

ESSAYS ON DECISION THEORY

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Abstract

This dissertation studies forward looking in dynamic choice. In the first chapter, we propose a dynamic choice model for an error-prone decision maker choosing among risky options. Our axioms yield a representation of the decision maker's behavior in which the decision maker rationally anticipates her future mistakes, and might be averse to making choices. The resulting model provides a natural welfare criterion even though the decision maker makes mistakes. We introduce comparative measures of error-proneness and choice attitude. We characterize the logit quantal response model as a special case of our model that exhibits constant measure of error-proneness and choice neutrality. We show that different from standard risk, when risk is induced by mistakes, the expected value of a future decision problem might increase even if its options become worse.

In the second chapter, we dispense with the assumption of rational anticipation of future mistakes. We propose a model of backward induction with an error-prone decision maker who has limited understanding of her own future choices. To an outside observer, her behavior appears stochastic and her choices become imperfect signals of her payoffs. Our axioms yield a two-parameter representation of the decision maker's behavior; one parameter characterizes her attitude towards complexity; i.e., her willingness to choose more complicated subtrees over simpler ones, the other her error-proneness. Our model nests fully rational backward induction as a limit of these parameters. We introduce and analyze a measure of complexity aversion and a measure of error-proneness. We show through examples how different decision trees

induce different choice behavior in the context of product assortment and advertising problems.

In the last chapter, we present a negative result showing that if the forward looking satisfies three simple conditions, then the decision maker must have ignored her possible mistakes when looking forward. If one of the conditions, *monotonicity* is replaced with *strict monotonicity*, then forward looking can never satisfy all three conditions at the same time. We show in specific models of mistakes why these conditions are incompatible.

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Chapter 1

Rational Anticipation of Mistakes and Revealed Choice Aversion

1.1 Introduction

People make mistakes when choosing. Consider a static decision problem consisting of options $a = \{p_1, \dots, p_n\}$. If we observe an error-prone decision maker choosing from a repeatedly, we will find that sometimes she chooses p_i , but sometimes she chooses some other option. In other words, her choice appears to be random: she behaves as if she chooses each option in a with some probability.

Using random choice models to describe an error-prone decision maker's behavior is not new. Perhaps one of the most popular models is from McKelvey and Palfrey (1995, 1998). In their model, each option p_i has utility $u(p_i)$, but the decision maker can only observe a noisy signal of it, $u(p_i) + \varepsilon_i$. She chooses the option with the

highest signal value. Since ε_i 's are random error terms, her choice contains random mistakes.

In general, it is difficult for choice models of mistakes to answer welfare questions. Let us take McKelvey and Palfrey (1995, 1998) as an example. Suppose we want to know what is the expected utility that the decision maker gets by choosing from a . The existing approach calculates $\sum w_i \times u(p_i)$, where w_i is the actual choice probability of p_i due to the error terms. The idea is simple. It takes $u(p_i)$ as the true utility of p_i , and then calculates the expected utility. The problem is how do we know that the function u is a vNM utility index? When we estimate u from the choice data, there is no guarantee that such a u function is a vNM utility index. If it is not the vNM utility index, calculating its expected value is not a natural way to think about welfare questions. If it is, in what sense it is?

This issue is particularly important when the decision problem is dynamic. In a dynamic decision problem, the error-prone decision maker needs to look forward. In order to make a current-stage decision, she needs to take into account her future mistakes to form some belief about the expected utility that she will get from each continuation. In this case, identifying the vNM utility index (not just a utility function) is inevitable.

Having this in mind, we propose an axiomatic, dynamic model of mistakes in which the decision maker chooses among risky options. The decision maker's error-prone choice is also interpreted as random to capture that she does not make the same mistakes over time.

We analyze how the decision maker chooses from a decision problem. A decision problem is a set of lotteries from which the decision maker makes a choice. Each lottery in turn is a probability distribution over (continuation) decision problems; that is, after the risk associated with the chosen lottery resolves, the decision maker will face another decision problem to choose from. Choosing and risk resolution alternate until the decision maker reaches an outcome.

In the resulting model, for each option we identify simultaneously *(i)* its Luce value that describes the decision maker's propensity to choose it, and *(ii)* its vNM utility index. As a byproduct, we also identify the decision maker's subjective benefit/cost of making a choice from a decision problem. In the model, each option has a vNM utility index. The utility of a lottery follows the standard expected utility formula. The utility of a decision problem follows the standard expected utility formula weighted by the actual choice probability of each option, added (subtracted) by a benefit (cost) term that depends on the size of the decision problem. To relate utility to error-prone choice, a conversion function converts utility into Luce value. The higher is an option's Luce value, the more likely it will be chosen.

The model is derived from simple axioms. The primitive of our model is a *random choice rule* $\rho(\cdot, \cdot)$ that describes how the decision maker chooses. For example, $\rho(\cdot, a)$ describes the choice probability of each option in the decision problem a . We impose axioms on ρ .

The decision maker we want to model has a stable underlying preference. Her random choice is solely attributed to mistakes. The first axiom we consider allows us to identify her preference even though she makes random mistakes. If the decision

maker chooses the lottery p from the problem $\{p, r_1, \dots, r_n\}$ more often than chooses q from $\{q, r_1, \dots, r_n\}$ for all r_1, \dots, r_n , then she reveals that she prefers p over q . The axiom, *Luce Independence*, ensures that she reveals her preference consistently; that is, if p is chosen more often than q in one case, then p is chosen more often than q in all cases.

Now that the underlying preference is uncovered, we impose the standard *vNM Independence* axiom. We identify technical axioms that help us pin down the decision maker's vNM utility index under the presence of random choice.

Lastly, to have some structure on the benefit/cost term associated with making choices, we consider the following simple axiom, *Preference for Lottery-Choice Swaps*. This axiom captures that when a decision maker is averse to making choices, she would find it more painful to choose from larger decision problems. Therefore, when a larger decision problem is replaced with a comparable lottery, she benefits more; when a smaller decision problem is replaced with a comparable lottery, she benefits less. A comparable lottery p_a of a decision problem a is a lottery such that the probability distribution over (future) decision problems that p_a induces is identical to the distribution induced by the decision maker's own choice when the problem is a . Then, a choice-averse decision maker would prefer $\frac{1}{2}p_a + \frac{1}{2}\delta_b$ over $\frac{1}{2}p_b + \frac{1}{2}\delta_a$ when a is a larger problem than b , where δ_c is the degenerate lottery that yields problem c for sure, $c = a, b$. The decision maker prefers $\frac{1}{2}p_a + \frac{1}{2}\delta_b$ because in this lottery, she only has to worry about choosing from b , not from a , whereas in $\frac{1}{2}p_b + \frac{1}{2}\delta_a$, she might need to choose from the larger decision problem a .

Theorem 1.2.1 of this chapter establishes that these three axioms together with other technical axioms result in the following representation of a random choice rule: there exists a utility function U such that for each lottery p ,

$$U(p) = \sum_{a_i \in \text{supp}(p)} p(a_i)U(a_i)$$

which is the standard expected utility function. Then, for a decision problem $a = \{p_1, \dots, p_n\}$, the decision maker assigns utility

$$U(a) = \sum_{p_i \in a} \rho(\{p_i\}, a)U(p_i) + \psi(|a|)$$

to it. The term $\sum_{p_i \in a} \rho(\{p_i\}, a)U(p_i)$ reflects her rational anticipation of her future mistakes, because this term is exactly the expected utility she would get from a had she chosen from a . The added term $\psi(|a|)$ is a monotone function that measures her attitude towards choice. Lastly, there is a conversion function ϕ such that

$$\rho(\{p_i\}, a) = \frac{\phi(U(p_i))}{\sum_j \phi(U(p_j))} \tag{1.1}$$

The strictly increasing function ϕ describes the relation between the utility and the decision maker's error-prone choice. It converts utility into Luce value (Luce (1959)). We call this representation of a random choice rule an *Anticipated-Mistakes Rule* (AMR). The AMR has a natural way to assess the expected utility that the decision maker gets from choosing from a decision problem, since the function U is indeed the vNM utility index.

The AMR, unlike the other ones in the literature, distinguishes a Luce rule from a logit model (see Luce (1959), McFadden (1974)). An AMR is always a Luce rule, but it is a logit model if and only if

$$\phi(u) = e^{u/\lambda}$$

(unique up to a positive scalar multiplication) for some positive λ , and $\psi = 0$. The logit model has been widely used in applications mostly because it is a Luce rule and hence is more tractable. Thus, our representation generalizes the logit model by allowing other functional forms of ϕ , but still ensures that the generalized models are all Luce rules, which maintains its tractability.

In an AMR, the parameter ϕ and ψ quantify error-proneness and choice attitude respectively. Decision maker 2 is said to be more error-prone than decision maker 1 if decision maker 2 always chooses inferior options more often. In Section 1.3, we show that this happens if and only if $\frac{\phi'_2}{\phi_2} \leq \frac{\phi'_1}{\phi_1}$; that is, the rate of change of ϕ_1 is greater than the rate of change of ϕ_2 .

From the representation, it can be seen that the lower the value of ψ is, the more choice-averse the decision maker should be. We show in Section 1.3 that the converse is true. Compared to the existing measures of error-proneness and choice attitude, our measure of error-proneness is more general than the one in Ke (2015a), and our measure of choice aversion (seekingness) is more general than the one in Fudenberg and Strzalecki (2015).

Through the characterization of error-proneness and choice attitude, we show that the widely-used logit quantal response model, introduced in McKelvey and Palfrey

(1995, 1998), can be characterized as a special case of our model that exhibits constant measure of error-proneness and neutrality to choice. In general, neither the quantal response model is a special case of AMR, nor the other way around. The two classes of models intersect at the case of logit quantal response model ($\phi(u) = e^{u/\lambda}$ and $\psi = 0$). To put it another way, only for the logit quantal response model, the original approach of calculating the expected utility in McKelvey and Palfrey (1995, 1998) is valid. It is valid under the assumption of constant measure of error-proneness and neutrality to choice.

Mistakes generate risk. In finance, risk from mistakes is classified as an important case of the *operational risk*, defined in one of the most influential banking regulations, Basel II. Despite the importance and prevalence of mistakes, it is not well-understood how risk from mistakes differs from standard financial risk. In Section 1.4, we point out one major difference between these two types of risks: risk from mistakes might not be *monotonic*. To see this, suppose a principal assigns a task to an agent. The error-prone agent could choose either action 1 that leads to a good outcome, or action 2 that leads to an extremely bad outcome. Note that with standard financial risk, if a lottery's possible outcomes improve, the lottery itself must improve. In our case, if the principal improves the extremely bad outcome, the expected payoff she gets from the task performed by the agent might in fact decrease. The reason is simple: When the outcome of action 2 is bad enough, the agent, according to our model, would choose action 1 with high probability. In contrast, when the outcome of action 2 is improved, the agent often mistakenly chooses action 2, which lowers the overall expected payoff.

1.1.1 Related Literature

The role of mistakes has been studied in the literature. Sah and Stiglitz (1986) investigate the impact of mistakes in economic systems with different architectures. In particular, they analyze the performance of two decision making processes, one is a two-unit polyarchy and the other is a two-unit hierarchy. They identify the conditions under which one outperforms the other. McKelvey and Palfrey (1995, 1998) introduce the quantal response model to analyze mistakes in both static and dynamic games. Their model has been successful in explaining the experimental data. The main differences between their model and ours are all discussed in the introduction. In other fields, for example, Hassan and Mertens (2014) study a macroeconomic model in which the decision makers make small correlated errors when forming expectations about future productivity. The authors show that even if the errors are small, they amplify in equilibrium and significantly reduce the price informativeness. Our model has a different focus. Our purpose is to derive a model of mistakes from reasonable axioms, and to emphasize how risk from mistakes (described by our model) differs from standard risk.

This chapter is closely related to Fudenberg and Strzalecki (2015) and Ke (2015a). Fudenberg and Strzalecki (2015) formulate an extension of the Luce's random choice model to dynamic problems. In their model, an alternative of a decision problem is a current-period consumption good plus a continuation decision problem. The decision maker has random tastes and she rationally anticipates them. She enjoys the option value of the future problems. In the representation there is also an additively separable term describing the decision maker's choice attitude, as in this chapter.

Fudenberg and Strzalecki's work is the first to introduce the notion of choice aversion in the choice theory literature. Our representation differs from theirs in three ways. First, our model generalizes the logit model, while theirs is a logit model. Second, although the decision maker looks forward rationally in both their model and ours, the meaning of rational forward looking is different. In their case, a decision maker anticipates future taste shocks, and hence a larger (future) decision problem has a higher option value. In our case, the decision maker understands her future choice probability of each alternative, and evaluates a future decision problem by the expected utility that she will get from choosing. Lastly, our function that characterizes the choice attitude is more general than Fudenberg and Strzalecki's in that we only require the function's monotonicity. In Fudenberg and Strzalecki, the function is a log function.

Ke (2015a) studies a relaxation of fully rational backward induction in which the decision maker chooses from a set of decision (sub)trees. In fully rational backward induction, the decision maker identifies each decision problem with its best option, and chooses the option that has the highest value with certainty. In Ke, the decision maker is averse to complex options and hence does not identify a decision problem with its best option. Moreover, the decision maker makes mistakes when choosing; that is, she cannot choose the option that has the highest value with certainty.

The major difference between that paper and the current chapter is that the decision maker in the current chapter rationally anticipates her future choices, while the rational anticipation is exactly the assumption that Ke (2015a) wants to relax;

that is, the decision maker in Ke (2015a) does not understand how she will choose in the continuation problems.

1.2 A Dynamic Model of Mistakes

Confronting a decision problem, the decision maker makes a series of choices to reach an outcome. We use a decision problem to describe such a choice situation. A decision problem is a finite set of options. Each option is a lottery. Facing a decision problem, the decision maker first chooses a lottery. A lottery is a probability measure over a finite set of decision problems. After a lottery is chosen, its risk resolves. Then, the decision maker either faces an outcome and stops, or faces a new decision problem and continue to make a choice. Decision maker's choice and risk resolution continue to alternate for finitely many times until the decision maker receives an outcome.

Formally, let D_0 be the set of outcomes, and $\mathcal{L}_1 := \Delta(D_0)$ be the set of simple lotteries on D_0 , where $\Delta(\cdot)$ denotes the set of simple lotteries. Next, let $D_1 := K(\mathcal{L}_1)$ be the collection of nonempty finite subsets of \mathcal{L}_1 , where $K(\cdot)$ denotes the collection of nonempty finite subsets of a set. Confronting some $a \in D_1$, the decision maker only needs to make one choice to reach an outcome; that is, she chooses a lottery from a , and then she gets an outcome after the lottery's risk resolves. Recursively, we define $\mathcal{L}_{k+1} := \Delta(D_k)$ and $D_k := K\left(\bigcup_{i=1}^k \mathcal{L}_i\right)$. Facing some $b \in D_k$, the decision maker at most chooses k times in order to reach an outcome, and each of her choices is followed by risk resolution of the chosen lottery. Let $\mathcal{D} := \bigcup_{i=1}^{\infty} D_i$ be the set of all possible decision problems, and $\mathcal{L} := \bigcup_{i=1}^{\infty} \mathcal{L}_i$ be the set of all possible options/lotteries. For

any lottery $p \in \mathcal{L}$, we use $\text{supp}(p)$ to denote p 's support. Clearly, $D := \mathcal{D} \cup D_0$ is the union of p 's support, for all $p \in \mathcal{L}$. Thus, a typical decision problem is $a = \{p_1, \dots, p_n\} \in \mathcal{D}$ where $p_i \in \mathcal{L}$ and $\text{supp}(p_i) \subset D$. Clearly, $\mathcal{D} = K(\mathcal{L})$ and $\mathcal{L} = \Delta(D)$. As usual, $\sum_{i=1}^n \alpha_i p_i$ is a lottery such that the probability assigned to $a \in D$ is $\sum_{i=1}^n \alpha_i p_i(a)$, where $\alpha_i \in [0, 1]$, $\sum \alpha_i = 1$ and $p_i \in \mathcal{L}$.

The decision maker makes mistakes, but she does not make the same mistake over time. To the modeler who observes the decision problem and her choice, her choice appears to be random. The decision maker's random choice is defined as follows.

Definition 1.2.1 *A function $\rho : \mathcal{D} \times \mathcal{D} \rightarrow [0, 1]$ is a random choice rule (RCR) if $\rho(a, a) = 1$ and $\rho(c, d) = \sum_{b \in c} \rho(\{b\}, d)$.*

According to the definition, $\rho(a, b)$ is the probability that any option of a is chosen when the decision problem is b .

The decision maker in our model has a stable underlying preference. Since she is error-prone, she cannot reveal her preference deterministically. However, she can reveal her preference statistically. Based on an RCR, we define the decision maker's preference relation as follows.

Definition 1.2.2 *For $\forall p, q \in \mathcal{L}$, $p \succeq q$ (p is preferred to q) if $\rho(\{p\}, \{p\} \cup a) \geq \rho(\{q\}, \{q\} \cup a)$ for any $a \in \mathcal{D}$ such that $p, q \notin a$.*

The decision maker reveals that she prefers p to q if p is chosen over a more often than q over a , for all a that does not contain p or q . Below we impose axioms on a random choice function. Many of the axioms are imposed on the underlying preference directly. The first three axioms are taken from the literature.

Axiom 1.2.1 (*Positivity*) If $p \in a \in \mathcal{D}$, then $\rho(\{p\}, a) > 0$.

Axiom 1.2.2 (*Luce Independence*) For $a, b, c, d \in \mathcal{D}$ such that $(a \cup b) \cap (c \cup d) = \emptyset$, $\rho(a, a \cup c) \geq \rho(b, b \cup c)$ implies $\rho(a, a \cup d) \geq \rho(b, b \cup d)$.

Axiom 1.2.3 (*vNM Independence*) For $p, q, r \in \mathcal{L}$, $\alpha \in (0, 1)$, $p \succ q$ implies $\alpha p + (1 - \alpha)r \succ \alpha q + (1 - \alpha)r$.

Positivity is from McFadden (1974). In our setting, it implies that every option of a decision problem could have some chance to be (mistakenly) chosen, no matter how small the probability is. *Luce Independence* is from Gul, Natenzon and Pendorfer (2014). This axiom ensures that the decision maker reveals her preference consistently; that is, if she chooses some $p \in \mathcal{L}$ over a set of lotteries a more often than q over a , then she always chooses p more often than q over any other set of lotteries that does not contain p, q . This axiom implies that the preference relation is complete. The third axiom, *vNM Independence*, is the well-known axiom used in expected utility theory. We apply it to the decision maker's underlying preference to identify the vNM utility index.

Two technical assumptions, *Continuity* and *Unboundedness*, are needed to pin down the representation.

Axiom 1.2.4 (*Continuity*) For $p, q \in \mathcal{L}$, $a \in \mathcal{D}$, $\rho(\{\alpha p + (1 - \alpha)q\}, \{\alpha p + (1 - \alpha)q\} \cup a)$ is continuous in α .

Axiom 1.2.5 (*Unboundedness*) For $a \in \mathcal{D}$, $\alpha \in (0, 1)$, there exist $p, q \notin a$ such that $\rho(\{p\}, \{p\} \cup a) < \alpha$ and $\rho(\{q\}, \{q\} \cup a) > \alpha$.

Since we have imposed the *vNM Independence*, it is natural to consider the vNM continuity axiom. However, the standard vNM continuity axiom is not sufficient. In particular, it does not imply that $\rho(\{\alpha p + (1 - \alpha)q\}, \{\alpha p + (1 - \alpha)q\} \cup a)$ would change smoothly as α changes. Our *Continuity* axiom is a natural extension of vNM continuity to a model with random choice. *Unboundedness* is a random-choice version of the standard unboundedness assumption. *Unboundedness* implies that, given a , we can find p so that $\rho(\{p\}, \{p\} \cup a)$ is arbitrarily close to 0, and q so that $\rho(\{q\}, \{q\} \cup a)$ is arbitrarily close to 1.

Our next axiom is a simple consistency requirement. It states that a degenerate lottery would be identified with its only associated decision problem, and a degenerate decision problem would in turn be identified with its only option/lottery.

Axiom 1.2.6 (*Degenerate-Choice Indifference*) For $p \in \mathcal{L}$, $\delta_{\{p\}} \sim p$.

Given a degenerate lottery $\delta_{\{p\}}$, the decision maker would face the decision problem $\{p\}$ with certainty. Meanwhile, $\{p\}$ is a decision problem that the decision maker only needs to trivially choose p . *Degenerate-Choice Indifference* says that the decision maker ignores the degeneracy of lotteries and decision problems. Hence, $\delta_{\{p\}}$ and p are indifferent to her.

Our decision maker might either find it difficult to make a choice, or enjoy making a choice. The last axiom allows the decision maker to be either choice-averse or choice-seeking. Let $\delta_b \in \mathcal{L}$ denote the degenerate lottery such that $\delta_b(b) = 1$, $b \in \mathcal{D}$. For any $b = \{p_1, \dots, p_n\} \in \mathcal{D}$, we can find a lottery denoted by p_b such that

$$p_b(a) = \sum_{i=1}^n \rho(\{p_i\}, b) \times p_i(a)$$

for $a \in D$; that is, the probability that p_b assigns to each $a \in D$ (either an outcome or a decision problem) is equal to the probability that a would be presented to the decision maker after (i) she chooses from b and (ii) the chosen lottery's risk resolves. We call p_a a *comparable lottery* of decision problem a . We use an example to illustrate what a comparable lottery is.

Example 1.2.1 *Suppose $a, b \in \mathcal{D}$ are decision problems. Consider a decision problem $c = \{\frac{1}{2}a + \frac{1}{2}b, \delta_a\}$. Let $q = \rho(\{\delta_a\}, c)$ be the probability that the decision maker chooses δ_a over $\frac{1}{2}a + \frac{1}{2}b$. Confronting c , the decision maker makes a choice. If she chooses δ_a , she expects to choose from a in the next stage. If she chooses $\frac{1}{2}a + \frac{1}{2}b$, she expects to choose from a in the next stage with $\frac{1}{2}$ probability, and from b with $\frac{1}{2}$ probability. Thus in the next choice stage, with probability $q + \frac{1}{2}(1 - q) = \frac{1}{2}(1 + q)$, the decision maker confronts a , and with probability $\frac{1}{2}(1 - q)$, the decision maker confronts b . According to the definition, the comparable lottery of c , p_c , is a lottery that assigns $\frac{1}{2}(1 + q)$ to a and $\frac{1}{2}(1 - q)$ to b .*

Axiom 1.2.7 *(Preference for Lottery-Choice Swaps) For $a, b \in \mathcal{D}$, if $|a| \geq |b|$, then*

- (a) *(Choice Aversion) $\frac{1}{2}p_a + \frac{1}{2}\delta_b \succeq \frac{1}{2}p_b + \frac{1}{2}\delta_a$;*
- (b) *(Choice Seekingness) $\frac{1}{2}p_b + \frac{1}{2}\delta_a \succeq \frac{1}{2}p_a + \frac{1}{2}\delta_b$.*

To understand this axiom, first note that when δ_a 's risk resolves (trivially), the decision maker needs to choose from a . Then, if a decision maker understands the probability with which she chooses each option of a , she would notice that δ_a and p_a induce the same probability distribution over future decision problems and outcomes. Thus, in terms of the probability distribution over future decision problems and outcomes, $\frac{1}{2}p_b + \frac{1}{2}\delta_a$ and $\frac{1}{2}p_a + \frac{1}{2}\delta_b$ are identical.

However, the decision maker might not be indifferent between making the choice by herself (δ_a) and letting the choice be randomly determined by a lottery (p_a). In other words, if she is choice-averse, she might prefer p_a to δ_a , and if she is choice-seeking, she might prefer δ_a to p_a . This is true because with p_a , she does not need to choose from a , and the resulting distribution over future decision problems and outcomes is the same as the one she would face had she chosen from a on her own.

Ideally, if a decision maker prefers not to choose (choice-averse), then the larger the size of a is, the more she prefers to switch from δ_a to p_a . If she enjoys choosing (choice-seeking), the larger the size of a is, the less willing she would be to switch to p_a .

Preference for Lottery-Choice Swaps summarizes the arguments above. In the axiom, $\frac{1}{2}p_a + \frac{1}{2}\delta_b$ is a lottery such that with $\frac{1}{2}$ probability the decision maker would face p_a , and with $\frac{1}{2}$ probability she would need to choose from decision problem b . The other lottery $\frac{1}{2}p_b + \frac{1}{2}\delta_a$ swaps the lottery and choice; that is, with $\frac{1}{2}$ probability the decision maker would need to choose from decision problem a , and with $\frac{1}{2}$ probability she would face the lottery p_b . Since p_a is the comparable lottery of a and p_b is the comparable lottery of b , for a rationally forward-looking decision maker who understands her future choice probability, there is only one difference between $\frac{1}{2}p_a + \frac{1}{2}\delta_b$ and $\frac{1}{2}p_b + \frac{1}{2}\delta_a$: with $\frac{1}{2}$ chance choosing from a or b . The axiom says that if a decision maker is choice-averse, she would prefer to have $\frac{1}{2}$ probability choosing from the smaller decision problem. The opposite applies to a choice-seeking decision maker.

