(z) \left(\zeta_{k-1}(z) - \zeta_k(z)\right), \text{ if } k = 1, ..., N-1; \\
f_N(x) &= f_N(x_0) + \sum_{\{z \in Z : z \prec x\}} \Pi(z) \zeta_{N-1}(z).
\end{aligned}$$

Step 4. Now the proof is completed by showing that since an appropriately weighted sum of the $f_k(x)$ cannot become unboundedly large, we can bound $P^*[Z]$. So let $\xi_1 = \gamma'_{N-1}$, $\xi_2 = \gamma'_{N-2} + \gamma'_{N-1}$, etc..., i.e., for each $k = 1, ..., N$, let $\xi_k = \sum_{j=0}^{k-1} \gamma'_{N-1-j}$. Note that $\xi_k \geq 0$ for each k, and $\xi_N = 0$. Now

$$\begin{aligned}
\sum_{x \in Z} \sum_{k=1}^{N} \xi_k f_k(x) &= \sum_{k=1}^{N} \xi_k f_k(x_0) + \xi_1 \sum_{x \in Z} \Pi(x) \zeta_0(x) + \sum_{x \in Z} \sum_{k=1}^{N-1} (\xi_k - \xi_{k+1}) \Pi(x) \zeta_k(x) \\
&= \sum_{k=1}^{N} \xi_k f_k(x_0) + \sum_{x \in Z} \Pi(x) \sum_{k=0}^{N-1} \gamma'_{N-1-k} \zeta_k(x) \qquad (4) \\
&\leq \sum_{k=1}^{N} \xi_k f_k(x_0) - c \sum_{x \in Z} \Pi(x), \text{ by (3).}
\end{aligned}$$

But $\xi_k \leq \sum_{j=1}^{N-1} |\gamma_j|$ for all k; so

$$\begin{aligned}
\sum_{k=1}^{N} \xi_k f_k(x_0) &\leq \left(\sum_{j=1}^{N-1} |\gamma_j| \right) \sum_{k=1}^{N} f_k(x_0) \\
&\leq \left(\sum_{j=1}^{N-1} |\gamma_j| \right) \sum_{x \in Y} \Pi(x)
\end{aligned}$$

$$\leq \left(\sum_{j=1}^{N-1} |\gamma_j| \right) NP^* [Y],$$

while $\sum_{x \in Z} \Pi(x) \geq P^* [Z]$. Since the left hand side of equation (4) is non-negative by construction, we have $cP^* [Z] \leq \left(\sum_{j=1}^{N-1} |\gamma_j| \right) NP^* [Y]$. Substituting for c gives the expression in the Proposition. ■

In the case where $\gamma = \overline{\gamma}^p$, this result is essentially a special case of Proposition 4.2 of Kajii and Morris [1995] (although there is a tighter bound in that paper).

4.2 Equilibrium

This section reports generalizations of (incomplete information) results in Morris, Rob and Shin [1995] and Kajii and Morris [1995] applied to general interaction games. Many proofs are abbreviated, where the arguments are essentially unchanged.

4.2.1 Existence

P6 (Finite Action Set): A is a finite set.

P7 (Finite Interactions): $P(X) > 0 \Rightarrow X$ is finite.

Remark 1 *If interaction game* $\left(\mathcal{X}, P, A, \{u_x\}_{x \in \mathcal{X}}\right)$ *satisfies* **P6** *and* **P7***, then there exists an equilibrium.*[8]

Throughout section 4.2, these two properties (**P6** and **P7**) are assumed, as are N-ary interaction (**P1**) and interaction ind't payoffs (**P3***). The latter two assumptions are inessential: more complicated results could be proved without them. Under assumption (**P3***), we can write $u_x(a, \pi)$ for player x's payoff from any interaction in which he chooses action a, his opponents choose $\mathbf{a}_{X \setminus x}$ and $\pi = \widehat{\pi}\left(\mathbf{a}_{X \setminus x}\right)$.