Empirically, consumers' aversion to choice (sometimes called overchoice) has been well documented (see Chernev (2003), Gourville and Soman (2005) among others). Recently it is also examined in the choice theory literature by Fudenberg and Strzalecki (2015) where a function $\kappa \log |a|$ is proposed to describe the decision maker's choice attitude. Our axioms are different from their work, and the resulting characterizing function for choice attitude is different from theirs too.

In Theorem 1.2.1, we show that the axioms will induce the following model in which the decision maker rationally anticipates her own future mistakes. Due to the axioms, we can identify (i) the vNM utility function that describes risk aversion as usual, (ii) the Luce value function that describes the decision maker's propensity to choose a particular option, and (iii) a benefit/cost function associated with making choices.

Definition 1.2.3 *An RCR ρ is a Anticipated-Mistakes Rule (AMR) if there exist a function $U : \mathcal{L} \cup D \rightarrow \mathbb{R}$, a surjective strictly increasing continuous function $\phi : U(\mathcal{L} \cup D) \rightarrow \mathbb{R}_{++}$, a monotone function $\psi : \mathbb{N} \rightarrow \mathbb{R}$ with $\psi(1) = 0$ such that for $p \in \mathcal{L}$, $a = \{p_1, \dots, p_n\} \in \mathcal{D}$,*

$$U(p) = \sum_{a_i \in \text{supp}(p)} p(a_i)U(a_i) \tag{1.2}$$

$$U(a) = \sum_{p_i \in a} \rho(\{p_i\}, a)U(p_i) + \psi(|a|) \tag{1.3}$$

and

$$\rho(\{p_i\}, a) = \frac{\phi(U(p_i))}{\sum_j \phi(U(p_j))} \quad (1.4)$$

When U, ψ, ϕ satisfy the equations above, we say that (U, ψ, ϕ) represents ρ . Although U depends on ψ and ϕ according to (1.2) and (1.3), if we restrict U 's domain to the set of outcomes, it is independent of ψ and ϕ . Thus, the function ψ and the function ϕ uniquely extend the utility of outcomes to the utility of all lotteries and decision problems.

In the representation, a lottery's utility is given by equation (1.2), the standard expected utility function. As for a decision problem a , equation (1.3) says that the decision maker forms a correct expectation of the actual expected utility she would get if she chooses from a , which is equal to $\sum_{p_i \in a} \rho(\{p_i\}, a)U(p_i)$. On top of this expected utility term, $\psi(|a|)$ is added to describe the decision maker's attitude towards making choices. The function ψ could either be increasing or decreasing. When it is increasing, the utility of a decision problem a is more penalized as a 's size grows.

The decision maker's error-prone behavior is characterized by equation (1.4). Fixing ϕ , the higher an option's utility is, the more likely it would be chosen. Fixing U , consider for example $\phi(u) = u^k$. The higher is k , the more likely the decision maker would end up choosing the better options; that is, the decision maker is less error-prone when making choices. In the limiting case in which k is arbitrarily large, the best option of a decision problem would be chosen for sure. At the other extreme, when ϕ becomes a constant function in the limit, the decision maker chooses uniformly randomly. Allowing different patterns of error-proneness is important. In

poker games, both professional players and amateur players make mistakes. However, the professional players make fewer mistakes, while the amateur players make more. Applying our model, this can be captured by equipping the professional players with a less error-prone AMR, and the amateur players with a more error-prone one.

According to (1.4), an AMR is a Luce rule and $\phi(U(p_i))$ is the Luce value of p_i . However, an AMR is significantly different from the logit model. In a static decision problem, there is no reason why we should distinguish between a logit model and a Luce rule. For any Luce rule, we can find a logit model that provides the same choice prediction, and vice versa. Here we can (and should) make a distinction because we are considering dynamic choice. In our resulting model, the function ϕ converts utility into Luce value. A standard logit model requires this conversion function to be $\phi(u) = \exp\{u/\lambda\}$ for some positive constant λ . In our model, ϕ could be other functions, which allows for richer behavior patterns of error-proneness. Our model thus breaks the link between Luce rule and logit model.

Lastly, let us point out that understanding the future choice probabilities does not imply that the decision maker would not make mistakes in the future. Consider a decision maker who is grilling a beef steak for dinner. There are three styles of cooking: well-done, medium-rare and rare. The goal is to get the steak medium-rare. To achieve that, the steak needs to be turned at the right time while grilling. Before cooking, the decision maker may be able to understand with what probability she would turn the steak how much early/late. Nonetheless, she still cannot avoid mistakes when she cooks.

Our main result is the representation theorem below that establishes the relation between the axioms and the AMR.

Theorem 1.2.1 *An RCR ρ satisfies Axiom 1–6 and γ_a (γ_b) if and only if it is an AMR with a decreasing (increasing) ψ .*

The sufficiency of the theorem is not difficult. The necessity proof consists of three parts. First, we show that the preference we define satisfies the three standard vNM axioms. Thus, we can find the utility function defined for all lotteries that satisfies (1.2). Then, noting that the utility of a decision problem a could be different from $\sum_{p_i \in a} \rho(\{p_i\}, a)U(p_i)$, we show that the difference between $U(a)$ and $\sum_{p_i \in a} \rho(\{p_i\}, a)U(p_i)$ only depends on and is monotone in $|a|$ by *Preference for Lottery-Choice Swaps*. Thus, we obtain equation (1.3). Finally, we show that a richness assumption used in Gul, Natenzon and Pesendorfer (2014) is satisfied due to *Positivity, Continuity* and *Unboundedness*. Therefore, due to Gul, Natenzon and Pesendorfer’s Theorem 1, *Luce Independence* and richness imply the existence of a value function V that satisfies the Luce rule. Since U and V represent the same preference, we can find a strictly increasing function ϕ that converts U into V , which leads to our last equation (1.4).

The uniqueness result follows from the well-known fact that U is unique up to a positive affine transformation, and that Luce value function is unique up to a positive scalar multiplication. We omit the proof.

Proposition 1.2.1 *If both (U, ψ, ϕ) and $(\tilde{U}, \tilde{\psi}, \tilde{\phi})$ represent the RCR ρ , there exists $\alpha_1, \alpha_2 > 0$ and β such that $\tilde{U} = \alpha_1 U + \beta$, $\tilde{\psi} = \alpha_1 \psi$ and $\tilde{\phi}(u) = \alpha_2 \phi(\alpha_1 u + \beta)$.*

1.3 Measuring Error-Proneness and Choice Attitude

In an AMR, the function ϕ characterizes the decision maker's error-prone behavior, and the function ψ characterizes her aversion toward choice. In this section, we introduce measures for both of them.

1.3.1 Error-Proneness

In a related model in Ke (2015a), a measure of error-proneness is introduced. Suppose decision maker i has RCR ρ_i , $i = 1, 2$. In Ke's notion of error-proneness, ρ_2 is more error-prone than ρ_1 if there exists a function $h : (0, \frac{1}{2}] \rightarrow R_{++}$ such that $h(t) \leq t$, $h(\frac{1}{2}) = \frac{1}{2}$ and

$$\rho_1(\{x\}, \{x, y\}) = h(\rho_2(\{x\}, \{x, y\}))$$

for $\forall \{x, y\} \in D_1$ with $\rho_2(\{x\}, \{x, y\}) \leq \frac{1}{2}$; that is, ρ_1 chooses the worse option x with lower probability *through a function* h .

Ideally, a more general notion of error-proneness would be defined without h , which requires only that ρ_1 chooses the worse option with lower probability than ρ_2 ; that is,

$$\rho_1(\{x\}, \{x, y\}) \leq \rho_2(\{x\}, \{x, y\})$$

Below we introduce this more general notion of error-proneness. Recall that $p \in \mathcal{L}_1$ is a lottery that directly leads to an outcome, and a decision problem $a \in D_1$ is a

set of such lotteries; that is, confronting $a \in D_1$, the decision maker only needs to choose once to reach an outcome.

Definition 1.3.1 *RCR* ρ_2 is more error-prone than ρ_1 if for any $\{p, q\} \in D_1$ such that $\rho_2(\{p\}, \{p, q\}) \geq \rho_2(\{q\}, \{p, q\})$, $\rho_2(\{p\}, \{p, q\}) \leq \rho_1(\{p\}, \{p, q\})$.

The definition says, if decision maker 2 is more error-prone than decision maker 1, then whenever decision maker 2 reveals that she prefers p over q , decision maker 1 not only prefers p over q as well, but also chooses the preferred option p with higher probability. We focus on $\{p, q\} \in D_1$ to ensure that the key inequalities in the definition cannot be attributed to choice attitude. The result below relates our notion of error-proneness to ϕ .

Proposition 1.3.1 *Suppose each ρ_i is an AMR, $i = 1, 2$. Then ρ_2 is more error-prone than ρ_1 if and only if there exist (U_i, ψ_i, ϕ_i) 's representing ρ_i 's such that $U_1(p) = U_2(p)$ for all $p \in \mathcal{L}_1$, and $\frac{\phi_2(u)}{\phi_1(u)}$ is decreasing in u .*

Proof. Suppose $U_1(p) = U_2(p)$ for all $p \in \mathcal{L}_1$, and $\frac{\phi_2(u)}{\phi_1(u)}$ is decreasing in u . For any $p, q \in \mathcal{L}_1$ such that $\rho_2(\{p\}, \{p, q\}) \geq \rho_2(\{q\}, \{p, q\})$, i.e.,

$$\frac{\phi_2(U_2(p))}{\phi_2(U_2(p)) + \phi_2(U_2(q))} \geq \frac{\phi_2(U_2(q))}{\phi_2(U_2(p)) + \phi_2(U_2(q))}$$

we know that $U_2(p) \geq U_2(q)$. Define $u_h := U_1(p) = U_2(p)$ and $u_l := U_1(q) = U_2(q)$.

Since $\frac{\phi_2(u_h)}{\phi_1(u_h)} \leq \frac{\phi_2(u_l)}{\phi_1(u_l)}$, we have

$$\frac{\phi_1(u_h)}{\phi_1(u_h) + \phi_1(u_l)} \geq \frac{\phi_2(u_h)}{\phi_2(u_h) + \phi_2(u_l)}$$

as desired.

Now suppose we know that ρ_2 is more error-prone than ρ_1 . Since $\rho_i(\{p\}, \{p, q\}) \geq \rho_i(\{q\}, \{p, q\}) \iff U_i(p) \geq U_i(q)$, the hypothesis $\rho_2(\{p\}, \{p, q\}) \geq \rho_2(\{q\}, \{p, q\}) \Rightarrow \rho_2(\{p\}, \{p, q\}) \leq \rho_1(\{p\}, \{p, q\})$ then implies that

$$U_2(p) \geq U_2(q) \Rightarrow U_1(p) \geq U_1(q) \tag{1.5}$$

By the Corollary B.3 of Ghirardato, Maccheroni and Marinacci (2004), $U_1(p) = \alpha U_2(p) + \beta$ if $p \in \mathcal{L}_1$, for some $\alpha > 0$ and β . Due to Proposition 1.2.1, we can without loss of generality pick a U_2 such that $\alpha = 1$ and $\beta = 0$. Now for any $p, q \in \mathcal{L}_1$ such that $U_2(p) \geq U_2(q)$, similarly defining u_h and u_l , we must have $\frac{\phi_1(u_h)}{\phi_1(u_h) + \phi_1(u_l)} \geq \frac{\phi_2(u_h)}{\phi_2(u_h) + \phi_2(u_l)}$, which implies that $\frac{\phi_2(u_h)}{\phi_1(u_h)} \leq \frac{\phi_2(u_l)}{\phi_1(u_l)}$. ■

Equation (1.5) seems to allow for the case where $U_2(p) > U_2(q)$ but $U_1(p) = U_1(q)$. However, Ghirardato, Maccheroni and Marinacci (2004) show that when U_1 and U_2 are affine functions on a lottery space, (1.5) implies identical preferences (over that lottery space).

Notice that ϕ_1 and ϕ_2 are both strictly increasing, and are both unique up to positive scalar multiplications. Therefore, the value of $\phi_1(u)$, $\phi_2(u)$ or $\frac{\phi_2(u)}{\phi_1(u)}$ has no significance, because they can be changed freely by scalar multiplication. The result above implies that when comparing error-proneness, it is the monotonicity of $\frac{\phi_2(u)}{\phi_1(u)}$ that matters.

Our next proposition clarifies what the monotonicity of $\frac{\phi_2(u)}{\phi_1(u)}$ implies.

Proposition 1.3.2 *Suppose that (U_i, ψ_i, ϕ_i) represents ρ_i , $i = 1, 2$, $U_1(p) = U_2(p)$ for all $p \in \mathcal{L}_1$, and ϕ_i 's are differentiable. Then ρ_2 is more error-prone than ρ_1 if and only if $\frac{\phi_1(u)}{\phi_1'(u)} \leq \frac{\phi_2(u)}{\phi_2'(u)}$ for all u .*

Proof. We first prove the necessity. Continuing the notations in the previous proof, suppose $u_h \geq u_l$, and we have $\frac{\phi_2(u_h)}{\phi_1(u_h)} \leq \frac{\phi_2(u_l)}{\phi_1(u_l)}$. Now $\frac{\phi_1(u_h)}{\phi_1(u_l)} \geq \frac{\phi_2(u_h)}{\phi_2(u_l)} \Rightarrow \frac{\phi_1(u_h) - \phi_1(u_l)}{\phi_1(u_l)} \geq \frac{\phi_2(u_h) - \phi_2(u_l)}{\phi_2(u_l)}$, which in turn implies

$$\frac{[\phi_1(u_h) - \phi_1(u_l)]/(u_h - u_l)}{\phi_1(u_l)} \geq \frac{[\phi_2(u_h) - \phi_2(u_l)]/(u_h - u_l)}{\phi_2(u_l)}$$

Let u_h converge to u_l . We have $\frac{\phi_1'(u)}{\phi_1(u)} \geq \frac{\phi_2'(u)}{\phi_2(u)}$ for all u .

Conversely, consider $\frac{\phi_2(u)}{\phi_1(u)}$. We have

$$\begin{aligned} \frac{\partial(\phi_2(u)/\phi_1(u))}{\partial u} &= \frac{\phi_2'(u)\phi_1(u) - \phi_1'(u)\phi_2(u)}{\phi_1(u)^2} \\ &= \frac{\phi_2'(u)}{\phi_2(u)} \frac{\phi_2(u)}{\phi_1(u)} - \frac{\phi_1'(u)}{\phi_1(u)} \frac{\phi_2(u)}{\phi_1(u)} \\ &\leq 0 \end{aligned}$$

■

The proposition says if ρ_2 is more error-prone than ρ_1 , then the rate of change of the function ϕ_2 , $\frac{\phi_2'}{\phi_2}$, should be lower than that of ϕ_1 . Due to this proposition, it is natural to let $\frac{\phi(u)}{\phi'(u)} > 0$ be the *measure of error-proneness*. Then, a function ϕ exhibits constant measure of error-proneness if and only if

$$\phi(u) = e^{u/\lambda} \tag{1.6}$$

(up to positive scalar multiplications), for some $\lambda > 0$. When (1.6) holds, the measure of error-proneness is exactly λ . A higher λ implies more error-proneness. The measure of error-proneness could be very useful in game theory, as it is a natural tool to model players with different levels of skill.

1.3.2 Choice Attitude

Recall that the comparable lottery of a decision problem b is p_b . The lottery p_b and b are comparable in the sense that the probability that p_b assigns to each decision problem or outcome a is equal to the probability that the decision maker would confront a in the next stage if she chooses from b by herself.

The notion of choice aversion is simple. When a decision maker prefers a 's comparable lottery p_a over a itself, then the decision maker is averse to making choices.

Definition 1.3.2 *A decision maker is*

- (i) *choice-averse if $p_a \succeq \delta_a$, for all $a \in D$;*
- (ii) *choice-seeking if $\delta_a \succeq p_a$, for all $a \in D$;*
- (iii) *choice-neutral if $p_a \sim \delta_a$, for all $a \in D$.*

It follows immediately that a decision maker is choice-averse if and only if $\psi < 0$, is choice-seeking if and only if $\psi > 0$, and is choice-neutral if and only if $\psi = 0$.

Next, we compare the choice attitude of decision maker 1 and 2. Recall that D_0 is the set of possible outcomes, $p \in \mathcal{L}_1$ is a lottery over outcomes, and $a \in D_1$ is a set of such lotteries. If the decision maker chooses a lottery $p \in a \in D_1$, then she does not need to make any further choice to reach an outcome. A lottery $q \in \mathcal{L}_2$ is a

probability measure over decision problems in D_1 and outcomes in D_0 . Thus, if the lottery $q \in \mathcal{L}_2$ is chosen, the decision maker might expect to make one more choice to reach an outcome.

To fix error-proneness, suppose that decision maker 1 and 2 have the same choice behavior over any $a \in D_1$; that is, when no future choices are expected and hence choice attitude does not matter, they behave identically. Now consider the choice between $p \in \mathcal{L}_1$ and $q \in \mathcal{L}_2$ where p is a lottery that does not yield any future choices, and q is a lottery that yields at most one more choice in the future. If we observe that whenever decision maker 2 prefers q over p , decision maker 1 also reveals the same preference, then decision maker 1 is always less averse to making choices. In this case, we say that decision maker 2 is more choice-averse than decision maker 1. We say that ρ_1 and ρ_2 coincide on D_1 if $\rho_1(\cdot, a) = \rho_2(\cdot, a)$ for all $a \in D_1$. An RCR ρ_i induces preference \succeq_i .

Definition 1.3.3 *RCR ρ_2 is more choice-averse than ρ_1 if ρ_1 and ρ_2 coincide on D_1 , and $q \succeq_2 p$ implies $q \succeq_1 p$, for all $p \in \mathcal{L}_1$ and $q \in \mathcal{L}_2$.*

The proposition below is intuitive. Fixing other parameters, when ρ_2 is more choice-averse than ρ_1 , ψ_2 should be less than ψ_1 .

Proposition 1.3.3 *For AMRs ρ_1 and ρ_2 , ρ_2 is more choice-averse than ρ_1 if and only if there exist (U_i, ψ_i, ϕ_i) 's representing ρ_i 's respectively, $i = 1, 2$, such that (i) $U_1(p) = U_2(p)$ for all $p \in \mathcal{L}_1$, (ii) $\phi_1(u) = \phi_2(u)$ for all $u \in U_i(\mathcal{L}_1)$, and (iii) $\psi_2(t) \leq \psi_1(t)$.*

Fudenberg and Strzalecki (2015) is the first to introduce choice attitude to the choice theory literature. Our characterization of choice attitude is different from theirs in both the choice-theoretic definition and the representation. In terms of representation, in Fudenberg and Strzalecki, choice attitude is characterized by a specific function $\kappa \log |a|$, where a more choice-averse decision maker would have a lower κ . In our model, choice attitude is characterized by ψ , which can be any monotone function.

Fudenberg and Strzalecki's choice aversion, Ke's (2015) complexity aversion and our choice aversion are all related to the general idea that the decision maker might want to avoid larger problems. Our notion of choice attitude is similar to Fudenberg and Strzalecki's. However, both Fudenberg and Strzalecki's and our notions of choice attitude are different from Ke's complexity aversion. A decision maker with little complexity aversion in Ke's model would identify a decision problem with its best option. Fudenberg and Strzalecki's model and our model do not have this feature, even when the choice aversion disappears.

1.3.3 Characterizing the Logit Decision Model of Quantal Response Equilibrium

The results above can help us understand a widely-used decision model in the quantal response equilibrium literature (McKelvey and Palfrey (1995, 1998)). To begin with, we introduce the decision model in the quantal response equilibrium literature. In a game with quantal response, players make random mistakes, as in our model. The random choice model used in a quantal response equilibrium assumes that the

decision maker chooses an option p_i from the decision problem $a = \{p_1, \dots, p_n\}$ with probability

$$\rho(\{p_i\}, a) = \Pr[U(p_i) + \varepsilon_i \geq U(p_j) + \varepsilon_j, \forall j]$$

where each option p_i has a fixed utility $U(p_i)$, and the random error terms $\varepsilon_1, \dots, \varepsilon_n$ follow some probability distribution. Let us call this random choice rule the *quantal response rule* (QRR). When the error terms follow *i.i.d.* Gumbel distribution, the equation above becomes the most widely-used model, the logit QRR, in the quantal response equilibrium literature:

$$\rho(\{p_i\}, a) = \frac{\exp\{U(p_i)/\lambda\}}{\sum_j \exp\{U(p_j)/\lambda\}}$$

To first understand how AMR differs from QRR, note that McFadden's (1974) result implies that a QRR is a Luce rule if and only if it is a logit QRR. Thus, in general a QRR is not necessarily a Luce rule. In contrast, an AMR must be a Luce rule. However, AMR is *not* a special case of QRR. Relating McFadden's work to our model, we can also conclude that an AMR is a QRR if and only if $\phi(u) = e^{u/\lambda}$ for some $\lambda > 0$. In this case, AMR coincides with the logit QRR. In general, an AMR allows for other functional forms of ϕ . Therefore, QRRs and AMRs are two different classes of models that intersect at the case where $\phi(u) = e^{u/\lambda}$ for some $\lambda > 0$, and $\psi(\cdot) = 0$; that is, they intersect at the most widely-applied case, the logit QRR.

Corollary 1.3.1 *A random choice rule is both a QRR and an AMR if and only if it is a logit QRR.*

Now, due to our previous results, it follows immediately that the logit QRR is characterized by an AMR that exhibits neutrality to choice ($\psi(\cdot) = 0$) and constant measure of error-proneness.

Corollary 1.3.2 *A random choice rule is a logit QRR if and only if it is an AMR that is choice-neutral and has constant measure of error-proneness.*

Recall that the decision maker in our model rationally anticipates all her future random mistakes. If we treat the probabilities associated with lotteries as some other players' choice probabilities, then our decision maker not only understands her future strategy, but also her opponents' strategies. Indeed in McKelvey and Palfrey (1998), their decision maker also correctly anticipates future mistakes that are either made by the decision maker herself or by other players in dynamic games.

Our model could be useful in applications. The intersecting case between AMRs and QRRs, the logit QRR, is almost the only case of QRRs that has been used in either theoretical applications or empirical estimations, mostly because the logit QRR is a Luce rule and hence is highly tractable. Traditionally, a Luce rule is automatically paired with a logit model, due to McFadden (1974). Hence, our model enriches the useful Luce rule by allowing for other patterns of error-proneness, that is, by relaxing $\phi(u) = e^{u/\lambda}$ to arbitrary strictly increasing function ϕ . Further empirical research might be able to find out better ϕ functions rather than exponentials to describe a decision maker's error-prone behavior in games.

Unlike the QRRs that have been criticized as being neither falsifiable nor identifiable, when we have abundant choice data, AMRs are falsifiable by testing our axioms and are identifiable due to the uniqueness result in Proposition 1.2.1 (see

Haile, Hortaçsu and Kosenok (2008) for a discussion of QRR’s falsifiability and identifiability).

As argued in the Introduction, compared to QRRs, our model is more suitable for answering welfare-related questions and analyzing dynamic choice problems.

1.4 Risk from Mistakes vs. Standard Risk and Operational Risk Management

Our model relates two types of risks in a unified framework. The risk associated with lotteries is the standard type that has been studied for long, especially in the finance literature. The other type of risk, risk from mistakes, is due to the decision maker’s random error-prone behavior. Clearly, people make mistakes everyday. Economists have classified this second type of risk as an important case of the *operational risk* in one of the most influential banking regulations, Basel II.

Despite its importance and prevalence, it is not entirely clear how risk from mistakes differs from standard financial risk. In this section, we illustrate a fundamental difference between them. To focus on the understanding of error-proneness, we consider only AMRs with $\psi = 0$ (choice-neutral).

One of the most important features of risk from mistakes (modeled by AMRs) that differs from standard risk is that the model for the former might not be *monotonic*.

Definition 1.4.1 *An RCR ρ is monotonic if $p_1 \succeq q_1$ and $p_2 \succeq q_2$ implies $\{p_1, p_2\} \succeq \{q_1, q_2\}$.*

A lottery-version of monotonicity is satisfied in models of standard financial risk. Consider a lottery that with equal probability yields outcome w which has utility 2, and outcome x which has utility $-n$, where n is a large number. If we replace w with y that has utility 2.1, and replace x with z that has utility 1, clearly the new lottery is better, since in either case, the outcome is improved.

However, in a model of risk from mistakes, monotonicity does not necessarily hold. An extremely bad option x with utility $-n$ is obviously inferior than w . Hence, the decision maker chooses it with low probability. As a result, the ex ante expected utility that the decision maker gets from $\{w, x\}$ could be higher than $\{y, z\}$, even though $w \succeq y$ and $x \succeq z$.

Indeed in an AMR, suppose $\phi(u) = e^u$ and $\psi(\cdot) = 0$. When n becomes arbitrarily large,

$$\begin{aligned}
 U(\{\delta_w, \delta_x\}) &= U(w) = 2 \\
 &> U(\{\delta_y, \delta_z\}) \\
 &= \frac{\exp\{1\} + 2.1 \exp\{2.1\}}{\exp\{1\} + \exp\{2.1\}} \\
 &\approx 1.825
 \end{aligned}$$

It is easy to prove that in the logit case where $\phi(u) = e^{u/\lambda}$, monotonicity is always violated (see Ke (2015b) for a more detailed discussion). When $\phi(u) = e^{u/\lambda}$, $\lim_{u \rightarrow -\infty} u\phi(u) = 0$. More generally, the following result holds.