[8]Existence fails in the following example satisfying **P6** but not **P7**. Let $\mathcal{X} = \mathcal{Z}$; $A = \{I, D\}$; $P(\mathcal{X}) = 1$ and $P(X) = 0$ for all $X \neq \mathcal{X}$; and

$$u_x(\mathbf{a}, \mathcal{X}) = \begin{cases} 1, \text{ if } \mathbf{a}_x = I \text{ and } \mathbf{a}_y = D \text{ for all } y > x \\ -1, \text{ if } \mathbf{a}_x = I \text{ and } \mathbf{a}_y = I \text{ for some } y > x \\ 0, \text{ if } \mathbf{a}_x = D \end{cases}.$$

4.2.2 Binary Action Co-ordination Games

The equilibrium characterizations that follow will have their strongest bite in a class of binary action co-ordination games; we introduce this class here in order to motivate the later analysis.

A binary action co-ordination (BC) game is defined as follows. Each player has two possible actions, i.e., A has two elements. The game is symmetric, so each player's payoff depends only on the proportion of his neighbours taking each of the two actions. Thus $g(a, \pi)$ is a player's payoff if he chooses action a, proportion $\pi(a)$ of his neighbours choose action a and therefore proportion $\pi(a') = 1 - \pi(a)$ choose the other action a'. Write $\rho(a, q)$ for the gain to a player from choosing action a (rather than the other action a') if proportion q of his neighbours choose action a; thus $\rho(a, q) = g(a, \pi) - g(a', \pi)$, where $\pi(a) = q$. (Note that by construction, $\rho(a, q) = -\rho(a', 1 - q)$). The game is said to be a BC game if $\rho(a, q)$ is increasing in q for both a. A BC game is completely characterized by the reduced form payoff function ρ and the number of players N.

Definition 3 *Action a is a uniform best response of the BC game $[\rho, N]$ if*

$$\sum_{k=1}^{N-1} \frac{1}{N}\rho\left(a, \frac{k}{N-1}\right) > 0.$$

Thus action a is a uniform best response if it is the unique best response when a player puts a uniform prior on the number of his opponents choosing action a. Note that for a generic choice of ρ, exactly one of the two actions will be uniform best response (since if $a' \neq a$, $\sum_{k=1}^{N-1} \frac{1}{N}\rho\left(a, \frac{k}{N-1}\right) = -\sum_{k=1}^{N-1} \frac{1}{N}\rho\left(a', \frac{N-1-k}{N-1}\right) = -\sum_{k=1}^{N-1} \frac{1}{N}\rho\left(a', \frac{k}{N-1}\right)$).

In the special case of two players, the uniform best response is the risk dominant action in the sense of Harsanyi and Selten [1988]. The importance of the uniform best response action was highlighted by Carlsson and van Damme [1993b] and Kim [1996]. As Carlsson and van Damme [1993b] and Kim [1996] discuss in detail, many evolutionary and other models predict the risk dominant action in two player games but disagree in their predictions in many player games. But the uniform best response, they showed, emerged under a natural form of incomplete information. In particular, consider the case where "all play a" is a strict Nash equilibrium for each action a (this is true if $\rho(a, 1) > 0$ for each action a). Thus with complete information, each action is consistent with equilibrium. But if each player observed a noisy signal of the true payoffs in a certain natural way, only the uniform best response was consistent with equilibrium.

The following interaction game example (with an incomplete information interpretation) provides a discrete state space analogue of that argument.

- **The Critical N-ary Interaction Game**. Let $\mathcal{X} = \mathcal{Z}$; $A = \{\overline{a}, \underline{a}\}$;

$$P(X) = \begin{cases} 1, \text{ if } X = \{x, x+1, ..., x+N-1\} \text{ for some } x \in \mathcal{X} \\ 0, \text{ otherwise} \end{cases};$$

if $x \in \{1, ..., N\}$,

$$u_x(\mathbf{a}_X, X) = \begin{cases} 1, \text{ if } \mathbf{a}_x = \overline{a} \\ 0, \text{ if } \mathbf{a}_x = \underline{a} \end{cases};$$

if $x \notin \{1, ..., N\}$,

$$u_x(\mathbf{a}_X, X) = \rho\left(\mathbf{a}_x, \frac{\#\{y \in X \backslash x : \mathbf{a}_y = \mathbf{a}_x\}}{N-1}\right)$$

where ρ is some BC game payoff function and $\overline{a}$ is the uniform best response for $[\rho, N]$.

This interaction game has payoffs given by BC game ρ for all but N players in an infinite population. Those N players have a dominant strategy to play the uniform best response $\overline{a}$.

The interaction game has a unique equilibrium where action $\overline{a}$ is played by every player. To see why, first note that each player in $\{1, .., N\}$ has a dominant strategy to play $\overline{a}$, and thus must play $\overline{a}$ in any equilibrium. Now consider player $N+1$. He is a participant in N positive weight interactions, $\{2, ..., N+1\}$, $\{3, ..., N+2\}$,... , $\{N+1, ..., 2N\}$. Each of those interactions has equal weight. Thus proportion $\frac{1}{N}$ of his interactions involve all players in $\{1, ..., N\}$, $\frac{1}{N}$ of his interactions involve all but one players in $\{1, ..., N\}$, etc... Since each player in $\{1, .., N\}$ chooses $\overline{a}$ in any equilibrium, the uniform best response property of $\overline{a}$ ensures that player $N+1$ must play action $\overline{a}$ in any equilibrium. A symmetric argument ensures that player 0 must play $\overline{a}$ in any equilibrium. Then an iterative argument shows that players $N+2$ and -1 must play $\overline{a}$ in any equilibrium. And so on.

This example generalizes the logic of the binary interaction example of section 2 to many player interactions. It illustrates the ability of payoffs of a small (finite) number of players to influence equilibrium outcomes for a large (infinite) number of players. The analysis that follows identifies more generally when this is, and is not, possible.

4.2.3 The Basic Lemma

The first question we want to address is: when is it possible to characterize equilibrium behaviour for some group of players independently of what other players do? We provide one set of sufficient conditions, combining the cohesion properties of the interaction system with the following property of payoffs, adapted from Morris, Rob and Shin [1995]. Given $\gamma \in \Gamma_N$, say that action a

is a strictly γ-dominant action of game $[g,N]$ if action a is the unique best response against any conjecture over other players' actions where the γ-weighted proportion of players choosing action a is non-negative. To define this formally, write $\Delta_K(A) = \left\{\pi \in \Delta(A) : \pi(a) \in \left\{0, \frac{1}{K}, \frac{2}{K}, .., 1\right\} \text{ for all } a \in A\right\}$.

Definition 4 *Action a is strictly γ-dominant in $[g,N]$ if*

$$\sum_{\pi \in \Delta_{N-1}(A)} \lambda(\pi)\, g(a,\pi) > \sum_{\pi \in \Delta_{N-1}(A)} \lambda(\pi)\, g(a',\pi)$$

for all $a' \neq a$ and for all $\lambda \in \Delta(\Delta_{N-1}(A))$ with

$$\sum_{k=0}^{N-1} \gamma_k \lambda\left[\left\{\pi : \pi(a) = \frac{k}{N-1}\right\}\right] \geq 0.$$

Thus action a is strictly $\overline{\gamma}^p$-dominant if it a best response whenever proportion p of interactions involve all other players choosing a also. If a is strictly $\overline{\gamma}^1$-dominant, then everyone playing a is a strict symmetric Nash equilibrium of $[g,N]$. If a is $\overline{\gamma}^0$-dominant, then action a is a dominant action in $[g,N]$.