Proposition 1.4.1 *Suppose AMR ρ is represented by (U, ψ, ϕ) . If $\lim_{u \rightarrow -\infty} u\phi(u) = 0$, then ρ is not monotonic.*

Proof. This proposition is a corollary of Theorem 1 in Ke (2015b). ■

The condition $\lim_{u \rightarrow -\infty} u\phi(u) = 0$ describes how the decision maker behaves when facing extremely bad outcomes. If the decision maker has a ϕ that converges to 0 fast enough as u gets arbitrarily low, then the decision maker's random choice rule violates monotonicity.

This result has some implication on operational risk management. Suppose a manager asks an agent to perform a task on a machine. The agent could possibly hit button A that leads to w or button B that leads to x . The manager has the opportunity to "improve" the machine by replacing w with y and x with z at no cost. It seems that the manager of course should implement the improvement. However, according to our discussion above, whether or not the manager should improve the machine depends on the agent's error-proneness. If the consequence of hitting the inferior option is bad enough, it might in fact be better to leave the bad option as it is and not improve the machine.

The machine in the example is not as abstract as one might think. Every now and then, some traders from big banks place wrong orders by mistakenly hitting one more zero or so. These mistakes could induce huge loss to the banks. A bank can certainly set up some protection against these mistakes. For example, a bank could completely ban the traders' large orders that exceed some threshold. Our model suggests that such protection, seemingly good for operational risk management, might actually lead to more mistakes of choosing the inferior options. Depending on the error-proneness that can be estimated easily, restriction should be placed only to part of the agents.

1.5 Appendix

PROOF OF THEOREM 1.2.1: We prove the case that involves Axiom 1.2.7a and decreasing ψ only. We first show the sufficiency. An AMR is a Luce rule according to (1.4), and hence satisfies *Luce Independence*. In a Luce rule, $p \succeq q$ if and only if $U(p) \geq U(q)$ and hence $\phi(U(p)) \geq \phi(U(q))$, as ϕ is strictly increasing. According to the definition, ϕ 's image is \mathbb{R}_{++} , thus *Positivity* holds. *vNM Independence* and *Continuity* are satisfied according to equation (1.2) and the fact that the Luce value of each p is $\phi(U(p))$ in which ϕ is continuous. Because $\psi(1) = 0$, $U(\delta_{\{p\}}) = U(\{p\}) = U(p)$, *Degenerate-Choice Indifference* is satisfied. Since ϕ is surjective, *Unboundedness* holds. As for *Preference for Lottery-Choice Swaps*, we have for $a = \{p_1, \dots, p_n\}$, $b = \{q_1, \dots, q_m\}$, $n \geq m$. Since $p_a(b) = \sum_{i=1}^n \rho(\{p_i\}, a) \times p_i(b)$,

$$\begin{aligned} U(p_a) &= \sum_{c \in \text{supp}(p_a)} \left[\left(\sum_{i=1}^n \rho(\{p_i\}, a) p_i(c) \right) \times U(c) \right] \\ &= \sum_{i=1}^n \rho(\{p_i\}, a) U(p_i) \end{aligned}$$

Thus

$$\begin{aligned} U\left(\frac{1}{2}p_a + \frac{1}{2}\delta_b\right) &= \frac{1}{2}U(p_a) + \frac{1}{2}U(b) \\ &= \frac{1}{2}U(p_a) + \frac{1}{2}[U(p_b) + \psi(|b|)] \\ &\geq \frac{1}{2}[U(p_a) + \psi(|a|)] + \frac{1}{2}U(p_b) \\ &= U\left(\frac{1}{2}p_b + \frac{1}{2}\delta_a\right) \end{aligned}$$

Next we show the necessity.

Lemma 1.5.1 \succeq is complete and transitive.

Proof. From *Luce Independence*, we know that \succeq is complete because for any $\rho(\{p\}, \{p\} \cup a)$ and $\rho(\{q\}, \{q\} \cup a)$ such that $p, q \notin a$, the former is either greater than or less than the latter. Say it's greater. *Luce Independence* implies that $\rho(\{p\}, \{p\} \cup b) \geq \rho(\{q\}, \{q\} \cup b)$ for any $b \in D$ such that $p, q \notin b$, and hence $p \succeq q$.

To show transitivity, suppose $\rho(\{p\}, \{p\} \cup a) \geq \rho(\{q\}, \{q\} \cup a)$, and $\rho(\{q\}, \{q\} \cup b) \geq \rho(\{r\}, \{r\} \cup b)$, $p, q, r \in \mathcal{L}$, $p, q \notin a$, $q, r \notin b$. If any two of p, q, r are the same object, clearly we have $p \succeq r$. Otherwise by *Luce Independence*, we know that $\rho(\{p\}, \{p, r\}) \geq \rho(\{q\}, \{q, r\})$ and $\rho(\{q\}, \{p, q\}) \geq \rho(\{r\}, \{p, r\})$. Due to *Positivity*, $\rho(\{r\}, \{p, r\}) > 0$. Thus by *Unboundedness*, we can find an $r' \in \mathcal{L}$ such that

$$\rho(\{r'\}, \{p, r'\}) < \rho(\{r\}, \{p, r\})$$

It is then clear that r' is distinct from p, q, r . Now we have $\rho(\{p\}, \{p, r'\}) \geq \rho(\{q\}, \{q, r'\}) \geq \rho(\{r\}, \{r, r'\})$, and hence $p \succeq r$. ■

Lemma 1.5.2 For $p, q, r \in \mathcal{L}$, $p \succ q \succ r$ implies that there exist $\alpha, \beta \in (0, 1)$ such that $\alpha p + (1 - \alpha)r \succ q \succ \beta p + (1 - \beta)r$.

Proof. Following the similar argument in the previous lemma, we can find an r' such that $\rho(\{p\}, \{p, r'\}) > \rho(\{q\}, \{q, r'\}) > \rho(\{r\}, \{r, r'\})$, where $r' \neq p, q, r$. Now by *Continuity*, since $p = 1 \cdot p + 0 \cdot r$, we can find an α near 1 such that $\rho(\{\alpha p + (1 - \alpha)r\}, \{\alpha p + (1 - \alpha)r, r'\}) > \rho(\{q\}, \{q, r'\})$. Notice that we can require

$\alpha p + (1 - \alpha)r$ to be distinct from r' since if $\alpha p + (1 - \alpha)r = r'$, we can find another α in the neighborhood, which has to be different from r' . Similar argument applies to β . ■

The lemma above shows that the vNM continuity is satisfied by the stochastic preference \succeq . Knowing that \succeq is complete and transitive, and satisfies *vNM Independence* and vNM continuity, by the mixture space theorem and a simple induction statement, we know that there exists a function $\tilde{U} : \mathcal{L} \rightarrow \mathbb{R}$ such that $\tilde{U}(p) = \sum_{a \in \text{supp}(p)} p(a)\tilde{U}(\delta_a)$. Define a function $U : \mathcal{L} \cup D \rightarrow \mathbb{R}$ such that $U(a) := \tilde{U}(\delta_a)$, if $d \in D$, and $U(p) := \tilde{U}(p)$, we have equation (1.2).

Now for each $a \in \mathcal{D}$, let $\tilde{\psi}(a) := \tilde{U}(\delta_a) - \tilde{U}(p_a)$. By Axiom 1.2.7a, *Preference for Lottery-Choice Swaps*, consider any $a, b \in \mathcal{D}$ such that $|a| = |b|$, we have $\frac{1}{2}p_a + \frac{1}{2}\delta_b \sim \frac{1}{2}p_b + \frac{1}{2}\delta_a$. Thus

$$\begin{aligned} \tilde{U}\left(\frac{1}{2}p_a + \frac{1}{2}\delta_b\right) &= \frac{1}{2}\tilde{U}(p_a) + \frac{1}{2}\tilde{U}(\delta_b) \\ &= \frac{1}{2}\tilde{U}(p_a) + \frac{1}{2}\tilde{U}(p_b) + \tilde{\psi}(b) \\ &= \tilde{U}\left(\frac{1}{2}p_b + \frac{1}{2}\delta_a\right) \\ &= \frac{1}{2}\tilde{U}(p_b) + \frac{1}{2}\tilde{U}(p_a) + \tilde{\psi}(a) \end{aligned}$$

Since the equations above holds for any a and b such that $|a| = |b|$, we know that there exists a function such that $\psi(|a|) = \tilde{\psi}(a)$. We have used $U(a)$ to denote $\tilde{U}(\delta_a)$, and hence we have equation (1.3) established.

For $|a| \geq |b|$, $\frac{1}{2}p_a + \frac{1}{2}\delta_b \succeq \frac{1}{2}p_b + \frac{1}{2}\delta_a$, which implies

$$\begin{aligned} \tilde{U}\left(\frac{1}{2}p_a + \frac{1}{2}\delta_b\right) &= \frac{1}{2}\tilde{U}(p_a) + \frac{1}{2}\tilde{U}(p_b) + \psi(|b|) \\ &\geq \tilde{U}\left(\frac{1}{2}p_b + \frac{1}{2}\delta_a\right) \\ &= \frac{1}{2}\tilde{U}(p_a) + \frac{1}{2}\tilde{U}(p_b) + \psi(|a|) \end{aligned}$$

and thus $\psi(|a|)$ is decreasing. $\psi(1) = 0$ can be easily seen by applying the equation (1.3) to *Degenerate-Choice Indifference* axiom.

Finally, we show that equation (1.4) holds and ϕ 's properties as stated are satisfied.

Lemma 1.5.3 *For any $a \in \mathcal{D}$, $\alpha \in (0, 1)$ there exists infinitely many $p \in \mathcal{L}$ such that $\rho(\{p\}, \{p\} \cup a) = \alpha$.*

Proof. By *Unboundedness*, there exists q_1 and r_1 such that $\rho(\{q_1\}, \{q_1\} \cup a) > \alpha$ and $\rho(\{r_1\}, \{r_1\} \cup a) < \alpha$. By *Continuity* and the Intermediate Value Theorem, we can find β_1 such that $\rho(\{\beta_1 q_1 + (1 - \beta_1)r_1\}, \{\beta_1 q_1 + (1 - \beta_1)r_1\} \cup a) = \alpha$. Let $p_1 := \{\beta_1 q_1 + (1 - \beta_1)r_1\}$.

Now consider $\text{supp}(q_1)$ and $\text{supp}(r_1)$. We can find $b \in \text{supp}(q_1) \cup \text{supp}(r_1)$ such that $\delta_b \succeq \delta_d$ for all $d \in \text{supp}(q_1) \cup \text{supp}(r_1)$, and $c \in \text{supp}(q_1) \cup \text{supp}(r_1)$ such that $\delta_d \succeq \delta_c$ for all $d \in \text{supp}(q_1) \cup \text{supp}(r_1)$. Applying *Unboundedness* again, we can find a lottery q_2 such that $1 > \rho(\{q_2\}, \{q_2\} \cup a) > \rho(\{\delta_b\}, \{\delta_b\} \cup a)$. We must have $1 > \rho(\{q_2\}, \{q_2\} \cup a)$ because of *Positivity*. It should be clear that $\text{supp}(q_2) \neq \text{supp}(q_1), \text{supp}(r_1)$. Similarly, we can find r_2 such that $0 < \rho(\{r_2\}, \{r_2\} \cup a) < \rho(\{\delta_c\}, \{\delta_c\} \cup a)$, and hence $\text{supp}(r_2) \neq \text{supp}(q_1), \text{supp}(r_1)$. By *Continuity*, we can

find $\beta_2 \in (0, 1)$ such that $\rho(\{p_2\}, \{p_2\} \cup a) = \alpha$, where $p_2 := \beta_2 q_2 + (1 - \beta_2) r_2$. Clearly p_2 is distinct from p_1 since both q_2 and r_2 have different supports from q_1 and r_1 .

We can recursively continue this process. Each q_i and r_i will invite new elements to their supports, and hence generate countably infinitely many distinct p_i such that $\rho(\{p_i\}, \{p_i\} \cup d) = \alpha$. ■

From the lemma above together with *Luce Independence*, applying Theorem 1 of Gul, Natenzon and Pesendorfer (2014), we know that there exist a surjective function $V : \mathcal{L} \rightarrow \mathbb{R}_{++}$ such that $p \succeq q$ if and only if $V(p) \geq V(q)$, and for any $a = \{p_1, \dots, p_n\}$, $\rho(\{p_i\}, a) = \frac{V(p_i)}{\sum_{j=1}^n V(p_j)}$.

Since both U and V represent \succeq on \mathcal{L} , we know that there exists a strictly increasing function $\phi : U(\mathcal{L}) \rightarrow \mathbb{R}_{++}$ such that $V(p) = \phi(U(p))$. Since V is surjective, ϕ must be surjective too. It's not difficult to see that $U(\mathcal{L}) = U(\mathcal{L} \cup D)$. Finally, because of *Continuity*, ϕ has to be continuous.

■

PROOF OF PROPOSITION 1.3.3: When (U_i, ψ_i, ϕ_i) 's represent ρ_i 's respectively, $i = 1, 2$, and (i) (ii) and (iii) hold, it is clear that $U_2(q) \leq U_1(q)$ and $U_2(p) = U_1(p)$ for all $p \in \mathcal{L}_1, q \in \mathcal{L}_2$. Hence, whenever $U_2(q) \geq U_2(p)$, $U_1(q) \geq U_1(p)$ too, for all $p \in \mathcal{L}_1, q \in \mathcal{L}_2$.

Now suppose ρ_2 is more choice-averse than ρ_1 . First, note that $\mathcal{L}_1 \subset \mathcal{L}_2$, and consider $p, q \in \mathcal{L}_1$. The hypothesis in the definition of choice-aversion comparison then implies that $p \succeq_2 q \Rightarrow p \succeq_1 q$, for all $p, q \in \mathcal{L}_1$. Again by the Corollary B.3 of

Ghirardato, Maccheroni and Marinacci (2004), we know that

$$U_1(p) = \alpha_1 U_2(p) + \beta$$

if $p \in \mathcal{L}_1$, for some $\alpha_1 > 0$ and β . We pick $\alpha_1 = 1$ and $\beta = 0$. Next, since $\rho_1(\cdot, a) = \rho_2(\cdot, a)$ for all $a \in D_1$, it is not difficult to see that $\phi_2(u) = \alpha_2 \phi_1(u)$ by applying (1.4), for all $u \in U_i(\mathcal{L}_1)$. We can pick $\alpha_2 = 1$. Finally, consider $\delta_a \in \mathcal{L}_2 \setminus \mathcal{L}_1$ such that $a \in D_1$. Since $\delta_a \succeq_2 p$ implies $\delta_a \succeq_1 p$, for $p \in \mathcal{L}_1$,

$$\begin{aligned} U_2(\delta_a) &= U_2(a) \\ &= \sum_{p_i \in \mathcal{L}_1} \rho_2(\{p_i\}, a) U_2(p_i) + \psi_2(|a|) \\ &= \sum_{p_i \in \mathcal{L}_1} \rho_1(\{p_i\}, a) U_1(p_i) + \psi_2(|a|) \\ &\leq U_1(\delta_a) \\ &= \sum_{p_i \in \mathcal{L}_1} \rho_1(\{p_i\}, a) U_1(p_i) + \psi_1(|a|) \end{aligned}$$

Thus, $\psi_1(t) \geq \psi_2(t)$, for all $t \in \mathbb{N}$.

■

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Chapter 2

Boundedly Rational Backward Induction

2.1 Introduction

Rubinstein (1990) identifies backward induction as one topic in decision theory and game theory that is most in need of a model of bounded rationality. In dynamic economic problems, economists use solutions suggested by fully rational backward induction to predict a decision maker's behavior. However, it is well-acknowledged that such solutions would have less predicting power when the problems are more complicated. Indeed, fully rational backward induction implies a non-losing strategy for one of the players in chess, which is clearly misleading.

To see how we can relax fully rational backward induction, consider a decision maker who needs to make a series of choices to reach an outcome. Her valuation

for outcomes is fixed, and she fully understands the structure of the choice problem. We can describe such a choice situation by a *decision tree*. Fully rational backward induction has two implications here. First, the value of a decision tree is equal to the maximum of its subtree values. Second, the decision maker chooses the best subtree with certainty. Hence, a boundedly rational model of backward induction could relax either implication or both.

In particular, we want to propose a relaxation of fully rational backward induction that addresses the following question: what value should be assigned to a decision tree if the decision maker does not know how she will choose at the future stages. The value of the tree should still depend on its outcomes, but how should they be related? This issue appears in Jéhiel (1995) for example, where the decision maker can only look forward j stages in a decision tree. Since the decision tree has more than j stages, the question is what values should be assigned at the end of the j stages. Beyond them, the decision maker is agnostic about her own future behavior. Therefore, we need a model of how the decision maker behaves in this situation.

We formulate a coherent model of a decision maker who makes mistakes and is agnostic about (cannot predict) her own future choice behavior by interpreting the decision maker's choice as random. Random mistakes provide a reason why the decision maker is unable to know her future behavior even though she fully understands the structure of the decision tree.

Our resulting model deviates from fully rational backward induction in its both components. First, to evaluate a decision tree, instead of the maximum, the decision maker has a general aggregating function that aggregates the subtree values. The

aggregating function could be interpreted as if the decision maker assigns a uniform prior to the subtrees, which reflects her lack of understanding of her future choice behavior. Second, the decision maker makes random mistakes when choosing among the subtrees. The first departure enables us to identify a measure of *complexity aversion*, and the second enables us to identify a measure of *error-proneness*.

Our goal is not to model specific heuristics for a particular class of problems nor to study the decision maker’s actual reasoning process. Rather, we present a framework for analyzing how the decision maker’s choices may vary with the presentation of the decision problem; that is, how changes further down the decision tree affect the decision maker’s choice at a current decision node. As we show in Section 4 below, our model provides new explanations for the menu effect and framing effect in the context of product assortment and advertising problems.

Our model is derived from simple axioms on the decision maker’s choices. The primitive is a *random choice rule* that describes how the decision maker chooses among the available subtrees in any finite decision tree. Decision trees are defined recursively: depth-1 decision trees are finite sets of outcomes; depth-2 decision trees are finite sets consisting of outcomes and depth-1 decision trees and so on. A typical decision tree $a = \{a_1, \dots, a_n\}$ is a set of subtrees. Implicitly, we assume that the modeler can observe the decision tree and can observe the decision maker’s behavior in a variety of decision problems repeatedly.

We present axioms that relate how the decision maker chooses in some decision tree $a = \{a_1, \dots, a_n\}$ to how she would have chosen in each of the subtrees a_1, \dots, a_n . If a decision maker chooses a more often from $\{a, d_1, \dots, d_n\}$ than b from

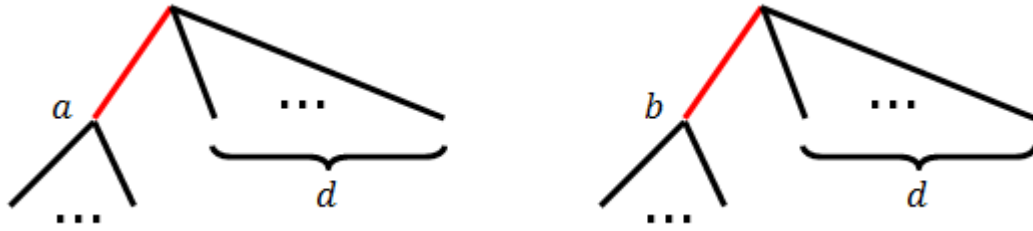


Figure 2.1: If for any d , the decision maker chooses a more often in the left-hand-side decision tree than b in the right-hand-side decision tree, then the decision maker reveals statistically that she prefers a to b .

$\{b, d_1, \dots, d_n\}$ for all d_1, \dots, d_n ; that is, if

$$P(\{a\}, \{a, d_1, \dots, d_n\}) \geq P(\{b\}, \{b, d_1, \dots, d_n\})$$

for all d_1, \dots, d_n , we say that the decision maker prefers a to b (see Figure 2.1). This terminology is appropriate: an error-prone decision maker cannot reveal her preference deterministically but can reveal it statistically. Our first axiom, *Independence*, requires the consistency of this preferences. It allows us to identify a complete preference relation from the decision maker's error-prone choices.

The second axiom *Dominance* states that the decision maker prefers a to b if and only if she prefers $\{a, d_1, \dots, d_n\}$ to $\{b, d_1, \dots, d_n\}$. *Independence* and *Dominance* together allow us to identify the decision maker's unchanging true objectives from her imperfect attempts at achieving them. The next two axioms describe the manner in which our decision maker can depart from rationality.

Stochastic Set Betweenness requires that if the decision maker prefers a to b and a, b are disjoint, then $a \cup b$ must be ranked between a and b in her preference. A

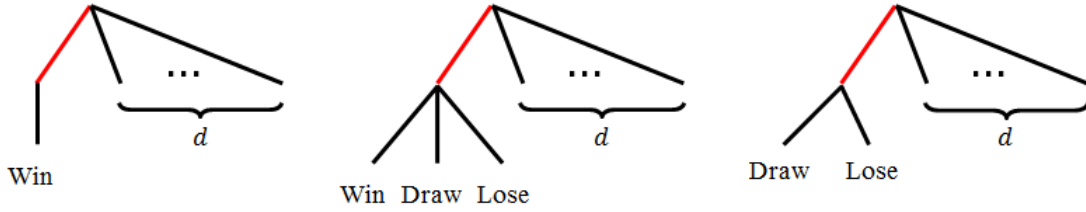


Figure 2.2: *Stochastic Set Betweenness* implies that the decision maker would choose $\{win\}$ more often in the first decision tree than $\{win, draw, lose\}$ in the second, which is in turn chosen more often than $\{draw, lose\}$ in the last decision tree.

fully rational decision maker would prefer a to b if and only if the best option in a is better than the best option in b , in which case $a \cup b$ and a would have the same best option and hence would be indifferent. *Stochastic Set Betweenness* relaxes this last requirement to capture the fact that the decision maker will make mistakes when making her choices and her choices reveal some awareness of her own error-proneness.

Figure 2.2 provides an example of this axiom.

Finally, the key axiom of this chapter, *Preference for Accentuating Swaps* implies that if the decision maker prefers b to a , then she will prefer $\tilde{d} = \{\{b, c_1, \dots, c_m\}, \{a, d_1, \dots, d_n\}\}$ to $d = \{\{a, c_1, \dots, c_m\}, \{b, d_1, \dots, d_n\}\}$ as long as $m \leq n$. To see what this means, note that in the decision problem d , the subtree a is more visible than the subtree b , because a in d is presented at a smaller simpler subtree, and b in d is presented at a larger subtree. In \tilde{d} , the places of a and b are reversed. Hence, \tilde{d} renders the better subtree more visible while d emphasizes the inferior subtree a . Accentuating the better subtree in this fashion improves the original tree and increases the probability that the decision maker chooses it. For a concrete example of this axiom, see Figure 2.3.

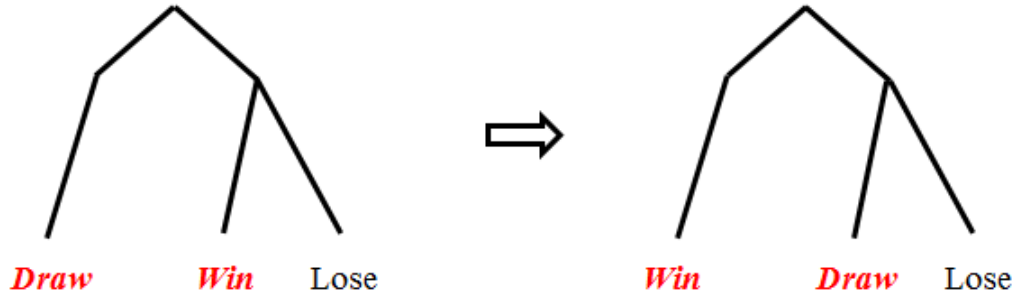


Figure 2.3: The left-hand-side decision tree is swapped into the right-hand-side one. After the swap, *win* becomes more salient and *draw* becomes less so. Thus the decision maker would choose the right-hand-side decision tree with higher probability.

Theorem 2.2.1 establishes that these four axioms, together with other technical conditions, yield the following representation of the random choice rule: there exists a *value function* V on the set of all decision subtrees such that

$$P(\{a_i\}, a) = \frac{V(a_i)}{\sum_{j=1}^n V(a_j)}$$

where $a = \{a_1, \dots, a_n\}$. Thus, the random choice rule P is a Luce rule (see Luce (1959)). Subtrees with higher Luce values would be chosen more often. Moreover, there is an *aggregator* f such that V satisfies

$$V(a) = f^{-1} \left(\frac{1}{n} \sum f(V(a_j)) \right) \tag{2.1}$$

for all $a = \{a_1, \dots, a_n\}$. The aggregating function in (2.1) relates decision tree a 's value to its subtree values. Intuitively, this aggregating function is a general notion of

average, which ranges from the maximum to the minimum.¹ The uniform weight $1/n$ is interpreted as the decision maker's equal attention to each subtree, and it captures the fact that the decision maker is agnostic about her future choice behavior. We call a random choice rule that has the representation described above a *Boundedly-Rational Backward-Induction Rule* (BBR). An alternative interpretation of BBR in which the decision maker receives noisy signals about the subtree values is provided in Section 4.2.

The two parameters V and f quantify the extent to which the decision maker's behavior differs from that of a fully rational decision maker. To see this, first consider the choice between an outcome and a decision tree. Outcomes are the simplest choice objects in our setting. For two decision makers, if decision maker 2 always chooses the outcome more often than decision maker 1 facing the same problem, then decision maker 2 is said to be more complexity-averse than decision maker 1. In Theorem 2.3.1, we show that f_2 is a concave transformation of f_1 if and only if BBR (V_2, f_2) is more complexity-averse than (V_1, f_1) in this sense.

Next, consider a decision maker defined by BBR (V_1, f) and another one defined by (V_2, g) such that for some $\lambda \in (0, 1)$, $[V_1(x)]^\lambda = V_2(x)$ for any outcome x . Note that the two decision makers have the same ranking of outcomes; that is, given any binary depth-1 decision problem $\{x, y\}$, the first decision maker chooses x more often than y if and only if the second one chooses x more often than y . However, the second decision maker makes more mistakes; that is, in any binary decision problem, she

¹This average is called *Kolmogorov-Nagumo* mean.

chooses the preferred outcome less often. Theorem 2.3.2 extends this observation to develop a comparative measure of error-proneness.

With the measure of complexity aversion and error-proneness, we identify limiting cases of BBR. In particular, fully rational backward induction is the limiting cases of BBR where both complexity aversion and error-proneness disappear. We also show through examples how complexity aversion and error-proneness interact. We show that in some decision problems, they are complements, and in some others, they are substitutes.

In Section 4, we present BBR's implications in dynamic choice problems. In the context of product assortment problems, we show a menu effect induced by BBR. We find that consistent with the empirical findings, reducing a store's product assortment properly could increase the store's sales. In particular, if the value of a product is below some threshold, it is better for the store to exclude it from the product assortment.

Fixing the set of outcomes to be presented, we analyze two simple presentation strategies to study the framing effect induced by BBR. The first strategy repeats an outcome in multiple subtrees, which we call the *strategy with recurrence*, and the second strategy singles out an outcome from many others, which we call the *strategy with emphasis*. These two strategies resemble some common features of advertising. We first show that both strategies outperform our benchmark case (presenting all the outcomes together). Then we show that when the decision maker's complexity aversion is above some threshold, the strategy with emphasis does better than the

strategy with recurrence. Conversely, when the consumer’s complexity aversion is sufficiently low, the strategy with recurrence dominates the strategy with emphasis.

2.1.1 Related Literature

Jéhiel (1995) examines the implication of limited foresight in a special class of repeated games. In his model, the agents can only look forward j steps. To obtain the decision maker’s value function beyond the j^{th} step, Jéhiel equips the agents with a particular heuristic, the average payoff from the j steps. The heuristic is reasonable for the games he studies. Gabaix and Laibson (2005) study a reasoning procedure where the decision maker evaluates the alternatives as if the game ends right away. Based on this heuristic, the procedure determines the optimal number of steps that the decision maker looks forward endogenously. Our work does not focus on a specific heuristic; rather, we aim to identify a model that can be applied to all finite decision trees from simple and general axioms.

Theorem 1 of Gul, Natenzon and Pesendorfer (2014) establishes that when the choice environment is rich enough, the Luce rule is the only random choice rule that satisfies *Independence*. The richness assumption is a random-choice version of the Savage’s small event continuity. Our model incorporates this axiom, and extends the Luce rule to model how changes further down a decision tree would affect the decision tree’s Luce value. Gul, Natenzon and Pesendorfer also study dynamic choice. The decision maker in their model can identify all the duplicates and treat the duplicates as a single choice object. In our case, duplicates should not be treated as a single choice object since the decision maker makes random mistakes when choosing. If

a choice problem consists of more duplicates of some choice object, then the other alternatives should have smaller probability to be chosen.

An axiom similar to our *Stochastic Set Betweenness* was first proposed by Bolker (1966). He uses it to propose a generalization of expected value. His analysis deals neither with random choice nor with dynamic problems. Gul and Pesendorfer (2001) propose a stronger axiom *Set Betweenness* to model temptation and self-control. In their model, a decision maker might prefer a smaller choice set to a larger one because the larger one contains tempting options. In our model, a smaller choice set might be preferred because it's simpler. Their axiom applies to the case with nonempty intersection, while ours does not.

Fudenberg and Strzalecki (2015) formulate an alternative extension of the Luce's random choice model to dynamic problems. In their model, a choice problem is a set of current-period choices. Each current-period choice yields current-period consumption and a choice problem for the next period. The utility of a current period choice has three components: a deterministic utility derived from backward induction, a random component reflecting possible taste shocks and a term that depends only on the number of alternatives available in the next period. This last term, when the relevant coefficient is positive, reflects the decision maker's choice aversion. When the coefficient is negative, the term captures a preference for flexibility beyond the option value associated with the continuation choice problem.

Fudenberg and Strzalecki introduce the notion of choice aversion, and one of their main findings is that choice aversion is associated with a preference for delaying decisions. Axiom 2.2.5 of our model rules out the type of preference for delay

that Fudenberg and Strzalecki consider. Fudenberg and Strzalecki do not restrict attention to boundedly rational backward induction, nor is their model derived from axioms that describe how choice frequencies may vary with the presentation of decision problems. The decision maker in their model fully understands her future choices, while in ours, the decision maker does not.

2.2 Model

In our model, a decision maker makes a series of choices to reach an outcome. A *decision tree* describes this choice situation. Let D_0 be the set of possible outcomes. A depth-1 decision tree is a nonempty finite subset of D_0 . When the decision maker confronts a depth-1 decision tree $a \subset D_0$, she chooses an outcome $x \in a$ from it. Let $D_1 := K(D_0)$ be the set of all depth-1 decision trees, where $K(\cdot)$ denotes the collection of all nonempty finite subsets of a set. Recursively, we define the set of depth- k decision trees as $D_k := K(D_{k-1} \cup D_0)$. Let $D := \bigcup_{i=1}^{\infty} D_i$ be the set of all decision trees. A decision tree $a \in D$ is a finite set of *subtrees*. A subtree could either be an outcome or itself a decision tree. Let $\mathcal{D} := D \cup D_0$ denote the set of all decision subtrees.

Confronting a decision tree $b \in D$, the decision maker chooses among b 's subtrees with randomness. Let \mathcal{L} be the set of finite-support probability measures on \mathcal{D} endowed with the topology of weak convergence. The probability measure $P(b) \in \mathcal{L}$ describes the choice probability that each subtree of b is chosen. With some abuse of notation, we use $P(a, b)$ to denote the probability that $P(b)$ assigns to the set $a \in D$;

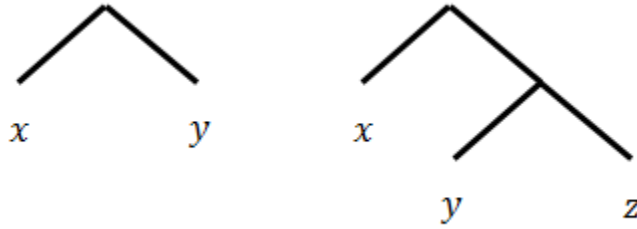


Figure 2.4: The left-hand-side decision tree is a depth-1 decision tree $\{x, y\} \in D_1$. The right-hand-side one is a depth-2 decision tree $\{x, \{y, z\}\} \in D_2$, where x, y, z are outcomes.

that is, the probability that any subtree in a is chosen when the decision tree is b . We call the function $P : D \rightarrow \mathcal{L}$ a *random choice rule* (RCR) if $P(a, a) = 1$ for all $a \in D$.

The decision maker's choice deviates from the optimal choice implied by fully rational backward induction. Fully rational backward induction implies that (i) the decision maker evaluates a tree by its best subtree, and (ii) she always chooses the best subtree with certainty. Here, we have in mind a decision maker who (i) does not identify a tree with its best subtree, and (ii) makes random mistakes when choosing. To obtain such a choice model, we consider the following simple axioms on the random choice rule.

The first axiom is from Gul, Natenzon and Pesendorfer (2014). In our context, it imposes some independence property on the way that the decision maker makes mistakes.

Axiom 2.2.1 (*Independence*) For $a, b, c, d \in D$ such that $(a \cup b) \cap (c \cup d) = \emptyset$, $P(a, a \cup c) \geq P(b, b \cup c)$ implies $P(a, a \cup d) \geq P(b, b \cup d)$.

One role of this axiom is to uncover the decision maker's underlying preference, despite her random choice mistakes. The decision maker might prefer subtree a to b , but she cannot reveal her preference deterministically due to the random mistakes. However, if we observe that the decision maker always chooses a over d more often than b over d for all d that does not intersect a and b , then the decision maker reveals statistically that she prefers a to b .

Definition 2.2.1 For any $a, b \in \mathcal{D}$, we say that the decision maker prefers a to b (and write $a \succeq b$) if $P(\{a\}, \{a\} \cup d) \geq P(\{b\}, \{b\} \cup d)$ for all $d \in \mathcal{D}$ such that $a, b \notin d$.

Independence guarantees that we can uncover a complete preference from the decision maker's random choice. Then, the remaining axioms are imposed on the uncovered preference.

The next axiom is a monotonicity assumption. It ensures that the decision maker's preference is not changing over time. In particular, it states that replacing a subtree with a better one makes the decision tree itself better. This axiom rules out temptation and related phenomena that have been studied extensively in the literature following Strotz's (1955) work.

Axiom 2.2.2 (*Dominance*) For $a = \{a_1, a_2, \dots, a_n\}$, $a' = \{a'_1, a_2, \dots, a_n\}$, $a_1 \succeq a'_1$ implies $a \succeq a'$, and $a_1 \succ a'_1$ implies $a \succ a'$.

The first part of *Dominance* ($a_1 \succeq a'_1$ implying $a \succeq a'$) is also satisfied by a fully rational decision maker. The second part ($a_1 \succ a'_1$ implying $a \succ a'$) incorporates some departure from the fully rational behavior. Suppose $d = \{a, b\}$ and $d' = \{a, c\}$, where $a \succ b \succ c$. A fully rational decision maker would be indifferent between d



Figure 2.5: *Dominance* implies that the choice probability of a is higher than a' if and only if a_1 is preferred to a'_1 .

and d' since they have the same best subtree. In contrast, *Dominance* implies that $d \succ d'$; that is, our decision maker has some awareness of her own error-proneness and more often avoids decision trees with inferior subtrees.²

The two axioms below encapsulate our model of complexity-averse and error-prone decision making. The first, *Stochastic Set Betweenness*, considers two decision trees a and b that have no subtree in common. For example, suppose a is $\{win\}$, b is $\{draw, lose\}$ and the decision maker prefers $\{win\}$ over $\{draw, lose\}$. *Stochastic Set Betweenness* requires that $\{win\}$ is chosen more often than $\{win, draw, lose\}$ which in turn is chosen more often than $\{draw, lose\}$ (see Figure 2.6).

Axiom 2.2.3 (*Stochastic Set Betweenness*) For all $a, b \in D$, $a \cap b = \emptyset$ and $a \succeq b$ imply $a \succeq a \cup b \succeq b$.

When $a \succeq b$, a fully rational decision maker should be indifferent between a and $a \cup b$ since they both contain the same best subtree from a . *Stochastic Set Betweenness* allows the decision maker to strictly prefer a over $a \cup b$, reflecting her aversion to

²However, the decision maker does not have full awareness of her error-prone behavior. See a discussion in Section 5.

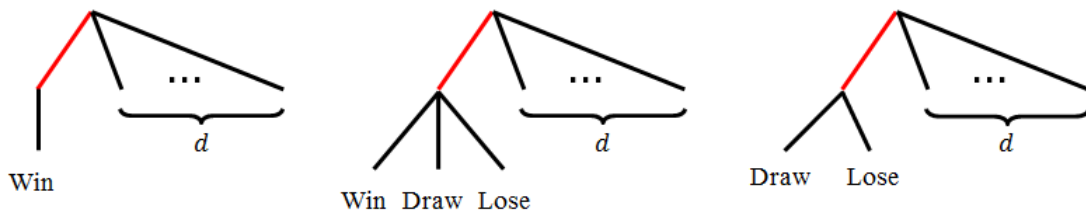


Figure 2.6: *Stochastic Set Betweenness* implies that the decision maker would choose $\{win\}$ more often in the first decision tree than $\{win, draw, lose\}$ in the second, which is in turn chosen more often than $\{draw, lose\}$ in the last decision tree.

more complex decision trees. In the literature, Bolker (1966) is the first to use this type of condition to derive a generalization of expected value.³ Gul and Pesendorfer (2001) use a related axiom to model temptation. Our axiom is weaker than the Gul-Pesendorfer version since we require that a and b have empty intersection. To see why this is important, assume that $a = \{win, lose\}$, $b = \{win, lose^*\}$ where $lose$ and $lose^*$ are two similar unattractive outcomes. If the decision maker struggles with complex decision trees, then it may well be that $\{win, lose, lose^*\}$ is worse than both $\{win, lose\}$ and $\{win, lose^*\}$. Therefore the Gul-Pesendorfer version of set betweenness would be violated.

The next axiom is built upon a simple idea: if a tree has fewer subtrees, then each of those subtrees commands more attention. To see what attention has to do with choice, let us first introduce a notion of a “swap.” Let $|\cdot|$ denote the cardinality of a set.

³We thank Larry Epstein for referring this paper to us.

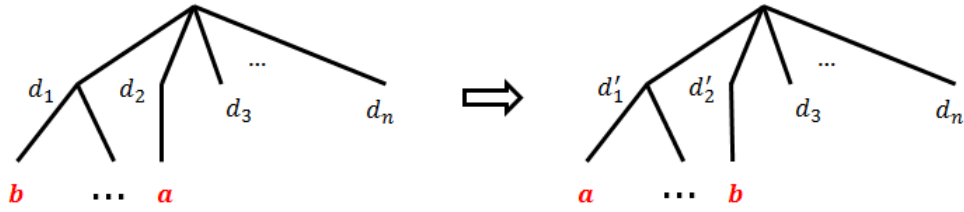


Figure 2.7: A swap of b for a converts d into $\Delta_a^b(d)$.

Definition 2.2.2 For $d = \{d_1, d_2, \dots, d_n\}$ such that $a \in d_2 \setminus d_1$, $b \in d_1 \setminus d_2$ and $|d_1| \geq |d_2|$, a swap of b for a is

$$\Delta_a^b(d) := d \setminus \{d_1, d_2\} \cup \{d'_1, d'_2\}$$

where $d'_1 := d_1 \setminus \{b\} \cup \{a\}$, $d'_2 := d_2 \setminus \{a\} \cup \{b\}$.

In the definition, the subtree b originally belongs to a larger tree (d_1) than the one (d_2) containing a . We assume that the subtrees from a smaller tree command more attention. Therefore, the swap of b for a accentuates b and masks a . Figure 2.7 illustrates the definition. If b is preferred to a , we call this swap an *accentuating swap* to emphasize the fact that the better subtree b is now more visible. When we write $\Delta_a^b(d)$ to denote the swap of b for a , implicit we have $d_1, d_2 \in d$, $a \in d_2 \setminus d_1$, $b \in d_1 \setminus d_2$ and $|d_1| \geq |d_2|$.

Axiom 2.2.4 (*Preference for Accentuating Swaps*) If $b \succeq a$, then $\Delta_a^b(d) \succeq d$.

To understand the motivation for *Preference for Accentuating Swaps*, consider a decision tree $d = \{d_1, d_2\}$ where $d_1 = \{lose, win\}$, $d_2 = \{draw\}$. Had the decision maker been fully rational, it does not matter which one of the three outcomes is

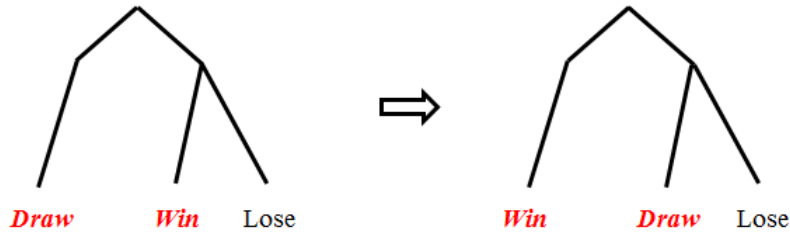


Figure 2.8: The left-hand-side decision tree is swapped into the right-hand-side one. After the swap, *win* becomes more salient and *draw* becomes less so. Thus the decision maker would choose the right-hand-side decision tree with higher probability.

presented in what subtree. However, when the boundedly rational decision maker tries to see through the tree, the outcomes in d_1 might command less attention than the one in d_2 , simply because there are more outcomes to be looked at in d_1 than d_2 . Suppose *win* is preferred to *draw*. An accentuating swap of *win* for *draw* makes the better outcome more salient and the worse outcome less (see Figure 2.8). Therefore the swapped decision tree might appear to be better, and be chosen more often than the original tree.

This axiom captures what we observe in many situations. For example, stores usually try to present better products at more visible places. One way to do so is to single out a few better products from many other alternatives. By doing that, the store can attract more consumers, given the same set of products.

This axiom certainly has its limitation. It says that swapping a good subtree from a bigger tree for a bad subtree from a smaller tree should constitute an improvement. In other words, only the sizes of trees matter, which greatly simplifies our problem. More generally, one might want to have an axiom saying that swapping a good subtree from a more complicated tree for a bad subtree from a simpler tree should constitute



Figure 2.9: The right-hand-side decision tree extends a into $\{a\}$ by adding a trivial choice.

an improvement. To achieve this, one needs to first define what complicated and simple mean in terms of the decision maker's choice. We leave this generalization for future works.

The remaining axioms are technical conditions that help pin down the model. The axiom below is a consistency requirement. It states that adding a trivial choice to subtree a is irrelevant (see Figure 2.9). As a result, the decision maker is indifferent between a and $\{a\}$.

Axiom 2.2.5 (*Consistency*) For $a \in \mathcal{D}$, $a \sim \{a\}$.

The last axiom is *Continuity*. The idea is simple. Suppose we already have a value function that assigns values to trees. The notion of continuity that we need is that, for any decision tree, we want small perturbations of its subtree values to have small impact on its own value (see Figure 2.10). Of course, we do not have the value function to begin with. To impose this notion of continuity, we first define the following distance function on the space of subtrees, \mathcal{D} . For any decision subtrees $a, b \in \mathcal{D}$, we let

$$\nu(a, b) := |P(\{a\}, \{a, b\}) - P(\{b\}, \{a, b\})|$$

be the distance between a, b . In other words, a and b are close whenever the decision maker considers them to be close substitutes. Next, analogous to the definition of the Hausdorff distance, we extend the distance function from the space of subtrees to the space of decision trees as follows.

$$\mu(c, d) := \begin{cases} \max \left\{ \max_{c_i \in c} \min_{d_j \in d} \nu(c_i, d_j), \max_{d_j \in d} \min_{c_i \in c} \nu(d_j, c_i) \right\}, & \text{if } |c| = |d| \\ 1, & \text{if } |c| \neq |d| \end{cases} \quad (2.2)$$

Hence, c and d are close if c 's and d 's subtrees are pairwise close in terms of ν . Unlike the standard Hausdorff distance, we only measure the distance between c and d that have the same cardinality. When they do not have the same cardinality, we consider them "far apart."⁴ To see why, suppose we have three indifferent outcomes $x \sim y \sim z$. Had we not required the second line in (2.2), we will find that $\mu(\{x\}, \{y, z\}) = 0$ according to the first line in (2.2). But clearly $P(\{x\})$ and $P(\{y, z\})$ are different probability measures.

Axiom 2.2.6 (*Continuity*) *The function P is continuous.*

In our notion of continuity, the function μ depends on P while P is required to be continuous with respect to μ . This circularity creates no problems; as in standard metric spaces, the metric itself is continuous with respect to the topology it induces. In our case, the distance $\mu(c, d)$ depends on the subtrees of c and d , rather than c and d themselves. Thus, like the *Dominance* axiom, *Continuity* builds a connection

⁴With the other axioms, μ is a pseudometric that only violates $\mu(c, d) = 0 \Rightarrow c = d$, compared to a metric. Without the other axioms, μ might also violate the triangle inequality, and is called a pseudosemimetric.

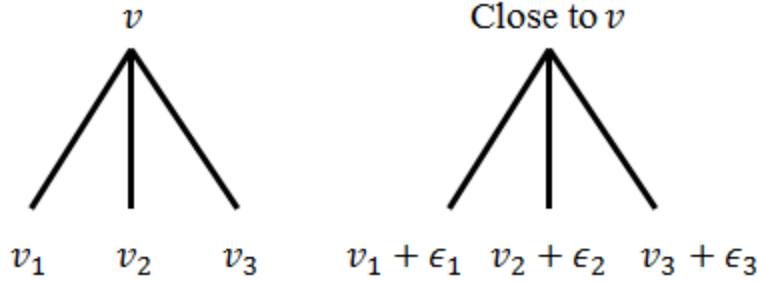


Figure 2.10: Suppose we already have the value function that evaluates subtrees. The continuity condition that we need is that small perturbations of a decision tree’s continuation values should have small impact on the decision tree’s value.

between decision trees and their subtrees. The function P defined on depth-1 decision trees imposes a continuity requirement on P defined on the set of depth-2 decision trees, and so on.

Our main theorem establishes that in a rich choice environment, the only model that can satisfy all these axioms is the following model. In the model, two functions fully describe the decision maker’s behavior: a value function V that describes the decision maker’s propensity to choose a particular subtree $a \in b$ from the decision tree b , and a function f that relates the value of the decision tree b to its subtree values.

Definition 2.2.3 *An RCR P is a Boundedly-Rational Backward-Induction Rule (BBR) if there exist a value function $V : \mathcal{D} \rightarrow \mathbb{R}_{++}$ and a strictly increasing continuous function $f : V(\mathcal{D}) \rightarrow \mathbb{R}$ such that for any $a = \{a_1, \dots, a_n\}$,*

$$P(\{a_i\}, a) = \frac{V(a_i)}{\sum_{j=1}^n V(a_j)} \tag{2.3}$$

for all $i = 1, \dots, n$ and

$$V(a) = f^{-1} \left(\frac{1}{n} \sum_{i=1}^n f(V(a_i)) \right) \quad (2.4)$$

As shown in (2.4), V depends on f . However, if we restrict the domain of V to the set of outcomes D_0 , then it is independent of the function f . In other words, through equation (2.4), the function f uniquely extends the valuation of outcomes to all finite decision trees. The value $V(a) > 0$ is called the Luce value of a , and the BBR is a dynamic extension of the Luce rule (Luce (1959)) (or called the logit model) that has been widely used in the industrial organization literature. When V and f satisfy the equations above, we say that (V, f) represents P .

The BBR relaxes both components of fully rational backward induction. First, fully rational backward induction requires tree a 's value to be equal to the maximum of a 's subtree values. In a BBR, the aggregator (2.4) aggregates a 's subtree values to evaluate a . Intuitively, this aggregator is some general notion of *average* instead of maximum.⁵ A rapidly increasing (i.e. convex) f ensures that the aggregator is close to the maximum function. Second, fully rational backward induction requires that a subtree with the highest value would be chosen with certainty. In a BBR, the decision maker is not able to do so. Her error-prone choice follows (3.1). The higher a tree's value is, the more likely it would be chosen.

This representation could be understood as follows. Given a decision tree, the decision maker fully understands its structure and outcomes. However, at the current stage the decision maker does not know how she would choose at the future stages.

⁵This average is called *Kolmogorov-Nagumo* mean.

Hence, she evaluates each tree as if she assigns a uniform prior (or an equal amount attention) to its subtrees, and uses a function f to aggregate the subtree values. A tree is treated as a closer substitute for its better outcomes when f is more convex. Random mistakes are useful in this interpretation: had the decision maker never made a mistake, knowing all the subtree values should naturally imply that she is able to use the maximum function to evaluate a tree, instead of some average. As expected, the axioms are also related to this interpretation. We will come back to this point in Section 5.

An alternative interpretation of BBR would be provided in Section 4.2, in which the decision maker receives random noisy signals about the subtree values. More complicated trees have more downward-biased signals in general. The decision maker bases her choice on the signals.

To state Theorem 2.2.1, we first define when the choice environment is rich. The richness condition we have here is similar to the one in Gul, Natenzon and Pesendorfer (2014).

Definition 2.2.4 *We say that (D_0, P) is rich if $\forall a, b \in D, q \in (0, 1), \exists x \in D_0$ such that $x \notin b$ and $P(\{x\}, \{x\} \cup a) = q$.*

Richness in our setting implies that for any given probability and any set of subtrees a , we can find an outcome that would be chosen with the required probability when put together with a . Moreover, we can find countably many such outcomes, because the definition requires that the desired outcome does not belong to b , for any predetermined $b \in D$. Richness is easy to satisfy when the outcome set contains lotteries. The example below yields a rich (D_0, P) .