Fix an interaction game and write $\Psi(a,\gamma)$ for the set of players for whom action a is strictly γ-dominant, i.e.,

$$\Psi(a,\gamma) = \{x \in \mathcal{X} : a \text{ is strictly } \gamma\text{-dominant in } [u_x, N]\ \}$$

Lemma 1 *Suppose $Y \subseteq \Psi(a,\gamma)$ and Y is γ-cohesive; then there exists an equilibrium $\boldsymbol{\alpha}$ of the interaction game with $\boldsymbol{\alpha}_x(a) = 1$ for all $x \in Y$.*

Proof. Consider the modified interaction game where all players in Y are required to play action a with probability one. Let $\boldsymbol{\alpha}$ be an equilibrium of the modified game (an equilibrium exists by remark 1). I will show that $\boldsymbol{\alpha}$ is an equilibrium of the original game. By construction, $\boldsymbol{\alpha}_x$ is a best response at all $x \in \overline{Y}$. But if $x \in Y$, then, since Y is γ-cohesive, the condition for a to be a best response is satisfied. ■

This result is an extension (to interaction games) of Lemma 5.2 of Kajii and Morris [1995] which in turn builds on theorem B of Monderer and Samet [1989]. By Proposition 1, the largest p-cohesive group contained in $\Psi(a,\gamma)$ is $C^\gamma(\Psi(a,\gamma))$. Thus the following Proposition follows from Lemma 1.

Proposition 3 *Interaction game $\left(\mathcal{X}, P, A, \{u_x\}_{x\in\mathcal{X}}\right)$ has an equilibrium $\boldsymbol{\alpha}$ with $\boldsymbol{\alpha}_x(a) = 1$ for all $x \in C^\gamma(\Psi(a,\gamma))$.*

4.2.4 Robustness

We first give an informal definition of robustness. An action a is said to be robust in complete information game $[g,N]$, if in *any* N-ary interaction game where almost all players' payoffs are given by g, there is an equilibrium where almost all players choose action a. To make this definition precise, we must define "almost all players". It is convenient to focus on "unbounded" interaction games with an infinite mass of interactions (i.e., games where the bounded interactions property (**P2**) does *not* hold). In this case, a property is said to hold for "almost all" players if it is true for a co-finite group of players; a group is co-finite if its complement is finite; we earlier (see page 22) defined a group of players to be finite if the mass of interactions involving players in that group was finite.

Definition 5 *Action a is robust in $[g, N]$ if every unbounded N-ary interaction game where almost all players' payoffs are given by g has an equilibrium where a is played by almost all players.*[9]

Proposition 4 *If action a is strictly γ-dominant in $[g, N]$ for some $\gamma \in \Gamma_N^+$, then a is the unique robust action in $[g, N]$.*

Proof. Suppose almost all players' payoffs are given by g. Write Y for the finite group for whom payoffs are *not* given by g. By Proposition 2, $\overline{\overline{Y} \cap C^\gamma\left(\overline{Y}\right)}$ is finite. The interaction game is unbounded, so since Y and $\overline{\overline{Y} \cap C^\gamma\left(\overline{Y}\right)}$ are both finite, $C^\gamma\left(\overline{Y}\right)$ must be co-finite. But by Proposition 3, there is an equilibrium where a is played by all players in $C^\gamma\left(\overline{Y}\right)$, i.e., by almost all players. Thus a is robust.

It is also straightforward to show, using a version of the N-ary critical interaction example, that if action a is strictly γ-dominant for some $\gamma \in \Gamma_N^+$, then no *other* action is robust. ■

4.2.5 Back to Binary Action Co-ordination Games

In general games, we know little about the existence of strictly γ-dominant actions with $\gamma \in \Gamma_N^+$. But we can give an exact characterization in BC co-ordination games.

Lemma 2 *Action a is robust in a generic binary action co-ordination game if and only if it is a uniform best response.*

Proof. The genericity qualifier is to ensure the existence of a uniform best response. Given Proposition 4, it is sufficient to show that $\overline{a}$ is a uniform best response if and only if it is strictly γ-dominant for some $\gamma \in \Gamma_N^+$.

[9] Exactly the same results follow if robustness is defined with respect to *bounded* interaction games and the "almost all" statements replaced with (ε, δ) characterizations as in Kajii and Morris [1995]. Thus we could alternatively say: action a is *robust* in $[g, N]$ if, for any $\delta > 0$, there exists $\varepsilon > 0$ such that the following holds; take any bounded N-ary interaction game where at most mass ε interactions involve players whose payoffs are not given by g; there exists an equilibrium where at most mass δ interactions involve some player not choosing a.