Example 2.2.1 Let D_0 be all the 50-50 lotteries over monetary prizes. Let δ_u denote the degenerate lottery that yields prize u with probability 1. For each 50-50 lottery $\frac{1}{2}\delta_{u_1} + \frac{1}{2}\delta_{u_2}$ that has 1/2 probability returning u_1 and 1/2 probability u_2 , let $U(\frac{1}{2}\delta_{u_1} + \frac{1}{2}\delta_{u_2}) := \exp\{\frac{1}{2}u_1 + \frac{1}{2}u_2\}$. Consider an RCR P such that $P(a, a \cup \{\frac{1}{2}\delta_{u_1} + \frac{1}{2}\delta_{u_2}\}) = \frac{1}{1+U(\frac{1}{2}\delta_{u_1} + \frac{1}{2}\delta_{u_2})}$, $\frac{1}{2}\delta_{u_1} + \frac{1}{2}\delta_{u_2} \notin a$. Because U is continuous and monotone, we can adjust u_1 and u_2 to find infinitely many 50-50 lotteries that has the same U value. Thus, we have infinitely many outcomes that satisfies that condition in the definition of richness.

Our main result below establishes the relation between the axioms and the representation. Richness is required in the necessity, but not the sufficiency. To put it another way, when the choice environment is sparse, there might be other choice models that satisfy our axioms. However, this can be viewed merely as an artifact of the sparse choice environment.

Theorem 2.2.1 *If (D_0, P) is rich, then the RCR P satisfies Axioms 1–6 if and only if it is a BBR.*

The sufficiency of the theorem can be easily verified. As for the necessity, by Theorem 1 in Gul, Natenzon and Pesendorfer (2014), *Independence* and richness ensure the existence of the function V such that the Luce formula (3.1) holds. The more challenging part of the proof is relating $V(a)$ to the $V(a_i)$'s for any $a = \{a_1, \dots, a_n\}$ and ensuring that (2.4) holds.

The construction of the function f is similar to how one would calibrate a vNM utility function from the data on a decision maker's certainty equivalents for 50-50

gambles (see Machina (1987)). Choose any $a, b \in \mathcal{D}$ such that $V(b) > V(a)$ and set $f(V(a)) = 0$ and $f(V(b)) = 1$. Let

$$f(V(\{a, b\})) = \frac{1}{2}f(V(a)) + \frac{1}{2}f(V(b)) = \frac{1}{2}$$

Note that when calibrating a utility function, we use a similar equation in which $V(a)$ and $V(b)$ are replaced with monetary prizes x and y , $V(\{a, b\})$ is replaced with the certainty equivalent of the 50-50 gambles between x and y , and f is replaced with the utility function. Then, we consider $d = \{b, \{a, b\}\}$ and set

$$f(V(d)) = \frac{1}{2}f(V(\{a, b\})) + \frac{1}{2}f(V(b)) = \frac{3}{4}$$

We can continue in this fashion and define f on some subset of the reals.

This construction works only because of our axioms. For example, if the representation is to hold, we must have $b \succ \{a, b\} \succ a$, because in the representation f is strictly increasing. This is guaranteed by *Stochastic Set Betweenness* and *Dominance*. More importantly, consider two decision trees $\{\{a, b\}, \{c, d\}\}$ and $\{\{a, c\}, \{b, d\}\}$. If P is a BBR, it must be true that

$$\{\{a, b\}, \{c, d\}\} \sim \{\{a, c\}, \{b, d\}\} \tag{2.5}$$

because

$$\begin{aligned} f(V(\{\{a, b\}, \{c, d\}\})) &= f(V(\{\{a, c\}, \{b, d\}\})) \\ &= \frac{1}{4}f(V(a)) + \dots + \frac{1}{4}f(V(d)) \end{aligned}$$

Preference for Accentuating Swaps ensures that (2.5) holds. To see why, consider $\{\{a, b\}, \{c, d\}\}$ and suppose $b \succeq c$. According to the axiom, a swap of b for c should be preferred to the original tree; that is, $\{\{a, c\}, \{b, d\}\} \succeq \{\{a, b\}, \{c, d\}\}$. However, we can swap b and c back, and apply the axiom again to conclude that $\{\{a, b\}, \{c, d\}\} \succeq \{\{a, c\}, \{b, d\}\}$. Thus we have (2.5).

Now we obtain that f satisfies

$$f(V(\{a', b'\})) = \frac{1}{2}f(V(a')) + \frac{1}{2}f(V(b'))$$

for a countable subset of V 's image. Next, *Dominance* implies that this subset must be dense in V 's image. Hence, together with *Continuity*, f can be extended to V 's image. The construction so far only deals with binary decision trees. The last step requires the interaction between all the axioms except *Independence* to show that (2.4) holds not only for binary trees, but also for all finite decision trees.

Proposition 2.2.1 below establishes the uniqueness of the BBR representation. In particular, the proposition shows that V is unique up to a positive scalar multiplication, and fixing V , f is unique up to a positive affine transformation. From here on, for simplicity, when (D_0, P) is rich and P is a BBR, we say that P is a rich BBR.

Proposition 2.2.1 *Suppose P is a rich BBR. Fixing V , f is unique up to a positive affine transformation. More generally, both (V, f) and (\tilde{V}, \tilde{f}) represent P if and only if there exist $\alpha_1, \alpha_2 > 0$ and $\beta \in \mathbb{R}$ such that $V(a) = \alpha_1 \tilde{V}(a)$ and $f(\alpha_1 \tilde{v}) = \alpha_2 \tilde{f}(\tilde{v}) + \beta$.*

2.3 Complexity Aversion and Error-Proneness

Our model describes a decision maker whose behavior falls short of fully rational backward induction on both dimensions, assigning correct values to trees and choosing the best subtree with certainty. In this section, we quantify the extent to which a BBR deviates from fully rational backward induction by providing comparative measures of complexity aversion and error-proneness. We take limits of these measures to find limiting cases of BBR, and we show by examples how complexity aversion and error-proneness interact.

2.3.1 Complexity Aversion

Confronting a depth-1 decision tree $a \in D_1$, the decision maker chooses an outcome $x \in a \subset D_0$. An outcome is the least complex choice object in our framework.

Consider two decision makers, labeled 1 and 2, who exhibit the same choice behavior when confronting any depth-1 decision tree. Suppose that compared to decision maker 1, decision maker 2 is always less likely to choose a nondegenerate decision subtree over an outcome. Then, we say that decision maker 2 is more *complexity-averse* than decision maker 1. To formalize this idea, recall that for an RCR P and a decision tree $a \in D$, $P(a) \in \mathcal{L}$ is the probability measure that describes

how the decision maker chooses. We say that an RCR P_1 and RCR P_2 coincide on the depth-1 decision trees if $P_1(a) = P_2(a)$, $\forall a \in D_1$. Let \succeq_i be the preference that P_i induces.

Definition 2.3.1 *RCR P_2 is more complexity-averse than RCR P_1 if P_1 and P_2 coincide on the depth-1 decision trees, and for any $x \in D_0$, $a \in D$, $a \succeq_2 x$ implies $a \succeq_1 x$.*

We say that the function f_2 is more concave than f_1 if $f_2 = g \circ f_1$ for some strictly increasing and concave function g . The following theorem establishes that the concavity of f is the measure of the decision maker's complexity aversion.

Theorem 2.3.1 *Suppose RCR P_1 and P_2 are rich BBRs. Then P_2 is more complexity-averse than P_1 if and only if there exist (V_1, f_1) and (V_2, f_2) that represent P_1 and P_2 respectively such that $V_1(x) = V_2(x)$ for all $x \in D_0$, and f_2 is more concave than f_1 .*

Theorem 2.3.1 suggests that the function f in a BBR describes a decision maker's complexity aversion the same way that a utility function describes a decision maker's risk aversion. The resulting complexity aversion is not the same as being averse to trees with more subtrees. It describes how the decision maker thinks of a tree. A decision tree is treated as a closer substitute for its worse outcomes if f is more concave, and vice versa. Indeed the aggregator converges to $\min\{V(a_i)\}$ as f gets more and more concave, and it converges to $\max\{V(a_i)\}$ case as f gets more and more convex.

Now that we have obtained a comparative measure of complexity aversion, let us introduce a subclass of BBRs that exhibit some constant measure of complexity aversion, like we do in expected utility theory. These BBRs will be used in the applications later. They are characterized by the following simple observable behavior.

Definition 2.3.2 *Suppose $a = \{w, x\}$, $b = \{y, z\}$ are depth-1 decision trees. We say that a BBR P is homogeneous if $P(\{x\}, a) \geq P(\{y\}, b)$ implies $P(\{x\}, \{x, a\}) \geq P(\{y\}, \{y, b\})$.*

The definition says that for a homogeneous BBR P , if x is chosen more frequently from a than y from b , then x should also be chosen more frequently over a than y over b . Proposition 2.3.1 below shows that such BBRs would have the following representation.

Definition 2.3.3 *An RCR P is a Constant-Complexity-Averse (CCA) BBR if there exists a function $V : \mathcal{D} \rightarrow \mathbb{R}_{++}$ and $\gamma \in \mathbb{R}$ such that for any $a = \{a_1, \dots, a_n\}$,*

$$P(\{a_i\}, a) = \frac{V(a_i)}{\sum_{j=1}^n V(a_j)}, i = 1, \dots, n$$

and either

$$V(a) = \left(\frac{1}{n} \sum_{i=1}^n [V(a_i)]^\gamma \right)^{1/\gamma}$$

or ($\gamma = 0$)

$$V(a) = \sqrt[n]{\prod_{i=1}^n V(a_i)}$$

Hence, the Luce value of a decision tree a is the γ -power mean of a 's subtree values $V(a_i)$'s. The following result establishes that the homogeneity condition is equivalent to constant complexity aversion.

Proposition 2.3.1 *A rich BBR P is homogeneous if and only if it is a CCA BBR.*

We use the term CCA to describe such BBRs because their f functions are similar to the CRRA utility functions with domain \mathbb{R}_{++} . If we mimick the definition of relative risk aversion, that is, $-v \frac{f''(v)}{f'(v)}$, we know that

$$-v \frac{f''(v)}{f'(v)} = 1 - \gamma$$

Recall that f_2 is more concave than f_1 if and only if $-\frac{f_2''}{f_2'} \geq -\frac{f_1''}{f_1'}$, if both f_1 and f_2 are twice differentiable. Since $v \in \mathbb{R}_{++}$, it is clear that if $\gamma_1 \geq \gamma_2$, RCR P_2 would be more complexity-averse than P_1 . Thus the CCA BBRs with the same outcome values are ordered with respect to the parameter γ .

2.3.2 Error-Proneness

Under richness and Independence, the RCR is a Luce rule. Hence, facing a binary set of outcomes $\{x, y\} \in D_1$, if a decision maker chooses x with lower probability than y , then she statistically reveals that $y \succeq x$. When comparing two decision makers, 1 and 2, who both prefer y over x , if decision maker 2 always chooses x with higher probability, then we say that decision maker 2 is more error-prone. Formally, we define it as follows.

Definition 2.3.4 *RCR P_2 is more error-prone than RCR P_1 if there exists a function $h : (0, \frac{1}{2}] \rightarrow \mathbb{R}_{++}$ such that $h(p) \leq p$, $h(\frac{1}{2}) = \frac{1}{2}$ and*

$$P_1(\{x\}, \{x, y\}) = h(P_2(\{x\}, \{x, y\})) \quad (2.6)$$

for $\forall \{x, y\} \in D_1$ with $P_2(\{x\}, \{x, y\}) \leq \frac{1}{2}$.

The equation (2.6) under $h(\frac{1}{2}) = \frac{1}{2}$ and $h(p) \leq p$ implies that $x \succeq_2 y$ if and only if $x \succeq_1 y$. Moreover, since $h(p) \leq p$, (2.6) implies that decision maker 2 is always more likely to choose the inferior outcome (through the function h) than decision maker 1. The theorem below characterizes our notion of error-proneness.

Theorem 2.3.2 *Suppose RCR P_1 and P_2 are rich BBRs. Then P_2 is more error-prone than P_1 if and only if there exist (V_1, f_1) and (V_2, f_2) that represent P_1 and P_2 respectively such that for any $x \in D_0$, $V_2(x) = [V_1(x)]^\lambda$ for some $\lambda \in (0, 1]$.*

Clearly, a smaller λ corresponds to a more error-prone decision maker. The key condition used to prove Theorem 2.3.2 is the following. Consider $x_1, x_2, y_1, y_2 \in D_0$ such that $P_2(\{x_1\}, \{x_1, x_2\}) = P_2(\{y_1\}, \{y_1, y_2\}) \leq \frac{1}{2}$. Say decision maker 2 is more error-prone than decision maker 1. Our definition immediately implies $P_1(\{x_1\}, \{x_1, x_2\}) \leq P_2(\{x_1\}, \{x_1, x_2\})$ and $P_1(\{y_1\}, \{y_1, y_2\}) \leq P_2(\{y_1\}, \{y_1, y_2\})$. More importantly, due to the h function, we must also have $P_1(\{x_1\}, \{x_1, x_2\}) =$

$P_1(\{y_1\}, \{y_1, y_2\})$, because

$$\begin{aligned} P_1(\{x_1\}, \{x_1, x_2\}) &= h(P_2(\{x_1\}, \{x_1, x_2\})) \\ &= h(P_2(\{y_1\}, \{y_1, y_2\})) \\ &= P_1(\{y_1\}, \{y_1, y_2\}) \end{aligned}$$

Thus from $P_2(\{x_1\}, \{x_1, x_2\}) = P_2(\{y_1\}, \{y_1, y_2\})$, decision maker 2 being more error-prone than decision maker 1 also implies $P_1(\{x_1\}, \{x_1, x_2\}) = P_1(\{y_1\}, \{y_1, y_2\})$. This property yields a functional equation for which the exponential is the solution.

2.3.3 Limiting Cases of BBR and the Interaction between Complexity Aversion and Error-Proneness

So far we have described BBR as two simultaneous deviations from the benchmark case, fully rational backward induction. By taking limits of the two measures we just derive, we elaborate the deviations one after another.

Fix some value function V only defined on the set of outcomes. Consider a collection of CCA BBRs in which each BBR is parametrized by two numbers, $\lambda > 0$ and γ . For the CCA BBR with parameters λ and γ , it assigns value $V_\lambda(x) = [V(x)]^\lambda$ to an outcome x , and has $f(v) = v^{\gamma/\lambda}$. Consider a simple decision tree $a = \{x_2, \{x_1, x_3\}\}$ where $V(x_i) = i$. According to (3.1) and (2.4), for each pair of λ and γ , we can define

$$V(\{x_1, x_3\}) := \left(\frac{1}{2}[V(x_1)]^\gamma + \frac{1}{2}[V(x_3)]^\gamma \right)^{1/\gamma}$$

and the equation below should hold

$$P(\{x_1, x_3\}, a) = \frac{[V(\{x_1, x_3\})]^\lambda}{[V(x_2)]^\lambda + [V(\{x_1, x_3\})]^\lambda}$$

Note that for each pair of λ and γ , the decision maker's value function is $V_\lambda(x) = [V(x)]^\lambda$ instead of V .

When both λ and γ are arbitrarily large, the choice behavior of BBR coincides with fully rational backward induction (with an equal-probability tie-breaking rule), because

$$\lim_{\gamma \rightarrow \infty} V(\{x_1, x_3\}) = \max\{V(x_1), V(x_3)\}$$

and

$$\lim_{\lambda \rightarrow \infty} P(\{x_1, x_3\}, a) = \begin{cases} 1, & \text{if } V(\{x_1, x_3\}) > V(x_2) \\ 0, & \text{if } V(\{x_1, x_3\}) < V(x_2) \\ \frac{1}{2}, & \text{if } V(\{x_1, x_3\}) = V(x_2) \end{cases} \quad (2.7)$$

Of course, since we have $V(x_1) < V(x_2) < V(x_3)$ and $V(\{x_1, x_3\}) = V(x_3)$ in this case, the decision maker will choose $\{x_1, x_3\}$ for sure at the first stage.

Next, let us bring in complexity aversion. Keep λ to be arbitrarily large, but consider a finite γ . The decision maker still always chooses the subtree with the highest value for sure as in (2.7). However, she might be averse to complex subtrees deterministically. For instance, if $\gamma = -1$, we know that

$$\begin{aligned} V(\{x_1, x_3\}) &= \left(\frac{1}{2}[1]^{-1} + \frac{1}{2}[3]^{-1} \right)^{-1} \\ &= 1.5 < 2 = V(x_2) \end{aligned}$$

Therefore in this limiting case, at the first stage, the decision maker chooses the safer bet x_2 with certainty, despite the fact that had she been confronted with $\{x_1, x_3\}$, she would have been able to choose x_3 for sure. In other words, had she not shied away from the complex subtree, she would have been better off.

Lastly, if we also let λ be finite, we will be back to the case with both complexity aversion and error-proneness.

From the analysis of the previous limiting case, we can see some interaction between complexity aversion and error-proneness. In this particular decision tree $\{x_2, \{x_1, x_3\}\}$, for a decision maker who never makes a mistake, the less complexity-averse she is, the more likely she will end up with the best outcome x_3 . In other words, complexity aversion and error-proneness can be complements. Indeed, from Figure 2.11, we can see that for a decision maker with high error-proneness, her expected payoff decreases as complexity aversion decreases.

Note that complexity aversion and error-proneness can sometimes be substitutes too. If we have another decision tree $\{x_1, \{x_2, x_3\}\}$. The more complexity-averse the decision maker is, the more likely that she can avoid the worst outcome x_1 . On the other hand, the less error-prone she is, the better off she would be.

2.4 BBR in Dynamic Choice Problems

In this section, we present several BBR's implications and properties in dynamic choice problems. Let us begin with a simple result. Suppose there is an outcome x_n presented either deep down a decision tree (e.g., $\{x_1, \{x_2, \dots, \{x_n\}\}\}$), or among

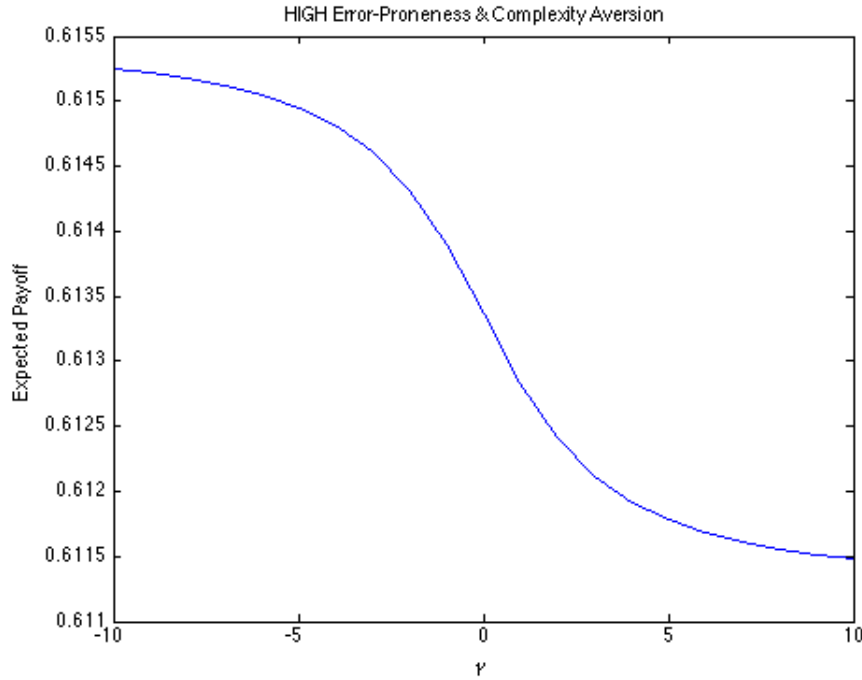


Figure 2.11: Complementarity: When the decision maker’s error-proneness is high, she benefits (having a higher expected payoff) from having a higher complexity aversion (lower γ). The numerical difference is small because when error-proneness is high, the decision maker’s behavior is very close to uniform random choice.

many alternative subtrees (e.g., $\{x_1, \{x_2, \dots, x_n\}\}$). Intuitively, such an outcome should have little contribution to the valuation of the decision tree as a whole. In other words, if we replace outcome x_n with some other outcome y , the value of the decision tree should not change much. This intuition indeed holds under a BBR, as stated below. We omit its proof.

Proposition 2.4.1 *Consider a BBR (V, f) , a sequence of outcomes $\{x_i\}$ such that $V(x_i) \in [\underline{v}, \bar{v}]$, $0 < \underline{v} < \bar{v}$, and an outcome $y \in D_0$. Then $\lim_{n \rightarrow \infty} V(\{x_1, \{x_2, \dots, \{x_n\}\}) -$*

$V(\{x_1, \{x_2, \dots, \{x_{n-1}, \{y\}\}\}\}) = 0$ and $\lim_{n \rightarrow \infty} V(\{x_1, \{x_2, \dots, x_n\}\}) - V(\{x_1, \{x_2, \dots, x_{n-1}, y\}\}) = 0$.

So far, given a decision tree $a = \{a_1, \dots, a_n\}$, our model predicts the probability with which the decision maker chooses each subtree a_i , but it has not yet predicted how she would continue to choose after choosing some $a_j \in D$. Thus, we have presented a theory that relates a decision maker's choices at the initial stage of a decision tree to how she would have chosen had she been asked to make choices in the subtrees of that tree. We have not addressed the decision maker's choice after the first stage of any decision tree.

There is a simple way to extend our model to the subsequent stages of choice: imposing *consequentialism*. Consequentialism means history independence. Suppose $b = \{b_1, \dots, b_n\}$ and $b_1 = \{a_1, \dots, a_m\}$. Under consequentialism, the probability $\pi(\{a_i\}, b)$ of a_i being chosen from the decision tree b is

$$\pi(\{a_i\}, b) = P(\{b_1\}, b) \times P(\{a_j\}, b_1) \tag{2.8}$$

Note that more generally, the second term on the right hand side of (2.8) could also depend on b_2, \dots, b_n . By assuming consequentialism, only the current tree matters. Consequentialism is a maintained hypothesis in the analysis below.

2.4.1 Menu Effect

Consider a decision tree and fix the choice path towards its unique best outcome. A fully rational decision maker's choice remains unaffected no matter how we change

the suboptimal paths. For example, if we add or remove a suboptimal path, the fully rational decision maker will still follow the optimal choice path with certainty. However, when the decision maker's choice follows BBR, all subtrees and outcomes affect her choice. In particular, adding or removing a suboptimal path could have a nontrivial impact.

In the marketing literature, it is widely acknowledged that product assortment affects a decision maker's choice (see Simonson (1999) for a review) in ways that are inconsistent with the standard utility maximizing model. Product assortment studies how adding or removing alternatives affects the decision maker's choice, that is, the menu effect. Our model provides a new stochastic framework for analyzing this question. When a consumer contemplates which store to go to or stands in front of a supermarket shelf, she is implicitly facing strategic agents who are "gaming" her complexity aversion.

Abundant evidence has shown that excluding some less appealing products from a store's assortment boosts its sales (see Broniarczyk, et al. (1998) and Boatwright and Nunes (2001)). However, standard theory predicts that a larger product assortment always induces weakly higher sales than a smaller one. To see how our model can accommodate such evidence, suppose the decision maker faces $\{not\ enter, \{good\ product, bad\ product, leave\}\}$; that is, she first chooses whether to enter a store or not, and then chooses which product to buy or leave the store. If the store eliminates *bad product*, the decision tree becomes $\{not\ enter, \{good\ product, leave\}\}$. This assortment reduction might decrease her probability of not entering. To see this,

according to *Stochastic Set Betweenness*, it is likely to be the case that

$$\{\text{good product, leave}\} \succ \{\text{good product, bad product, leave}\}$$

Formally, let the Luce value of *not enter* and *leave* be normalized to 1. Let the Luce values of *good product* and *bad product* be v_g and v_b ($v_g > v_b > 0$) respectively. Assume that the profit margins of them are $\alpha_g > 0$ and $\alpha_b > 0$ respectively. Then, $V_0 = f^{-1}(\frac{1}{3}(f(v_g) + f(v_b) + f(1)))$ and $V_1 = f^{-1}(\frac{1}{2}(f(v_g) + f(1)))$ are the Luce values of the original store and the store after assortment reduction respectively. The total profit for the original store would be

$$\beta_0 = \left(1 - \frac{1}{1 + V_0} - \frac{V_0}{1 + V_0} \frac{1}{v_g + v_b + 1}\right) \frac{v_g \alpha_g + v_b \alpha_b}{v_g + v_b}$$

The total profit for the store after assortment reduction would be

$$\beta_1 = \left(1 - \frac{1}{1 + V_1} - \frac{V_1}{1 + V_1} \frac{1}{v_g + 1}\right) \alpha_g$$

The following result establishes that if the Luce value of the bad product is below some threshold, then total profit of the store would increase if the bad product is removed from the product assortment.

Proposition 2.4.2 *For each $v_g > 0$, there exists a threshold \hat{v} such that if $v_b < \hat{v}$, then $\beta_1 > \beta_0$.*

The proof is simple. Assume for a moment that $v_b = 0$, then obviously $\beta_0 < \beta_1$. Since β_0 is continuous in v_b , we can find $(0, \hat{v})$ such that for all $v_b \in (0, \hat{v})$, $\beta_0 < \beta_1$.

Note that the menu effect caused by the assortment reduction is significantly different from the attraction affect or the compromise affect often studied in the marketing literature; it is inherently dynamic. Our RCRs are extensions of the Luce rule and therefore satisfy regularity: removing a subtree $a \in b$ from the decision tree b cannot decrease the choice probability of any of the remaining subtrees in b . However, removing a bad subtree from the tree a can make a more attractive and hence reduce the probability that the decision maker chooses any of the other subtrees.

In this particular setting (facing stores and products), one might demand a psychological interpretation for how the decision maker actually chooses. To do this, it is useful to introduce an alternative interpretation of BBR. In the industrial organization literature, researchers usually use the logit model instead of the Luce rule, although the two models are equivalent in terms of observable choice. To see why they are equivalent, given a set of choice objects $a = \{a_1, \dots, a_n\}$ and a utility function U , a logit model says that the probability that a_i is chosen is

$$P(\{a_i\}, a) = \frac{\exp\{U(a_i)\}}{\sum_j \exp\{U(a_j)\}} \quad (2.9)$$

Clearly, if we let each a_j has Luce value $\exp\{U(a_j)\}$, the resulting Luce rule would be equivalent to this logit model (2.9). However, the logit model has its own interpretation. In a logit model, the decision maker receives a noisy signal $U(a_j) + \varepsilon_j$ about each choice object's utility, where ε_j follows some *i.i.d.* Gumbel distribution. Then, she chooses the choice object with the highest signal value. It is well-known that in this case the decision maker's choice probabilities would exactly be (2.9).

A BBR can also be interpreted in this way. For example, suppose the decision maker's RCR is represented by (V, f) . Say there are two stores. One only has product x , and the other has products y, z . Let the utility function be $U := \log V$ and define $h(u) := f(\exp(u))$. Now the as-if interpretation becomes that the decision maker receives a signal about the utility of each store, and then she chooses the store with the highest signal value to go to. The first store has a signal $U(x) + \varepsilon$, and the second store has a signal

$$U(\{y, z\}) + \varepsilon' = h^{-1} \left(\frac{1}{2}h(U(y)) + \frac{1}{2}h(U(z)) \right) + \varepsilon'$$

where ε and ε' follows *i.i.d.* Gumbel distribution. The second store's signal is centered around some average product values of that store.

Back to the assortment reduction example, under this interpretation, the decision maker first receives two signals about the utility of entering and not entering the store. The signals contain errors, but the decision maker simply chooses the alternative that has a better signal. The signal for entering the store consists of two components: an error term and an average about the product utility in the store. Hence, having a bad product in the store will on average punish signal value.

2.4.2 Framing Effect

Consider a simple example. A seller in a store wants to present a fixed set of products to a decision maker. There is a singled-out place to present only one product, and a shelf to present the rest of the products. For a fully rational decision maker, how

the seller presents the products does not matter. The fully rational decision maker simply buys the best product from the store that contains it. However, this is not true for a decision maker whose choice follows BBR. As an immediate example, *Preference for Accentuating Swaps* implies that the decision maker will more likely choose a store that presents a better product at the singled-out place. Consistent with our observation, how the seller presents matters.

To put this in an abstract way, given a fixed set of outcomes, there are many different ways to organize the outcomes through decision trees. Say there are three outcomes x_1, x_2, x_3 . The mindless way to present them would be to show them altogether to the decision maker; that is, let the decision maker be confronted with $\{x_1, x_2, x_3\}$. Let us give a name to this presentation strategy, strategy N.

Obviously if we use other decision trees to present the outcomes, the choice probabilities would not be the same. Consider the following two decision trees

$$a = \{\{x_1, x_2\}, \{x_1, x_3\}\}$$

and

$$b = \{x_1, \{x_2, x_3\}\}$$

We can interpret the first decision tree a as follows. The outcomes are classified into two groups $\{x_1, x_2\}$ and $\{x_1, x_3\}$, and more importantly, x_1 recurs in both. Intuitively, by repeating x_1 in this way, the choice probability of x_1 should be higher. In the second case, decision tree b , outcome x_1 is presented in a simpler subtree. It is singled out and hence emphasized compared to the grouped ones x_2 and x_3 .

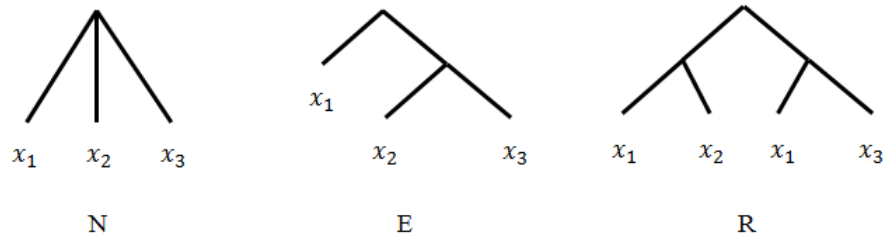


Figure 2.12: Strategy N, strategy E and strategy R.

The two abstract examples above capture some common features of advertising. For instance, when we search for certain search keywords in Google, an advertised website not only appears on the first page of the search results, but also recurs on all the other pages (up to ten or more). In contrast, an unadvertised website only appears once. The decision tree a captures the key element in this advertising strategy: it repeats the outcome x_1 . Due to this connection, we call decision tree a an *advertising strategy with recurrence* (strategy R). Decision tree b also captures some important feature of advertising. When we apply for a Chase’s credit card online or in a local branch, usually three advertised cards are singled out and introduced to us first, Chase Freedom, Slate, and Sapphire Preferred. Of course Chase has a lot of other cards, but the other cards are grouped together when presented. In other words, the first singled-out credit cards are emphasized. Decision tree b captures the key element in this advertising strategy: it emphasizes the outcome x_1 by singling it out. Because of this, we call decision tree b an *advertising strategy with emphasis* (strategy E). Note that many advertising strategies are combinations of recurrence and emphasis, including the actual Google’s and Chase’s strategies.

To see how these two examples, strategy R and E, differ from the benchmark case strategy N, we define $P_N := P(\{x_1\}, \{x_1, x_2, x_3\})$ and $P_E := P(\{x_1\}, \{x_1, \{x_2, x_3\}\})$ as the probabilities of x_1 being chosen in strategy N and strategy E respectively. The choice probability of x_1 under strategy R is

$$P_R : = P(\{\{x_1, x_2\}\}, a) \times P(\{x_1\}, \{x_1, x_2\}) \quad (2.10)$$

$$+ P(\{\{x_1, x_3\}\}, a) \times P(\{x_1\}, \{x_1, x_3\}) \quad (2.11)$$

Our first result confirms that both strategy R and E increase the choice probability of x_1 , no matter what the functions V and f are.

Proposition 2.4.3 *If the RCR P is a BBR, $P_N < \min\{P_E, P_R\}$.*

Proof. A BBR is a Luce rule. In a Luce rule, adding a subtree to a decision tree will strictly lower the probability that the decision maker chooses any of the existing subtrees. Hence,

$$P(\{x_1\}, \{x_1, x_2, x_3\}) < \min\{P(\{x_1\}, \{x_1, x_2\}), P(\{x_1\}, \{x_1, x_3\})\}$$

where the left-hand-side is P_N . Consider P_E . If $V(x_2) \geq V(x_3)$, then $P_E \geq P(\{x_1\}, \{x_1, x_2\})$ since $V(x_2) \geq V(\{x_2, x_3\})$. Same applies when $V(x_2) \leq V(x_3)$. Therefore we have $P_E > P_N$. Next consider P_R . From (2.10) we know that P_R is a weighted average of $P(\{x_1\}, \{x_1, x_2\})$ and $P(\{x_1\}, \{x_1, x_3\})$. Hence $P_R > P_N$. ■

Another question we could ask is when which strategy works better, recurrence or emphasis. To simplify the analysis, we consider only CCA BBRs. For a CCA

BBR

$$V(\{x_i, x_j\}) = \left(\frac{1}{2}[V(x_i)]^\gamma + \frac{1}{2}[V(x_j)]^\gamma \right)^{1/\gamma}$$

$$P_E = \frac{V(x_1)}{V(x_1) + V(\{x_2, x_3\})}$$

and

$$P_R = \sum_{i=1}^2 \frac{V(\{x_1, x_i\})}{V(\{x_1, x_2\}) + V(\{x_1, x_3\})} \frac{V(x_1)}{V(x_1) + V(x_i)}$$

Recall that a higher γ corresponds to lower complexity aversion. The following result implies that when the outcome being promoted (x_1) is not the worst one, there is a unique cutoff of complexity aversion such that if the complexity aversion is above the cutoff, then the strategy with emphasis works better, and vice versa.

Theorem 2.4.1 *Suppose RCR P is a CCA BBR, and $V(x_1) \geq \min\{V(x_2), V(x_3)\}$.*

Then $P_R \geq P_E$ if and only if $\gamma \geq 1$.

When γ is low (high complexity aversion), strategy E takes advantage of the decision maker's complexity aversion to steer her to outcome x_1 . In contrast, P_R is less affected by complexity aversion since the strategy R's first-stage subtrees are similarly complex. Note that the theorem also implies that the level of error-proneness does not matter qualitatively. Only the ordinal ranking of $V(x_i)$'s might matter.

For the case in which x_1 is the worst outcome, the following similar but slightly weaker result holds.

Proposition 2.4.4 *Suppose RCR P is a CCA BBR. For each value function V , there exist $\bar{\gamma} \geq 1 \geq \underline{\gamma}$ such that $P_R < P_E$ whenever $\gamma \leq \underline{\gamma}$ and $P_R > P_E$ whenever $\gamma \geq \bar{\gamma}$.*

The result above is similar to our previous theorem, except that now the outcome x_1 could be the worst, and that we have two cutoffs instead of one. What happens between the two cutoffs? It turns out that between the two cutoffs, our result still holds approximately. For more details, please refer to the Additional Results at the end of the chapter.

2.5 Axioms Revisited: The Lack of Understanding of Future Choices

In Section 2, we offer one interpretation of BBR: it is a choice model of an error-prone decision maker who does not understand how she would choose in the future. In the representation, this is reflected by the uniform weights ($1/n$) in the aggregating function

$$V(a) = f^{-1} \left(\frac{1}{n} \sum f(V(a_i)) \right)$$

In contrast, when we say that the decision maker understands her future choices, we mean that she uses the correct weights in her aggregating function, and possibly

some function other than f to aggregate the subtree values. In other words,

$$V(a) = g^{-1} \left(\sum_i P(\{a_i\}, a) \times g(V(a_i)) \right)^6 \quad (2.12)$$

Since the axioms and the representation are equivalent, some of our axioms must also depend on the lack of understanding of future choices. Which axioms depend on this?

Expectedly, *Preference for Accentuating Swaps* might not hold if the decision maker understands her future choices. For example, in the sketch of proof, we know that this axiom implies

$$\{\{w, x\}, \{y, z\}\} \sim \{\{w, y\}, \{x, z\}\}$$

However, if the aggregating function is (2.12), it's easy to see that this indifference condition might not hold.

More surprisingly, *Dominance* also fails. Consider a simple example where $V(w) = 0, V(x) = \exp\{2\}, V(y) = \exp\{1\}, V(z) = \exp\{2.1\}$, and $g(v) = \log v$. Since $V(w) < V(y)$ and $V(x) < V(z)$, if *Dominance* holds, $\{y, z\} \succ \{w, x\}$. However, we have $V(\{w, y\}) = 2$ and $V(\{y, z\}) < 2$ under (2.12). In other words, for a decision maker who understands her future choices, she knows that she never runs into the bad outcome w . Hence she identifies $\{w, x\}$ with x . But when facing $\{y, z\}$, she understands that she sometimes mistakenly chooses y . In this particular case, we can see that even though $V(z) > V(x)$, the fact that y is sometimes chosen drags

⁶In Ke (2015), a generalization of this model is characterized.

down the value that the decision maker can get out of $\{y, z\}$. The other axioms hold with or without understanding of future choices.

These findings do not mean that *Dominance* and *Preference for Accentuating Swaps* are not suitable in our case. When the decision maker does not understand her future choices, *Dominance* is one of the most natural axioms to impose: without knowing how choices would be made in the future, a simple rule of thumb for the current-stage choice is to choose the subtree that has better outcomes. Similarly, *Preference for Accentuating Swaps* is also a simple rule of thumb without looking into how future choices will actually be made. Also note that by imposing *Dominance*, the case where the decision maker understands her future choices would not be a special case of BBR. We leave the more general model that allows for both cases for future research.

2.6 Appendix and Additional Results

2.6.1 Appendix

Lemma 2.6.1 For $d = \{d_1, d_2, \dots, d_n\}$ such that $b \in d_1 \setminus d_2$, $a \in d_2 \setminus d_1$ and $|d_1| = |d_2|$, $\Delta_a^b(d) \sim d$.

PROOF OF LEMMA 2.6.1: Say $b \succeq a$. Then $|d_1| \geq |d_2|$ implies $\Delta_a^b(d) \succeq d$ by *Preference for Accentuating Swaps*. Let $d' := \Delta_a^b(d)$, $d'_1 := d_1 \setminus \{b\} \cup \{a\}$, and $d'_2 := d_2 \setminus \{a\} \cup \{b\}$. Notice that now $b \in d'_2 \setminus d'_1$ and $a \in d'_1 \setminus d'_2$. Clearly $|d'_1| = |d'_2|$, and hence $|d'_2| \leq |d'_1|$ implies $\Delta_b^a(d') \succeq d'$. It is not difficult to see that $\Delta_b^a(d') = d$. Therefore $\Delta_a^b(d) \sim d$.

■

PROOF OF THEOREM 2.2.1: First we show the necessity. Suppose (D_0, P) is rich and P is a BBR. According to (3.1), the RCR P is a Luce rule. In a Luce rule, we know that (i) $V(a) \geq V(b)$ implies $a \succeq b$, and (ii) Luce rule satisfies IIA and IIA implies *Independence*.

Dominance is satisfied because f is strictly increasing. *Continuity* is satisfied too. Small value of $\nu(a, b)$ is equivalent to that $V(a)$ and $V(b)$ are close. For two sets $c = \{c_1, \dots, c_n\}$, $d = \{d_1, \dots, d_n\}$, $\mu(c, d)$ being small implies that there is a bijection $\pi : \{1, \dots, n\} \rightarrow \{1, \dots, n\}$ such that

$$\max_i \nu(c_i, d_{\pi(i)})$$

is small too. Thus $V(c_i)$ and $V(d_{\pi(i)})$ are close. By f 's continuity, we know that $V(c)$ and $V(d)$ should be close and hence $\nu(c, d)$ would be small.

As for *Stochastic Set Betweenness*, consider any $a, b \in D$ such that $a \cap b = \emptyset$, say $a = \{a_1, \dots, a_m\}$, $b = \{b_1, \dots, b_n\}$. If $a \succeq b$, then $V(a) \geq V(b)$. Since $f(V(a)) = \frac{1}{m} \sum_{i=1}^m f(V(a_i))$, $f(V(b)) = \frac{1}{n} \sum_{i=1}^n f(V(b_i))$,

$$\begin{aligned} f(V(d_1 \cup d_2)) &= \frac{1}{n_1 + n_2} \left(\sum_{i=1}^m f(V(a_i)) + \sum_{i=1}^n f(V(b_i)) \right) \\ &= \frac{m}{m+n} f(V(a)) + \frac{n}{m+n} f(V(b)) \end{aligned}$$

Thus $V(a) \geq V(a \cup b) \geq V(b)$, and *Stochastic Set Betweenness* is satisfied. *Consistency* is satisfied since $V(\{a\}) = f^{-1}(f(V(a))) = V(a)$.

For $d = \{d_1, d_2, \dots, d_n\}$ such that $b \in d_1 \setminus d_2$, $a \in d_2 \setminus d_1$, $b \succeq a$ and $|d_1| \geq |d_2|$, let $d'_1 := d_1 \setminus \{b\} \cup \{a\}$ and $d'_2 := d_2 \setminus \{a\} \cup \{b\}$. We have

$$\begin{aligned} |d|[f(V(\Delta_a^b(d))) - f(V(d))] &= f(V(d'_1)) + f(V(d'_2)) - f(V(d_1)) - f(V(d_2)) \\ &= (f(V(b)) - f(V(a))) \left(\frac{1}{|d_2|} - \frac{1}{|d_1|} \right) \geq 0 \end{aligned}$$

Therefore *Preference for Accentuating Swap* is satisfied.

Next we prove the sufficiency. When (D_0, P) is rich and P satisfies *Independence*, P would be a Luce rule (see Gul, Natenzon and Pesendorfer (2014)); that is, there exists a function $V : \mathcal{D} \rightarrow \mathbb{R}_{++}$ that assigns each decision subtree $a \in \mathcal{D}$ a Luce value

$V(a) > 0$, and for $a = \{a_1, \dots, a_n\}$,

$$P(\{a_i\}, a) = \frac{V(a_i)}{\sum_{j=1}^n V(a_j)}$$

It's easy to see that $a \succeq b$ implies $V(a) \geq V(b)$.

For each $x \in D_0$, we have $V(x)$ already. We first prove that $V(D_0) = \mathbb{R}_{++}$. For any $v \in \mathbb{R}_{++}$, we can find an $x \in D_0$ such that $V(x) = v'$. If $v \neq v'$, by richness, we can find $y \in D_0$ such that $P(\{y\}, \{x, y\}) = \frac{v}{v+v'}$. Then $V(y) = v$. Not only so, for any v and any given finite set $a \subset D_0$, we can find $z \in D_0$ such that $V(z) = v$ and $z \notin a$.

A standard induction argument would show that P satisfies *Dominance* only if the following statement holds. For $a = \{a_1, \dots, a_n\}$ and $b = \{b_1, \dots, b_n\}$ such that $a_i \succeq b_i$, $a \succeq b$; if any of the former is strict, so is the latter. Let us call this statement *Dominance**.

Next we show that for all $a = \{a_1, \dots, a_n\} \in D$, by *Dominance*, there is a sequence of symmetric and strictly increasing function M_n 's such that $V(a) = M_n(V(a_1), \dots, V(a_n))$, where $M_n : \mathbb{R}_{++}^n \rightarrow \mathbb{R}_{++}$. The previous result shows that M_n 's domain is indeed \mathbb{R}_{++}^n . For any $(v_1, \dots, v_n) \in \mathbb{R}_{++}^n$, we can find $\{x_1, \dots, x_n\}$ such that $V(x_i) = v_i$. We can guarantee by richness that x_i 's are distinct even if some $v_i = v_j$. Now for any $a = \{a_1, \dots, a_n\}$ such that $V(a_i) = v_i$, it has to be true that $V(a) = V(\{x_1, \dots, x_n\})$, because we at the same time have all $V(a_i) \geq V(x_i)$ which by *Dominance** implies $V(a) \geq V(\{x_1, \dots, x_n\})$, and the other way around. Therefore we can let M_n maps (v_1, \dots, v_n) to $V(\{x_1, \dots, x_n\})$, which delivers a well-defined sequence of functions. Clearly M_n would be symmetric, meaning that

$M_n(v_1, \dots, v_n) = M_n(v_{\pi(1)}, \dots, v_{\pi(n)})$ for any permutation function π . Furthermore, the strictness in *Dominance* implies that M_n would be strictly increasing. *Consistency* implies that $M_1(v) = v$.

Notice that by *Dominance*, $\nu(a, b) = 0$ if $\nu(a_i, b_i) = 0$ for all i . It is then straightforward to translate *Continuity* into the following statement. For $\forall \varepsilon > 0$, $a = \{a_1, \dots, a_n\}$, there exists a $\delta > 0$ such that for all $b = \{b_1, \dots, b_n\}$, if $\max_i \nu(a_i, b_i) < \delta$, then $\nu(a, b) < \varepsilon$.

We show in this paragraph that M_n is continuous. Consider any $\varepsilon > 0$ and (v_1, \dots, v_n) , where $V(a_i) = v_i$, $a = \{a_1, \dots, a_n\}$. Now for $\varepsilon' = \frac{\varepsilon}{\varepsilon + 2V(a)}$, we can find a $1 > \delta' > 0$ such that if $\max_i \nu(a_i, b_i) < \delta'$, then $\nu(a, b) < \varepsilon'$. Notice that $\nu(a_i, b_i) < \delta'$ means that

$$\frac{|V(a_i) - V(b_i)|}{V(a_i) + V(b_i)} < \delta' \quad (2.13)$$

If $V(a_i) \leq V(b_i)$, (2.13) becomes $\frac{V(b_i) - V(a_i)}{V(b_i) + V(a_i)} < \delta'$, which is equivalent to

$$V(b_i) - V(a_i) < \frac{2V(a_i)}{1/\delta' - 1} := \delta_i$$

If $V(a_i) \geq V(b_i)$, we have

$$V(a_i) - V(b_i) < \frac{2V(a_i)}{1/\delta' + 1} < \delta_i$$

Thus now we know that if $\max |V(a_i) - V(b_i)| < \delta := \min \delta_i$, $\nu(a, b) < \varepsilon'$. And $\nu(a, b) < \varepsilon'$ implies

$$|V(a) - V(b)| < \frac{2V(a)}{1/\varepsilon' - 1} = \varepsilon$$

Therefore M_n is continuous.

Lemma 2.6.1 implies that for $x_i \in D_0$, $i = 1, \dots, 4$, where $V(x_i) = v_i$, $\{\{x_1, x_2\}, \{x_3, x_4\}\} \sim \{\{x_1, x_3\}, \{x_2, x_4\}\}$. Therefore we know that

$$M_2(M_2(v_1, v_2), M_2(v_3, v_4)) = M_2(M_2(v_1, v_3), M_2(v_2, v_4)) \quad (2.14)$$

By *Stochastic Set Betweenness*, for any $a, b \in D$, if $a \sim b$, then $a \sim a \cup b \sim b$, that is, if $V(a) = V(b)$, then $V(a) = V(a \cup b) = V(b)$. In particular, we know that

$$M_2(v, v) = v \quad (2.15)$$

This argument can be easily generalized to $M_n(v, \dots, v) = v$ by induction.

Consider $n = 2$. We have now shown the function M_2 is symmetric, strictly increasing, continuous and satisfies (2.15) and (2.14). According to Aczél (1948), we know that there exist a strictly increasing continuous function $f : V(\mathcal{D}) \rightarrow \mathbb{R}$ such that $M_2(v_1, v_2) = f^{-1}(\frac{1}{2}f(v_1) + \frac{1}{2}f(v_2))$. Thus for any $a = \{a_1, a_2\}$,

$$V(a) = f^{-1}\left(\frac{1}{2}f(V(a_1)) + \frac{1}{2}f(V(a_2))\right)$$

Notice that we already have $M_1(v) = v$, and hence $V(\{a\}) = f^{-1}(f(U(a)))$. Equation (2.4) is true for $n = 1, 2$ now.

To generalize (2.4) to the case with $n > 2$, we first prove the following lemma.

Lemma 2.6.2 For $d_i = \{d_{i,1}, \dots, d_{i,m}\}$, $d = \{d_1, \dots, d_n\}$ with $d_i \cap d_j = \emptyset$, $d \sim \bigcup_{i=1}^n d_i$.

PROOF OF LEMMA 2.6.2: For each $d_{i,j}$, we can find $d_{i,j}^{(t)} \in D_0$, $t = 2, \dots, n \times m$ such that $d_{i,j} \sim d_{i,j}^{(t)}$. Furthermore, by richness, we can make sure that none of the $d_{i,j}^{(t)}$ is the same as any $d_{i',j'}$ or any other $d_{i',j'}^{(t')}$. For simplicity, let us define $d_{i,j}^{(1)} := d_{i,j}$. Define $d_i^{(t)} := \{d_{i,1}^{(t)}, \dots, d_{i,m}^{(t)}\}$, and $d^{(t)} := \{d_1^{(t)}, \dots, d_n^{(t)}\}$. By *Dominance*, $d^{(t)} \sim d^{(t')}$. Then via *Stochastic Set Betweenness*, we have $\{d\} \sim \{d^{(1)}, \dots, d^{(nm)}\}$. Since $d \sim \{d\}$ by *Consistency*, we have $d \sim \{d^{(1)}, \dots, d^{(nm)}\} = d'$.