Let $\overline{a}$ be the uniform best response. Thus $\sum_{k=1}^{N-1} \rho\left(\overline{a}, \frac{k}{N-1}\right) > 0$. Now let $c = \frac{1}{2N} \sum_{k=1}^{N-1} \rho\left(\overline{a}, \frac{k}{N-1}\right) > 0$ and $\gamma_k = \rho\left(\overline{a}, \frac{k}{N-1}\right) - c$. By construction, $\gamma \in \Gamma_N^+$. Now suppose $\lambda \in \Delta\left(\Delta_{N-1}(A)\right)$ and $\sum_{k=0}^{N-1} \gamma_k \widehat{\lambda}_k \geq 0$, where $\widehat{\lambda}_k = \lambda\left[\left\{\pi : \pi(a) = \frac{k}{N-1}\right\}\right]$. Now $\sum_{k=0}^{N-1} \rho\left(\overline{a}, \frac{k}{N-1}\right) \widehat{\lambda}_k = \sum_{k=0}^{N-1} \gamma_k \widehat{\lambda}_k + c > 0$. Thus action $\overline{a}$ is strictly γ-dominant.

Conversely, suppose action $\overline{a}$ is strictly γ-dominant with $\gamma \in \Gamma_N^+$. Let $\lambda \in \Delta\left(\Delta_{N-1}(A)\right)$ satisfy $\lambda(\pi) = \frac{1}{N}$ for all $\pi \in \Delta_{N-1}(A)$. Clearly, $\sum_{k=0}^{N-1} \gamma_k \lambda\left[\left\{\pi : \pi(\overline{a}) = \frac{k}{N-1}\right\}\right] = \frac{1}{N} \sum_{k=0}^{N-1} \gamma_k > 0$, so by definition of strict γ-dominance, $\sum_{\pi \in \Delta_{N-1}(A)} \lambda(\pi)\, g(\overline{a}, \pi) > \sum_{\pi \in \Delta_{N-1}(A)} \lambda(\pi)\, g(a, \pi)$ if $a \neq \overline{a}$. Thus action $\overline{a}$ is a uniform best response. ■

4.3 Dynamics

In the investment example of section 2, we noted the close apparent connection between iterated deletion of dominated actions (with an incomplete information interpretation) and best response dynamics (with a local interaction interpretation). Here we show that the connection is precise. In particular, we will show that the sufficient condition for robustness that we identified is also a sufficient condition for an action to spread contagiously under a class of best response dynamics with inertia, in *some* interaction system; it is also a sufficient condition for it to be uninvadable by all such inertial best response dynamics, in *all* interaction systems.

In this section, we restrict attention to interaction games satisfying N-ary interaction (**P1**), unbounded interactions (i.e., not **P2**) and player ind't payoffs (**P3****). Write $\beta_x(\mathbf{a})$ for set of actions that are best responses for player x to pure strategy profile $\mathbf{a} \in A^{\mathcal{X}}$. We will be concerned with sequences of pure strategy profiles $\left\{\mathbf{a}^k\right\}_{k=1}^{\infty}$.

Definition 6 *Pure strategy profile sequence $\left\{\mathbf{a}^k\right\}_{k=1}^{\infty}$ is a best response sequence if [1] $\mathbf{a}_x^{k+1} = a$ and $\mathbf{a}_x^k \neq a$ for some $k \geq 1 \Rightarrow a \in \beta_x\left(\mathbf{a}^k\right)$; and [2] $\beta_x\left(\mathbf{a}^k\right) = \{a\}$ for all $k \geq K \Rightarrow \mathbf{a}_x^k = a$ for some $k > K$.*

Property [1] requires that each player at each date *either* plays the action played in the previous period *or* plays some best response. Property [2] requires that if action a is going to be the unique best response for player x forever, it is never abandoned (even though it might be played only rarely).