Notice that any $d_{i,j}^{(t)}$ and $d_{i',j'}^{(t')}$ can be swapped in d' without changing the cardinality of any subtree, since none of them is the same as another. With $|d'_i| = |d'_j|$, due to Lemma 2.6.1, we can perform any swap and end up having a new decision tree that is indifferent to d . In particular, we can swap for many times and obtain the following decision tree, $d'' = \{d''_{1,1}, \dots, d''_{n,m}\} \sim d'$ where $d''_{i,j} = \{d''_{i,j,1}, \dots, d''_{i,j,n}\}$ and $d''_{i,j,t} = \{d_{i,j}^{((t-1) \times m + 1)}, \dots, d_{i,j}^{(tm)}\}$. Now by *Stochastic Set Betweenness*, $d''_{i,j,t} \sim \{d_{i,j}^{(tm)}\} \sim d_{i,j}^{(tm)}$, and thus $d''_{i,j} \sim \{d_{i,j}\} \sim d_{i,j}$. Finally, $d \sim d' \sim d'' \sim \{d_{1,1}, \dots, d_{n,m}\} = \bigcup_{i=1}^n d_i$. ■

Now suppose (2.4) works for all $m < n$ for some $n > 2$. For any $d = \{d_1, \dots, d_{n+1}\}$, let us find by richness distinct x_1, \dots, x_{n-1} such that none of them belongs to d and each of them is indifferent to d . By *Stochastic Set Betweenness*, $d \sim d \cup \{x_1, \dots, x_{n-1}\} = d'$. By Lemma 2.6.2, $d' \sim \{\{d_1, \dots, d_n\}, \{d_{n+1}, x_1, \dots, x_{n-1}\}\} = d''$. Define $d''_1 := \{d_1, \dots, d_n\}$, and $d''_2 := \{d_{n+1}, x_1, \dots, x_{n-1}\}$. As $|d''| = 2$ and

$|d''_i| = n$, noting that $V(x_i) = V(d) = V(d'')$ we know that

$$\begin{aligned}
f(V(d'')) &= f(V(d)) \\
&= \frac{1}{2}f(V(d''_1)) + \frac{1}{2}f(V(d''_2)) \\
&= \frac{1}{2n} \left(\sum_{i=1}^{n+1} f(V(d_i)) + \sum_{i=1}^{n-1} f(V(x_i)) \right) \\
&= \frac{1}{2n} \left(\sum_{i=1}^{n+1} f(V(d_i)) + (n-1)f(V(d)) \right)
\end{aligned}$$

Thus we have

$$V(d) = f^{-1} \left(\frac{1}{n+1} \sum f(V(d_i)) \right)$$

■

PROOF OF PROPOSITION 2.2.1: The sufficiency is straightforward. If both (V, f) and (\tilde{V}, \tilde{f}) represent P , since the Luce value is unique up to a scalar multiplication,

$$V(a) = \alpha_1 \times \tilde{V}(a)$$

for all $a \in \mathcal{D}$, and $\alpha_1 > 0$.

As for f 's uniqueness, consider now $x, y \in D_0$. Define $v_1 := V(x)$, $v_2 := V(y)$ and $v_3 := V(\{x, y\})$, and similarly $\tilde{v}_1 := \tilde{V}(x)$, $\tilde{v}_2 := \tilde{V}(y)$ and $\tilde{v}_3 := \tilde{V}(\{x, y\})$. We have

$$f(v_3) = \frac{1}{2}f(v_1) + \frac{1}{2}f(v_2) \tag{2.16}$$

and

$$\tilde{f}(\tilde{v}_3) = \frac{1}{2}\tilde{f}(\tilde{v}_1) + \frac{1}{2}\tilde{f}(\tilde{v}_2)$$

Since we already have $V(a) = \alpha_1 \tilde{V}(a)$, let us define $\hat{f}(\tilde{v}) := f(\alpha_1 \tilde{v})$. Now (2.16) becomes

$$\hat{f}(\tilde{v}_3) = \frac{1}{2}\hat{f}(\tilde{v}_1) + \frac{1}{2}\hat{f}(\tilde{v}_2)$$

Thus

$$\hat{f}^{-1}\left(\frac{1}{2}\hat{f}(\tilde{v}_1) + \frac{1}{2}\hat{f}(\tilde{v}_2)\right) = \tilde{f}^{-1}\left(\frac{1}{2}\tilde{f}(\tilde{v}_1) + \frac{1}{2}\tilde{f}(\tilde{v}_2)\right) \quad (2.17)$$

Define $t_1 := \hat{f}(\tilde{v}_1)$ and $t_2 := \hat{f}(\tilde{v}_2)$. (2.17) becomes

$$\tilde{f} \circ \hat{f}^{-1}\left(\frac{1}{2}t_1 + \frac{1}{2}t_2\right) = \frac{1}{2}\tilde{f} \circ \hat{f}^{-1}(t_1) + \frac{1}{2}\tilde{f} \circ \hat{f}^{-1}(t_2)$$

Since \tilde{v}_1 and \tilde{v}_2 can be arbitrary on the domain, by Jensen's inequality, it must be true that

$$\tilde{f} \circ \hat{f}^{-1}(t) = \alpha'_2 t + \beta'$$

and hence $\tilde{f}(\tilde{v}) = \alpha'_2 \hat{f}(\tilde{v}) + \beta'$. Since both f and \tilde{f} are strictly increasing, $\alpha'_2 > 0$.

Reorganizing the equation with $\alpha_2 := \frac{1}{\alpha'_2}$ and $\beta_2 := -\frac{\beta'_2}{\alpha'_2}$, we get

$$f(\alpha_1 \tilde{v}) = \alpha_2 \tilde{f}(\tilde{v}) + \beta$$

■

PROOF OF THEOREM 2.3.1: We first prove the sufficiency. Suppose P_1 and P_2 can be represented by (V_1, f_1) and (V_2, f_2) respectively. Since $V_1(x) = V_2(x)$ for $x \in D_0$, P_1 must coincide with P_2 on depth-1 decision trees according to (3.1). Now for any $x \in D_0$, $a \in D_1$, that is $a = \{x_1, \dots, x_n\}$, let $v_i := V_1(x_i) = V_2(x_i)$. Since

$$f_2 = g \circ f_1,$$

$$\begin{aligned} f_2(V_2(a)) &= \frac{1}{n} \sum f_2(v_i) \\ g \circ f_1(V_2(a)) &= \frac{1}{n} \sum g \circ f_1(v_i) \end{aligned}$$

On the other hand, $f_1(V_1(a)) = \frac{1}{n} \sum f_1(v_i)$. By Jensen's inequality, as

$$\frac{1}{n} \sum g \circ f_1(v_i) \leq g \left(\frac{1}{n} \sum f_1(v_i) \right) = g(f_1(V_1(a)))$$

it's clear that $V_1(a) \geq V_2(a)$, and hence $a \succeq_2 x$ implies $a \succeq_1 x$. Now suppose we have proved that for any $m' \leq m$, $a \succeq_2 x$ implies $a \succeq_1 x$ for any $x \in D_0$ and $a \in \bigcup_{i=1}^{m'} D_i$. Now for a $b = \{b_1, \dots, b_n\} \in D_{m+1}$, by the induction hypothesis we have $V_1(b_i) \geq V_2(b_i)$, and thus

$$\begin{aligned} V_1(b) &= f_1^{-1} \left(\frac{1}{n} \sum f_1(V_1(b_i)) \right) \\ &\geq f_1^{-1} \left(\frac{1}{n} \sum f_1(V_2(b_i)) \right) \\ &\geq f_2^{-1} \left(\frac{1}{n} \sum f_2(V_2(b_i)) \right) \\ &= V_2(b) \end{aligned}$$

Next we prove the necessity. Since P_1 and P_2 coincide on the depth-1 decision trees, and they are both Luce rules, we can by Proposition 2.2.1 set $\alpha_1 = 1$ and find V_1 and V_2 such that $V_1(x) = V_2(x)$ for $x \in D_0$. Suppose (V_i, f_i) represents P_i . Define $g := f_2 \circ f_1^{-1}$. The function g is clearly strictly increasing. We know that

for any $x \in D_0$ and $a = \{x_1, \dots, x_n\} \in D_1$, $a \succeq_2 x$ implies $a \succeq_1 x$, where we again let $v_i := V_1(x_i) = V_2(x_i)$. In particular, by richness, we can find $y \in D_0$ such that $a \sim_2 y$. Say $V_1(y) = V_2(y) = v$; that is, $V_2(a) = f_2^{-1}(\frac{1}{n} \sum f_2(v_i)) = v$, and $V_1(a) \geq v$, which implies

$$\begin{aligned} f_1^{-1} \left(\frac{1}{n} \sum f_1(v_i) \right) &\geq f_2^{-1} \left(\frac{1}{n} \sum f_2(v_i) \right) \\ g \left(\frac{1}{n} \sum f_1(v_i) \right) &\geq \frac{1}{n} \sum f_2(v_i) \end{aligned}$$

Define $t_i := f_1(v_i)$. The inequality above becomes $\frac{1}{n} \sum g(t_i) \leq g(\frac{1}{n} \sum t_i)$, which implies that g is concave.

■

PROOF OF PROPOSITION 2.3.1: Suppose a rich CCA BBR is homogeneous.

Then from

$$\frac{V(x)}{V(x) + V(w)} \geq \frac{V(y)}{V(y) + V(z)}$$

we know that

$$V(x)/V(w) \geq V(y)/V(z)$$

By definition, $V(a) = (\frac{1}{2}[V(x)]^\gamma + \frac{1}{2}[V(w)]^\gamma)^{1/\gamma}$ and $V(b) = (\frac{1}{2}[V(y)]^\gamma + \frac{1}{2}[V(z)]^\gamma)^{1/\gamma}$.

Therefore

$$\begin{aligned} V(a) &= V(x) \left(\frac{1}{2} + \frac{1}{2} \left(\frac{V(w)}{V(x)} \right)^\gamma \right)^{1/\gamma} \\ &\leq V(x) \left(\frac{1}{2} + \frac{1}{2} \left(\frac{V(z)}{V(y)} \right)^\gamma \right)^{1/\gamma} \\ &= \frac{V(x)}{V(y)} V(b) \end{aligned}$$

which implies that

$$\frac{V(x)}{V(x) + V(a)} \geq \frac{V(y)}{V(y) + V(b)}$$

To show necessity, notice that $P(\{x\}, a) = P(\{y\}, b)$ implies that $P(\{x\}, \{x, a\}) = P(\{y\}, \{y, b\})$. Since

$$\frac{V(x)}{V(x) + V(w)} = \frac{V(y)}{V(y) + V(z)}$$

there exists a α such that $V(x) = \alpha V(y)$ and $V(w) = \alpha V(z)$. Since

$$\frac{V(x)}{V(x) + V(a)} = \frac{V(y)}{V(y) + V(b)}$$

we know that $V(a) = \alpha V(b)$ too. By richness, we can pick any α . Therefore f must be homogeneous of degree 1, and take the form $f(v) = \beta v^\gamma$ (see Wnuk (1984)).

■

PROOF OF THEOREM 2.3.2: First we show the sufficiency. For any $\{x, y\} \in D_1$ such that $V_2(x) \leq V_2(y)$, we can let h map $P_2(\{x\}, \{x, y\})$ to $P_1(\{x\}, \{x, y\})$. It is clear that $P_1(\{x\}, \{x, y\}) \leq P_2(\{x\}, \{x, y\})$ since $\lambda \in (0, 1)$. The only thing

that needs to be shown is that h is well-defined; that is, for $P_2(\{x\}, \{x, y\}) = P_2(\{x'\}, \{x', y'\})$, we have $P_1(\{x\}, \{x, y\}) = P_1(\{x'\}, \{x', y'\})$ too. Since $[V_1(x)]^\lambda = V_2(x)$,

$$\frac{V_2(x)}{V_2(x) + V_2(y)} = \frac{V_2(x')}{V_2(x') + V_2(y')}$$

implies that

$$\begin{aligned} V_2(x)/V_2(y) &= V_2(x')/V_2(y') \\ [V_2(x)]^{1/\lambda}/[V_2(y)]^{1/\lambda} &= [V_2(x')]^{1/\lambda}/[V_2(y')]^{1/\lambda} \\ V_1(x)/V_1(y) &= V_1(x')/V_1(y') \end{aligned}$$

Thus we know that $P_1(\{x\}, \{x, y\}) = P_1(\{x'\}, \{x', y'\})$.

Now consider the necessity. Under the hypothesis of the theorem, $P_2(\{x\}, \{x, y\}) \leq \frac{1}{2}$ if and only if $V_2(x) \leq V_2(y)$. Since $h(p) \leq p$, we know that if $V_2(x) < V_2(y)$, then $V_1(x) < V_1(y)$ too, for $x, y \in D_0$. And since $h(\frac{1}{2}) = \frac{1}{2}$, we know that $V_2(x) = V_2(y)$ implies that $V_1(x) = V_1(y)$. Thus there is a strictly increasing function ϕ such that $V_1(x) = \phi(V_2(x))$, $x \in D_0$. Now for any $x, y \in D_0$ such that $V_2(x) \leq V_2(y)$, by richness, we can find x_α and y_α such that $V_2(x_\alpha) = \alpha V_2(x)$ and $V_2(y_\alpha) = \alpha V_2(y)$.

Notice that

$$P_2(\{x\}, \{x, y\}) = P_2(\{x_\alpha\}, \{x_\alpha, y_\alpha\})$$

we must have

$$P_1(\{x\}, \{x, y\}) = P_1(\{x_\alpha\}, \{x_\alpha, y_\alpha\})$$

which implies that $V_1(x_\alpha) = \psi_x(\alpha)V_1(x)$, $V_2(y_\alpha) = \psi_y(\alpha)V_2(y)$ and $\psi_x(\alpha) = \psi_y(\alpha)$. Since x and y are arbitrary, there has to be a $\psi(\alpha) = \psi_x(\alpha)$ for any $x \in D_0$. Thus

$$\begin{aligned}\phi(V_2(x_\alpha)) &= \phi(\alpha V_2(x)) \\ V_1(x_\alpha) &= \psi(\alpha)V_1(x) \\ &= \psi(\alpha)\phi(V_2(x))\end{aligned}$$

Therefore we have

$$\phi(\alpha v) = \psi(\alpha)\phi(v)$$

To satisfy the equation above, according to Aczél (1966, p. 144–145), $\phi(v) = \alpha_1 v^{\lambda'}$; that is, $V_1(x) = \alpha_1[V_2(x)]^{\lambda'}$. With an abuse of notation, we can pick the representation via Proposition 2.2.1 so that $\alpha_1 = 1$. Then $h(p) \leq p$ implies that $\lambda' \geq 1$, and thus $\lambda := 1/\lambda' \in (0, 1]$.

■

PROOF OF THEOREM 2.4.1 AND PROPOSITION 2.4.4: In our example, we deal with only x_1 , x_2 and x_3 . Let us define $v_i := V(x_i)$ and

$$v_{i,j} = \left(\frac{1}{2}[V(d_i)]^\gamma + \frac{1}{2}[V(d_j)]^\gamma \right)^{1/\gamma}$$

Note that $v_{i,i} = v_i$. Sometimes to emphasize the parameter γ , we write $v_{i,j}(\gamma)$. Now

$$P_E = \frac{v_1}{v_1 + v_{2,3}} = \frac{v_{1,1}}{v_{1,1} + v_{2,3}}$$

and

$$P_R = \frac{v_{1,2}}{v_{1,2} + v_{1,3}} \times \frac{v_1}{v_1 + v_2} + \frac{v_{1,3}}{v_{1,2} + v_{1,3}} \times \frac{v_1}{v_1 + v_3}$$

Again sometimes to emphasize the parameter γ , we write $P_E(\gamma)$ and $P_R(\gamma)$.

We first prove the following lemmas. Define $\tilde{v}(v_x, v_y; \gamma) = (\frac{1}{2}v_x^\gamma + \frac{1}{2}v_y^\gamma)^{1/\gamma}$.

Lemma 2.6.3 $\tilde{v}(v_x, v_y; \gamma)$ is increasing in γ . It's strictly increasing if $v_x \neq v_y$.

PROOF OF LEMMA 2.6.3: Say $\gamma_1 \leq \gamma_2$. We have

$$\begin{aligned} \tilde{v}(v_x, v_y; \gamma_2) &= \left(\frac{1}{2}v_x^{\gamma_2} + \frac{1}{2}v_y^{\gamma_2}\right)^{1/\gamma_2} \\ &= \left(\frac{1}{2}v_x^{\gamma_1 \times (\gamma_2/\gamma_1)} + \frac{1}{2}v_y^{\gamma_1 \times (\gamma_2/\gamma_1)}\right)^{1/\gamma_2} \\ &\geq \left(\frac{1}{2}v_x^{\gamma_1} + \frac{1}{2}v_y^{\gamma_1}\right)^{(\gamma_2/\gamma_1) \times 1/\gamma_2} \\ &= \tilde{v}(v_x, v_y; \gamma_1). \end{aligned}$$

It is not difficult to see the strictness. ■

To prove the next lemma, first notice that \tilde{v} is homogenous of degree 1, that is,
 $\tilde{v}(\alpha v_x, \alpha v_y; \gamma) = \alpha \tilde{v}(v_x, v_y; \gamma)$.

Lemma 2.6.4 If $v_l \leq v_m \leq v_h$, $\gamma_1 \leq \gamma_2$, then

$$\frac{\tilde{v}(v_l, v_h; \gamma_2)}{\tilde{v}(v_l, v_h; \gamma_1)} \geq \frac{\tilde{v}(v_m, v_i; \gamma_2)}{\tilde{v}(v_m, v_i; \gamma_1)}, \quad i = l \text{ or } h$$

PROOF OF LEMMA 2.6.4: We prove the case of $i = h$. $i = l$ can be proven similarly. Obviously $\tilde{v}(v_m, v_h; \gamma_1) \geq \tilde{v}(v_l, v_m; \gamma_1)$ since \tilde{v} is increasing in its both

arguments. Suppose for some $\beta \geq 1$, $\tilde{v}(v_m, v_h; \gamma_1) = \beta \tilde{v}(v_l, v_h; \gamma_1)$. By the homogeneity,

$$\tilde{v}(v_m, v_h; \gamma_1) = \tilde{v}(\beta v_l, \beta v_h; \gamma_1)$$

Note that it has to be the case that $\beta v_l \leq v_m$, as we already have $\beta v_h \geq v_h$. Now γ_1 is increased to γ_2 . Due to the homogeneity, we still have $\tilde{v}(\beta v_l, \beta v_h; \gamma_2) = \beta \tilde{v}(v_l, v_h; \gamma_2)$.

If we can show that $\tilde{v}(v_m, v_h; \gamma_2) \leq \beta \tilde{v}(v_l, v_h; \gamma_2)$, then

$$\tilde{v}(v_m, v_h; \gamma_2) \leq \frac{\tilde{v}(v_m, v_h; \gamma_1)}{\tilde{v}(v_l, v_h; \gamma_1)} \tilde{v}(v_l, v_h; \gamma_2)$$

which implies the conclusion.

Suppose $\tilde{v}(v_m, v_h; \gamma_2) := v' > \beta \tilde{v}(v_l, v_h; \gamma_2) = \tilde{v}(\beta v_l, \beta v_h; \gamma_2) := v''$. Say $\tilde{v}(v_m, v_h; \gamma_1) = \tilde{v}(\beta v_l, \beta v_h; \gamma_1) = v = \tilde{v}(v, v; \gamma_1)$. We know that $v'' \geq v$ by Lemma 2.6.3. Clearly there is one and only one point (v_m, v_y) such that $\tilde{v}(\beta v_l, \beta v_h; \gamma_2) = \tilde{v}(v_m, v_y; \gamma_2)$ by continuity and strict monotonicity. We know that $v_y \geq \beta v_h$ since $\beta v_l \leq v_m$. Furthermore, we know that $v_y \geq v_h$. To see this, define curve $C(v, \gamma) := \{(v'_x, v'_y) : v'_x \geq v'_y \text{ and } \tilde{v}(v'_x, v'_y; \gamma) = v\}$. We can show that for every $C(v, \gamma)$, its slope at $(v'_x, v'_y) \in C(v, \gamma)$ is decreasing in γ because the slope is equal to

$$-\frac{\partial \tilde{v}(v'_x, v'_y; \gamma) / \partial v'_x}{\partial \tilde{v}(v'_x, v'_y; \gamma) / \partial v'_y} = -\left(\frac{v'_x}{v'_y}\right)^{\gamma-1}$$

Together with some standard arguments, it is not difficult to conclude that $v_y \geq v_h$. Now we have $\tilde{v}(v_m, v_h; \gamma_2) = v' > v'' = \tilde{v}(\beta v_l, \beta v_h; \gamma_2) = \tilde{v}(v_m, v_y; \gamma_2)$ but $v_y \geq v_h$, which is a contradiction. ■

The next lemma can be stated in a much more general way, but for our purpose, we merely need to prove the following version.

Lemma 2.6.5 For $\tilde{v}(v_x, v_y; \gamma)$, $\frac{\partial^2 \tilde{v}}{\partial v_x \partial v_y} \leq 0$ if $\gamma \geq 1$, $\frac{\partial^2 \tilde{v}}{\partial v_x \partial v_y} \geq 0$ if $\gamma \leq 1$.

PROOF OF LEMMA 2.6.5:

$$\frac{\partial \tilde{v}}{\partial v_x} = \frac{1}{2} v_x^{\gamma-1} \left(\frac{1}{2} v_x^\gamma + \frac{1}{2} v_y^\gamma \right)^{1/\gamma-1}$$

and hence

$$\frac{\partial^2 \tilde{v}}{\partial v_x \partial v_y} = \frac{1}{4} (1 - \gamma) v_x^{\gamma-1} v_y^{\gamma-1} \left(\frac{1}{2} v_x^\gamma + \frac{1}{2} v_y^\gamma \right)^{1/\gamma-2} \quad (2.18)$$

Clearly (2.18) is less than 0 if $\gamma \geq 1$, and vice versa. ■

Without loss of generality, let $v_2 \leq v_3$. First let us consider $v_2 \leq v_3 \leq v_1$.

Clearly P_E is decreasing in γ since $v_{2,3}$ is increasing in γ by Lemma 2.6.3. Consider

$$P_R = \frac{v_{1,2}}{v_{1,2}+v_{1,3}} \frac{v_1}{v_1+v_2} + \frac{v_{1,3}}{v_{1,2}+v_{1,3}} \frac{v_1}{v_1+v_3}. \text{ By Lemma 2.6.4, if } \gamma_1 \geq \gamma_2, \frac{v_{1,2}(\gamma_1)}{v_{1,2}(\gamma_2)} \geq \frac{v_{1,3}(\gamma_1)}{v_{1,3}(\gamma_2)}.$$

Therefore

$$\begin{aligned} \frac{v_{1,2}(\gamma_1)}{v_{1,2}(\gamma_1) + v_{1,3}(\gamma_1)} &= \frac{1}{1 + \frac{v_{1,3}(\gamma_1)}{v_{1,2}(\gamma_1)}} \\ &\geq \frac{1}{1 + \frac{v_{1,3}(\gamma_2)}{v_{1,2}(\gamma_2)}} \\ &= \frac{v_{1,2}(\gamma_2)}{v_{1,2}(\gamma_2) + v_{1,3}(\gamma_2)} \end{aligned}$$

Notice that $\frac{v_1}{v_1+v_2} \geq \frac{v_1}{v_1+v_3}$, therefore $P_R(\gamma_1) \geq P_R(\gamma_2)$; that is, P_R in this case is increasing in γ , which implies that $P_E - P_R$ is decreasing in γ . It's easy to verify that $P_E = P_R$ when $\gamma = 1$. Thus if $\gamma \geq 1$, $P_R \geq P_E$, and if $\gamma \leq 1$, $P_R \leq P_E$.

If $v_2 \leq v_1 \leq v_3$ and $\gamma \geq 1$, then

$$\begin{aligned}
P_R &= \frac{v_{1,2}}{v_{1,2} + v_{1,3}} \frac{v_1}{v_1 + v_2} + \frac{v_{1,3}}{v_{1,2} + v_{1,3}} \frac{v_1}{v_1 + v_3} \\
&\geq \frac{(v_1 + v_2)/2}{v_{1,2} + v_{1,3}} \frac{v_1}{v_1 + v_2} + \frac{(v_1 + v_3)/2}{v_{1,2} + v_{1,3}} \frac{v_1}{v_1 + v_3} \\
&= \frac{v_1}{v_{1,2} + v_{1,3}},
\end{aligned}$$

since when $\gamma \geq 1$, $v_{1,i} \geq \frac{1}{2}v_1 + \frac{1}{2}v_i$, $i = 2, 3$, by Lemma 2.6.3. By Lemma 2.6.5, it is easy to show that $v_1 - v_{1,2} \geq v_{1,3} - v_{2,3}$, and hence

$$\frac{v_1}{v_{1,2} + v_{1,3}} \geq \frac{v_1}{v_1 + v_{2,3}}.$$

Therefore if $\gamma \geq 1$, $P_R \geq P_E$. The other case of $\gamma \leq 1$ can be proven similarly.

Finally, to see why Proposition 2.4.4 is true, notice that when $v_1 \leq v_2 \leq v_3$, P_E is decreasing in γ because $v_{2,3}$ is increasing in γ by Lemma 2.6.3. Same is P_R . Since $P_R = \frac{v_{1,2}}{v_{1,2} + v_{1,3}} \times \frac{v_1}{v_1 + v_2} + \frac{v_{1,3}}{v_{1,2} + v_{1,3}} \times \frac{v_1}{v_1 + v_3}$ where by Lemma 2.6.4, when γ increases, the increase of $v_{1,3}$ is larger than the increase of $v_{1,2}$, which implies that $\frac{v_{1,2}}{v_{1,2} + v_{1,3}}$ will decrease and $\frac{v_{1,3}}{v_{1,2} + v_{1,3}}$ will increase. Since $\frac{v_1}{v_1 + v_3} \geq \frac{v_1}{v_1 + v_2}$, when γ increase, P_R would decrease.