These two weak properties characterize a class of best response dynamics with inertia. Many dynamics studied in the literature satisfy the two properties (at least with probability one). Three examples are the following: each player chooses a best response in each period (see, e.g., Morris [1997]); one randomly chosen player gets to revise in each period while all others stick with their previous action (see, e.g., Blume [1995]); each player revises or sticks with some probability (see, e.g., Anderlini and Ianni [1996]).

Definition 7 *Action a is contagious in $[g, N]$ if there exists an unbounded N-ary interaction system $(\mathcal{X}, P)$ with a finite group Z such that every best response sequence $\{\mathbf{a}^k\}_{k=1}^{\infty}$ with $\mathbf{a}_x^1 = a$ for all $x \in Z$ satisfies $\mathbf{a}_x^k = a$ for all sufficiently large k, for each $x \in \mathcal{X}$.*

Thus an action is contagious if there exists some interaction system such that every best response sequence leads that action to spread from some finite group to the whole population.

Definition 8 *Action a is uninvadable in $[g, N]$ if for every unbounded N-ary interaction system $(\mathcal{X}, P)$ and every finite group Z, there exists another finite group Y such that every best response sequence $\{\mathbf{a}^k\}_{k=1}^{\infty}$ with $\mathbf{a}_x^1 = a$ for all $x \notin Z$ satisfies $\mathbf{a}_x^k = a$ for all $x \notin Y$, for all k.*

Thus an action is uninvadable if once it is played by almost all players, it continues to be played by almost all players, in every interaction system and every best response sequence.

Note that if one action is contagious in $[g, N]$, then, by definition, no other action may be uninvadable in $[g, N]$.

Proposition 5 *If action a is strictly γ-dominant in $[g, N]$, for some $\gamma \in \Gamma_N^+$, then a is contagious and uninvadable in $[g, N]$.*

Proof. [1] *Contagiousness.* Fix the interaction system of the critical N-ary interaction example and let $Z = \{1, .., N\}$. Consider any best response sequence with $\mathbf{a}_x^1 = a$ for all $x \in Z$. By strict γ-dominance and property [1], $\mathbf{a}_x^k = a$ for all $x \in Z$ and all k. Thus by strict γ-dominance, $\beta_{N+1}(\mathbf{a}^k) = \{a\}$ for all k. By property [2], $\mathbf{a}_{N+1}^k = a$ for some $\widehat{k}$. By strict γ-dominance and property [1], $\mathbf{a}_{N+1}^k = a$ for all $k > \widehat{k}$. The argument iterates to ensure the result.

[2] *Uninvadability.* For any finite Z, let $Y = \overline{C^{\gamma}(\overline{Z})}$. Now suppose $\mathbf{a}_x^1 = a$ for all $x \notin Z$. By γ-dominance and property [1], $\mathbf{a}_x^k = a$ for all $x \in C^{\gamma}(\overline{Z})$ and all k. By Proposition 2, Y is finite. ■

Corollary In a generic binary action co-ordination game, the following four properties are *equivalent*: [1] action a is a uniform best response; [2] action a is robust; [3] action a is uninvadable; and [4] action a is contagious.

The results in this section tell us only about what happens for extreme local interaction systems. Morris [1997] characterizes (in a simpler environment) *which* local interaction systems allow a contagious action to spread.

5 Conclusion

Incomplete information, local interaction and random matching games can all be understood as special cases of a general class of interaction games. The distinguishing features of particular classes of games - for example, N-partite interaction for incomplete information games - are in many cases a distraction. A more abstract approach may both allow productive arbitrages across the different research areas and provide a better understanding of what is driving results. The equivalence allowed us to extend robustness results to many player games and interpret those results in a wider set of contexts. Morris [1997] represents a further attempt to exploit the equivalence.

One can think of further games that can be embedded in this class. Dynamic games, where each player gets to make many choices, are routinely interpreted as games between "agents" of those players, where each agent gets to make only one choice. If payoffs are additively separable through time, each agent's payoff depends only on interactions with a small subset of all agents (i.e., those acting in the same time period). But the characteristic feature of dynamic games - that players must anticipate the impact of their actions on others' actions - is not naturally embedded in the class of games described in this paper. However, there are two special cases where the analysis translates. First, there is the case where players make a sequence of choices at different points in time, without observing others' choices until the end of the game. In this case, Morris [1995] shows that the incomplete information argument of Carlsson and van Damme [1993a] translates to show that if players' clocks are not perfectly co-ordinated, they must play the risk dominant equilibrium in any two player two action co-ordination game. Second, there is the continuum of players case. In this case, again, individual players cannot influence others' actions. Burdzy, Frankel and Pauzner [1997] show that if there is symmetric noise concerning how payoffs evolve through time, the risk dominant equilibrium must be played always. They note the connection with the incomplete information argument of Carlsson and van Damme.

References

[1] Anderlini, L. and A. Ianni (1996). "Path Dependence and Learning from Neighbours," *Games and Economic Behavior* 13, 141-177.

[2] Aumann, R. (1987). "Correlated Equilibrium as an Expression of Bayesian Rationality," *Econometrica* 55, 1-18.

[3] Blume, L. (1995). "The Statistical Mechanics of Best-Response Strategy Revision," *Games and Economic Behavior* 11, 111-145.

[4] Burdzy, K., D. Frankel and A. Pauzner (1997). "Fast Equilibrium Selection by Rational Players Living in a Changing World," Working Paper #7-97, Foerder Institute for Economic Research, Tel Aviv University.

[5] Carlsson, H. and E. van Damme (1993a). "Global Games and Equilibrium Selection," *Econometrica* 61, 989-1018.

[6] Carlsson, H. and E. van Damme (1993b). "Equilibrium Selection in Stag Hunt Games," in *Frontiers of Game Theory* (K. Binmore, A. Kirman and A. Tani, Eds.). Cambridge, MA: MIT Press.

[7] Ellison, G. (1993). "Learning, Local Interaction, and Coordination," *Econometrica* 61, 1047-1071.

[8] Fudenberg, D. and J. Tirole (1991). *Game Theory.* Cambridge: M.I.T. Press.

[9] Harsanyi, J. (1967). "Game with Incomplete Information played by Bayesian Players. Part I: the Basic Model," *Management Science* 14, 159-182.

[10] Hartigan, J. (1983). *Bayes Theory.* New York: Springer-Verlag.

[11] Harsanyi, J. and R. Selten (1988). *A General Theory of Equilibrium Selection in Games.* Cambridge: M.I.T. Press.

[12] Ianni, A. (1996). "Learning Correlated Equilibria in Balanced Games."

[13] Kajii, A. and S. Morris (1995). "The Robustness of Equilibria to Incomplete Information," CARESS Working Paper #95-18, forthcoming in *Econometrica.*

[14] Kim, Y. (1996). "Equilibrium Selection in n-Person Coordination Games," *Games and Economic Behavior* 15, 203-227.

[15] Mailath, G., L. Samuelson and A. Shaked (1997). "Correlated Equilibria and Local Interactions," *Economic Theory* 9, 551-568.

[16] Monderer, D. and D. Samet (1989). "Approximating Common Knowledge with Common Beliefs," *Games and Economic Behavior* 1, 170-190.

[17] Morris, S. (1995). "Co-ordination and Timing," CARESS Working Paper #95-05, University of Pennsylvania.

[18] Morris, S. (1997). "Contagion," CARESS Working Paper #97-01, University of Pennsylvania.

[19] Morris, S., R. Rob and H. Shin (1995). "p-Dominance and Belief Potential," *Econometrica* 63, 145-157.

[20] Rubinstein, A. (1989). "The Electronic Mail Game: Strategic Behavior under Almost Common Knowledge," *American Economic Review* 79, 385-391.

[21] Shin, H. and T. Williamson (1996). "How Much Common Belief is Necessary for a Convention?" *Games and Economic Behavior* 13, 252-268.

[22] Sugden, R. (1995). "The Coexistence of Conventions," *Journal of Economic Behavior and Organization* 28, 241-256.

[23] Young, P. (1996). "The Economics of Conventions," *Journal of Economic Perspectives* 10, 105-122.