Now notice that

$$\lim_{\gamma \rightarrow +\infty} P_R = \frac{v_2}{v_2 + v_3} \frac{v_1}{v_1 + v_2} + \frac{v_3}{v_2 + v_3} \frac{v_1}{v_1 + v_3}$$

$$\lim_{\gamma \rightarrow -\infty} P_R = \frac{1}{2} \frac{v_1}{v_1 + v_2} + \frac{1}{2} \frac{v_1}{v_1 + v_3}$$

while

$$\lim_{\gamma \rightarrow +\infty} P_E = \frac{v_1}{v_1 + v_3}$$

$$\lim_{\gamma \rightarrow -\infty} P_E = \frac{v_1}{v_1 + v_2}$$

Therefore we know that

$$\lim_{\gamma \rightarrow -\infty} P_E \leq \lim_{\gamma \rightarrow -\infty} P_R \leq \lim_{\gamma \rightarrow +\infty} P_R \leq \lim_{\gamma \rightarrow +\infty} P_E$$

Note that when $\gamma = 1$, $P_E = P_R$ as usual. By the intermediate value theorem, there exist $\bar{\gamma}$ and $\underline{\gamma}$ such that $P_E(\bar{\gamma}) = \lim_{\gamma \rightarrow +\infty} P_R$ and hence for any $\gamma > \bar{\gamma}$, $P_E > P_R$. Similar argument can be applied to the other direction.

■

2.6.2 Additional Results

An Approximate Result of Advertising Strategy Comparison

Proposition 2.6.1 *Suppose RCR P is a CCA BBR, and $V(x_1) \leq V(x_2) \leq V(x_3)$.*

Then $\gamma \geq (\leq) 1$ implies $P_R > (<) P_E - (+) \frac{1}{16}$.

PROOF OF PROPOSITION 2.6.1: Suppose $v_1 \leq v_2 \leq v_3$. Consider first the case of $\gamma \geq 1$. Since $v_{1,2} \geq v_1$ and $v_{2,3} \geq v_{1,3}$, we know that

$$\frac{v_{1,2}}{v_{1,2} + v_{1,3}} \geq \frac{v_1}{v_1 + v_{2,3}}$$

Therefore $P_R \geq P'_R := \frac{v_1}{v_1+v_{2,3}} \frac{v_1}{v_1+v_2} + \frac{v_{1,3}}{v_{1,2}+v_{1,3}} \frac{v_1}{v_1+v_3}$. The inequality becomes an equality if and only if $v_1 = v_2$. Define $\Delta := \frac{v_1}{2v_1+v_2+v_3} \frac{v_2-v_1}{v_2+v_1}$. We prove that $P'_R + \Delta \geq P_E$. It is easy to verify that when $\gamma = 1$,

$$P'_R(1) + \Delta = P_E(1) = P_R(1) = \frac{2v_1}{2v_1 + v_2 + v_3}$$

where Δ is not affected by γ . Now consider γ increasing from 1 to $\gamma > 0$. Since $v_{2,3}$ increases, P_E changes from $\frac{v_1}{v_1 + \frac{v_2+v_3}{2}}$ to $\frac{v_1}{v_1 + \frac{v_2+v_3}{2}} - \delta$ for some nonnegative constant $\delta \geq 0$ ($\delta = 0$ if and only if $v_2 = v_3$). Similarly, P'_R changes from $\frac{v_1}{v_1+v_{2,3}} \frac{v_1}{v_1+v_2} + \frac{v_{1,3}}{v_{1,2}+v_{1,3}} \frac{v_1}{v_1+v_3}$ to

$$\left(\frac{v_1}{v_1 + v_{2,3}} - \delta\right) \frac{v_1}{v_1 + v_2} + \frac{v_{1,3}(\gamma)}{v_{1,2}(\gamma) + v_{1,3}(\gamma)} \frac{v_1}{v_1 + v_3}$$

According to Lemma 2.6.5, we know that $\frac{v_{1,3}(\gamma)}{v_{1,2}(\gamma)+v_{1,3}(\gamma)}$ is increasing in γ . Therefore

$$\begin{aligned} P_E(\gamma) &= \frac{v_1}{v_1 + \frac{v_2+v_3}{2}} - \delta \\ &= P'_R(1) + \Delta - \delta \\ &\leq \frac{v_1}{v_1 + v_{2,3}} \frac{v_1}{v_1 + v_2} - \delta + \frac{v_{1,3}(\gamma)}{v_{1,2}(\gamma) + v_{1,3}(\gamma)} \frac{v_1}{v_1 + v_3} + \Delta \\ &\leq \left(\frac{v_1}{v_1 + v_{2,3}} - \delta\right) \frac{v_1}{v_1 + v_2} + \frac{v_{1,3}(\gamma)}{v_{1,2}(\gamma) + v_{1,3}(\gamma)} \frac{v_1}{v_1 + v_3} + \Delta \\ &= P'_R(\gamma) + \Delta \end{aligned}$$

To summarize, we have now

$$P_R + \Delta \geq P'_R + \Delta \geq P_E$$

for all values of $v_1 \leq v_2 \leq v_3$ and $\gamma \geq 1$; that is $P_R \geq P_E - \Delta$ and the inequality becomes an equality if and only if $v_1 = v_2 = v_3$. Since

$$\begin{aligned} \Delta &= \frac{v_1}{2v_1 + v_2 + v_3} \frac{v_2 - v_1}{v_2 + v_1} \\ &\leq \frac{v_1(v_2 - v_1)}{2(v_1 + v_2)^2} \\ &= \frac{v_2/v_1 - 1}{2(v_2/v_1 + 1)^2} \end{aligned}$$

Define $t := v_2/v_1 - 1 \in [0, +\infty)$ and $T(t) := \frac{t}{2(t+2)^2}$. It is not difficult to analyze this function and conclude that

$$\max_{t \in [0, +\infty)} T(t) = \frac{1}{16}$$

reached when $t = 2$. Hence we prove one side of the case of $v_1 \leq v_2 \leq v_3$.

Notice that when $t = 2$, $v_1 < v_2$. Therefore our result for this case is not tight.

For the other side of the statement, we define $P''_R := \frac{v_{1,2}}{v_{1,2} + v_{1,3}} \frac{v_1}{v_1 + v_2} + \frac{v_{2,3}}{v_{1,2} + v_{2,3}} \frac{v_1}{v_1 + v_3}$, and $\Delta' := \frac{v_1}{2v_1 + v_2 + v_3} \frac{v_2 - v_1}{v_3 + v_1}$. Then similarly one can show that $P_R - \Delta' \leq P''_R - \Delta' \leq P_E$, if $\gamma \leq 1$. Since

$$\begin{aligned} \Delta' &= \frac{v_1}{2v_1 + v_2 + v_3} \frac{v_2 - v_1}{v_3 + v_1} \\ &\leq \frac{v_1(v_2 - v_1)}{2(v_1 + v_2)^2} \end{aligned}$$

the same argument continues, proving the case of $\gamma \leq 1$.

■

The Depth of Outcomes

Here we show that for a CCA BBR with $\gamma = 1$, to calculate the Luce value of a decision tree, the structure of the tree matters only through the depth of outcomes. For simplicity, let us only consider a decision tree a whose parent nodes all have a branching factor k , where a branching factor means the number of children nodes that a parent node has. The result can be easily extended to general finite decision trees. Again for simplicity, suppose each outcomes x_i that appears on a leaf node of a does not appears more than once. Then we must have either $x_i \in a$ or $x_i \in a^{(1)} \in \dots \in a^{(l_i-1)} \in a$ for some $l_i \geq 2$, but not both. In the former case, x_i 's depth $l_i = 1$. In the latter case, l_i is x_i 's depth. Since x_i only appears once, the depth is well-defined. If $x_i \in a$ or $x_i \in a^{(1)} \in \dots \in a^{(l_i-1)} \in a$, we say that x_i is an outcome of a .

Let $v_i := V(x_i)$. We want to prove that for a decision tree a that has outcomes x_1 through x_n and induces an n -tuple of depths (l_1, \dots, l_n) , the probability that the decision maker chooses each outcome is

$$\frac{k^{-l_i} v_i}{\sum k^{-l_j} v_j} \tag{2.19}$$

and the Luce value of a is

$$\sum k^{-l_j} v_j \tag{2.20}$$

Suppose $a \in D_1$ and satisfies our assumptions. We know that $|a| = k$, and each x_i has depth 1. Then $\frac{v_i}{\sum_{j=1}^k v_j}$ is the probability of x_i being chosen, which is equivalent to $\frac{\frac{1}{k}v_i}{\sum \frac{1}{k}v_j}$. Thus (2.19) holds. Clearly the Luce value of a is $\sum \frac{1}{k}v_j$. Thus (2.20) holds.

Now suppose for any $a \in D_m$ that satisfies our assumptions, $m \leq n$ for some $n \geq 1$, both (2.19) and (2.20) hold. Consider any $a \in D_{n+1}$ that satisfies our assumptions. Again $|a| = k$. Suppose $a = \{a_1, \dots, a_k\}$ where $a_i \in D_{m_i}$ for some $m_i \leq n$. The n -tuple of depths it induces is (l_1, \dots, l_n) . Obviously the depth of x_i in a exceeds its depth in a_j by 1, if x_i is an outcome of a_j . Since the Luce value of a is $\sum_{j=1}^k \frac{1}{k}V(a_j)$,

$$\sum_{j=1}^k \frac{1}{k}V(a_j) = \frac{1}{k} \sum k^{-l_i+1}v_i = \sum k^{-l_i}v_i$$

Thus (2.20) holds.

The probability of x_i being chosen is equal to $P(\{a_j\}, a)$ times the probability of x_i being chosen from a_j , if x_i is an outcome of a_j . We know that that latter is

$$\frac{k^{-(l_i-1)}v_i}{\sum_{\{t:x_t \in \dots \in d_j\}} k^{-(l_t-1)}v_t}$$

while the former is

$$\frac{\sum_{\{t:x_t \in \dots \in d_j\}} k^{-(l_t-1)}v_t}{\sum_{t=1}^n k^{-(l_t-1)}v_t}.$$

Combining the two, it is not difficult to see that (2.19) holds.

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Chapter 3

Forward Looking in Dynamic Choice - An Impossibility Result

3.1 Introduction

Confronting a dynamic economic problem, a decision maker needs to make a sequence of choices. Each choice that she makes not only affects what she receives at the current stage, but also what continuation problems that she would face in the future. Hence, to make a good decision at the current stage, the decision maker needs to look forward.

When looking forward in a dynamic problem, it is not enough to just understand the structure of it. The decision maker also needs to take her own future choice behavior into account. For example, if the decision maker predicts that she will

make mistakes in the future, then a complicated continuation problem that contains the best prize is not necessarily good.

Furthermore, it is not necessarily true that forward looking takes everything into account correctly. The decision maker might fail to understand the structure of the problem. For instance, the decision maker might have limited attention, and hence some choice options are unnoticed (Masatlioglu, Nakajima and Ozbay (2012), Brady and Rehbeck (2015)). The decision maker might also fail to understand her future choices due to her changing preference, which leads to dynamic inconsistency (Strotz (1955), Laibson (1997)). For instance, a naive decision maker believes that her future choices should be optimal with respect to her current preference, but in fact her future preference changes and she would optimize according to the future preference instead.

In this chapter, we analyze forward looking under a different assumption: the decision maker makes random mistakes when choosing. We assume that the decision maker understands the structure of the dynamic problem, and she does not have a changing preference, but she might and might not understand how exactly she makes mistakes.

We present a negative result showing that if the error-prone decision maker's forward looking satisfies three simple conditions, then she must have ignored her mistakes when looking forward. The first condition *monotonicity* states that a decision problem looks better if its prizes are better. The second condition, *betweenness*, assumes that a decision problem's value should be between its best prize and its worst prize. The reason is simple: even though the decision maker makes mistakes, she would

never end up with any prize better than the best or worse than the worst. Lastly, most existing models of error-prone choice behavior imply that if an option is bad enough, then the decision maker is able to avoid it even though she makes mistakes. The last condition *reducibility* captures this idea in forward looking. It states that if an option is bad enough, it will not affect the value of a decision problem because it would not be chosen.

Our result shows that if the decision maker's forward looking satisfies *monotonicity*, *betweenness* and *reducibility*, then she must identify a decision problem by its best prize. In other words, even though the conditions we consider allow for anticipation of mistakes, they together predict the ignorance of mistakes in forward looking. Moreover, no matter how the decision maker looks forward, *strict monotonicity*, *betweenness* and *reducibility* can never be satisfied at the same time. Most likely, one may believe that *monotonicity* is an innocent assumption, and it is the other conditions that go wrong. Under the most standard assumptions, error-prone choice following the logit model and forward looking exhibiting rational anticipation of mistakes (McKelvey and Palfrey (1998)), we show that, *monotonicity* fails in this case. The intuition is in line with *reducibility*. When an option is obviously inferior compared to the other options, it would be chosen with low probability. However, when it improves, although still inferior than the others, it will be chosen more often. It turns out that the overall effect of improving that inferior option could be negative.

In general, under rational anticipation of mistakes, *betweenness* holds, but either *monotonicity* or *reducibility* fails. The failure of *monotonicity* is not restricted to

the particular functional form of logit. We show that in a more general model of mistakes (Ke (2015a)), *monotonicity* not only fails in the logit case, but also many others.

3.2 An Impossibility Theorem

Consider a decision maker who is asked to evaluate a decision problem. The decision problem consists of several monetary prizes. It is understood that the decision maker needs to make a choice among those prizes at a future stage. We ask the decision maker to report the certainty equivalent of the decision problem; that is, what monetary prize is considered to be indifferent to the decision problem.

A monetary prize is a real number. A decision problem $X = (x_1, \dots, x_n)$ is an n -tuple of prizes, where n could be any positive integer. Let $\mathbb{X} = \bigcup_{n=1}^{+\infty} \mathbb{R}^n$ be the set of all decision problems. The reported certainty equivalent of the decision problem X is $I(X)$ where I is a function that maps \mathbb{X} to \mathbb{R} such that for any $x \in \mathbb{R}$

$$I((x)) = x$$

We call I the *foresight function*.

In the simplest case, this question is trivial: $I(X)$ should simply be equal to $\max x_i$. However, the decision maker might make mistakes when choosing. For example, in Caplin, Dean and Martin's (2011) experiments, the decision maker performs some simple addition and subtraction to figure out the exact monetary prize

associated with each option. Even though the calculation is simple enough, the decision maker sometimes fails to identify the option with the best prize.

When looking forward to evaluate a decision problem, the decision maker may and may not be aware of her mistakes. Even if she anticipates mistakes, her belief about the mistakes may not be "rational". However, there are a few conditions that seem natural to impose on the foresight function, under the presence of mistakes. To facilitate the explanation for those conditions, let us first introduce some existing models of how the decision maker makes mistakes.

3.2.1 Preliminaries: Models of Error-Prone Choice

There are several ways to model how the decision maker makes random mistakes. For a decision problem $X = (x_1, \dots, x_n)$, let $P(X)$ be the probability distribution that describes how the decision maker chooses. In other words, $P_i(X)$ is the probability that x_i is chosen when the decision problem is X . Of course, $P_i(X) \geq 0$ and $\sum_i P_i(X) = 1$ for any $X \in \mathbb{X}$. We call P the *random choice function*. Below are a few examples of models of random mistakes.

1. *Logit Model*: When facing a decision problem $X = (x_1, \dots, x_n)$, the decision maker chooses x_i with probability

$$P_i(X) = \frac{\exp\{u(x_i)\}}{\sum_j \exp\{u(x_j)\}} \quad (3.1)$$

where u is the utility function, and $\exp\{u(x_i)\}$ is also called x_i 's Luce value (see Luce (1959)). This choice probability formula can be generated as follows.

Confronting X , the decision maker receives a noisy signal $u(x_j) + \varepsilon_j$ for each x_j , where ε_j follows some *i.i.d.* Gumbel distribution. The decision maker chooses the prize with the highest signal. It is shown that such a procedure will induce the logit formula in (3.1) (see McFadden (1973)).

2. *Quantal Response Model (QRM)*: In McKelvey and Palfrey (1995, 1998), when facing a decision problem $X = (x_1, \dots, x_n)$, the decision maker receives a noisy signal $u(x_i) + \varepsilon_i$ for each prize x_i , where u is the utility function and ε_i is the noise term that satisfies some mild assumptions on the distribution. The decision maker chooses the prize that has the highest signal. Since the signal is random, the decision maker's choice is random from the observer's point of view. In this case, $P_i(X)$ is equal to the probability that x_i has the highest signal value. This model nests the logit model as a special case when the error terms follow some *i.i.d.* Gumbel distribution.
3. *Anticipated-Mistakes Rule (AMR)*: In Ke (2015a), the decision maker chooses each x_i from (x_1, \dots, x_n) with probability $\frac{\phi(u(x_i))}{\sum_{j=1}^n \phi(u(x_j))}$ where u is the utility function, and ϕ is a surjective strictly increasing function that maps $u(\mathbb{R})$ to \mathbb{R}_{++} . In an AMR, $\phi(u(x_i))$ is the Luce value of x_i . This model nests the logit model as a special case when the ϕ function is exponential.

3.2.2 Forward-Looking under Error-Prone Choice

We introduce three simple conditions that seem natural for the foresight function I to satisfy. The first condition is a monotonicity assumption.

Definition 3.2.1 *The foresight function satisfies monotonicity if for any $X = (x_1, \dots, x_{i-1}, x_i, x_{i+1}, \dots, x_n)$ and $X' = (x_1, \dots, x_{i-1}, x'_i, x_{i+1}, \dots, x_n)$*

$$x_i \geq x'_i \Rightarrow I(X) \geq I(X') \quad (3.2)$$

We say that the foresight function satisfies strict monotonicity if it satisfies monotonicity and the strict version of (3.2).

The second condition is based on the following idea. Even though the decision maker chooses with randomness, she can never get anything better than the best monetary prize, nor anything worse than the worst prize. Hence, the foresight function should be between the maximum function and the minimum function. Different forms of this condition have appeared in Bolker (1966), Gärdenfors (1994), Ahn (2008), and Ke (2015a).

Definition 3.2.2 *The foresight function I satisfies betweenness if for any $X = (x_1, \dots, x_n) \in \mathbb{X}$*

$$\min x_i \leq I(X) \leq \max x_i$$

The last condition we consider is called *reducibility*. In $X = (x_1, \dots, x_n)$, when some x_i gets worse and worse, any of the models introduced above (logit, QRM, and AMR) will predict that $P_i(X)$ converges to zero. In other words, when x_i is obviously inferior than the other prizes, it would never be chosen. We take this idea to the foresight function.

Definition 3.2.3 *The foresight function I satisfies reducibility if for any $X = (x_1, \dots, x_n) \in \mathbb{X}$*

$$\lim_{y \rightarrow -\infty} I((x_1, \dots, x_n, y)) = I(X)$$

Theorem 3.2.1 shows that in fact these simple conditions are incompatible. In particular, if I satisfies the three conditions above, the only possible solution is

$$I(X) = \max x_i$$

In other words, the decision maker evaluates a decision problem as if she never makes a mistake. In addition, if we replace the monotonicity condition with strict monotonicity, then there is no function I that can satisfy all the conditions.

Theorem 3.2.1 *The foresight function I satisfies monotonicity, betweenness and reducibility if and only if $I(X) = \max x_i$. Furthermore, there is no foresight function I that satisfies strict monotonicity, betweenness and reducibility.*

Proof. We first prove the second half: *strict monotonicity, betweenness and reducibility* are incompatible. Take any $x \in \mathbb{R}$. From *betweenness*, we know that

$$I((x, x)) = x$$

Take some fixed $y < x$. By *strict monotonicity*, we have

$$I((x, y)) < I((x, x)) = x \tag{3.3}$$

Lastly, due to *reducibility*, we have $I((x, z)) \rightarrow I((x)) = x$ as $z \rightarrow -\infty$. Hence, for some sufficiently small z , we must have $z < y$ but

$$I((x, z)) > I((x, y))$$

which violates *monotonicity*.

Next, we prove the first half: I satisfies *monotonicity*, *betweenness* and *reducibility* if and only if $I(X) = \max x_i$. It is easy to check that $I(X) = \max x_i$ satisfies those three conditions. To see the other direction, notice that now that we only have *monotonicity*, (3.3) will become

$$I((x, y)) \leq I((x, x)) = x$$

for some fixed $y < x$. For any $\varepsilon > 0$, we can find a sufficiently small $z(\varepsilon)$ such that we have $x - \varepsilon \leq I((x, z(\varepsilon))) \leq I((x, y)) \leq x$. Take ε to 0, we know that $I((x, y)) = x$. Since this holds for any $y < x$, and the same argument applies to the other $X \in \mathbb{X}$, we know that $I(X) = \max x_i$. ■

3.3 Failure of Monotonicity under Rational Anticipation of Mistakes

One might think that *reducibility* is the condition that leads to this impossibility result, and that assuming rational anticipation of mistakes, the logit model, the QRM and the AMR seem to satisfy *monotonicity* and *betweenness*, but not *reducibility*.

These conjectures are not entirely true. First of all, let us define what we mean by rational anticipation of mistakes. Suppose we have a utility function $u : \mathbb{R} \rightarrow \mathbb{R}$, as in the logit model, the QRM and the AMR. We say that I exhibits *rational anticipation of mistakes* if

$$I(X) = u^{-1} \left(\sum P_i(X) u(x_i) \right)$$

In other words, the utility of $I(X)$ would be equal to the expected utility that the decision maker can actually derive from the decision problem X . One among the three conditions is never violated in the logit model, QRM and AMR, if I exhibits rational anticipation of mistakes. Let us state this simple fact without proof.

Proposition 3.3.1 *When I exhibits rational anticipation of mistakes, betweenness holds for the logit model, the QRM and the AMR.*

This result is obvious, because in all these models, the utility derived from X is somewhere between the maximum and the minimum of $u(x_i)$. Hence, $I(X)$ should also be between $\max x_i$ and $\min x_i$.

Perhaps a more surprising observation is that *monotonicity* can be easily violated. Let us take the logit model as an example. Consider a fixed $x \in \mathbb{R}$ and $y \rightarrow -\infty$. Assume for a moment that u is the identity function. When the foresight function I exhibits rational anticipation of mistakes

$$I((x, y)) = \frac{x \exp\{x\}}{\exp\{x\} + \exp\{y\}} + \frac{y \exp\{y\}}{\exp\{x\} + \exp\{y\}}$$

It is not difficult to see that in this case

$$\lim_{y \rightarrow -\infty} I((x, y)) = x = I((x))$$

In other words, *reducibility* holds. It follows immediately that *monotonicity* fails. To see the intuition, consider $(-100, 1)$ and $(0, 1.01)$. In the first decision problem $(-100, 1)$, the decision maker rarely chooses the bad prize -100 , because compared to 1 , it is obviously inferior, even though the decision maker makes mistakes. On the other hand, facing $(0, 1.01)$, the logit model implies that the decision maker will often bump into the inferior prize 0 . As a result, her expected payoff from $(0, 1.01)$ is lower than $(-100, 1)$, even though $0 > -100$ and $1.01 > 1$. Indeed, a simple calculation shows that $I((-100, 1)) > 0.99$, while $I((0, 1)) < 0.99$.

Although we assume above that u is the identity function, the failure of *monotonicity* under the logit model does not depend on it, as stated in the result below.

Proposition 3.3.2 *Assume that P follows the logit model and I exhibits rational anticipation of mistakes. If u is continuously strictly increasing and $u(z) \rightarrow -\infty$ as $z \rightarrow -\infty$, then monotonicity does not hold.*

Proof. Under the assumptions

$$u(I((x, y))) = \frac{u(x) \exp\{u(x)\}}{\exp\{u(x)\} + \exp\{u(y)\}} + \frac{u(y) \exp\{u(y)\}}{\exp\{u(x)\} + \exp\{u(y)\}}$$

Let $v := \exp\{u(y)\}$. Then, $\lim_{y \rightarrow -\infty} u(y) \exp\{u(y)\} = \lim_{v \rightarrow 0} v \log v = 0$, by the L'Hôpital's rule. Therefore

$$\lim_{y \rightarrow -\infty} u(I((x, y))) = u(x)$$

Under the assumptions on u , we know that

$$\lim_{y \rightarrow -\infty} I((x, y)) = x$$

By Theorem 3.2.1 and Proposition 1, *monotonicity* fails. ■

One might continue to wonder if the failure of *monotonicity* merely comes from the specific functional form used by logit. This is again not true. Take the AMR as an example. Given X , the probability of choosing x_i is

$$P_i(X) = \frac{\phi(u(x_i))}{\sum_j \phi(u(x_j))}$$

When ϕ is an exponential, the AMR becomes a logit model. Again assume that u is continuously strictly increasing and $u(z) \rightarrow -\infty$ as $z \rightarrow -\infty$. The following result taken from Ke (2015a) states that some limiting behavior of ϕ is sufficient to let *monotonicity* be violated.

Proposition 3.3.3 (Ke (2015a)) *If $\lim_{u \rightarrow -\infty} u\phi(u) = 0$, then monotonicity does not hold.*

3.4 When Is Monotonicity a Good Assumption?

Despite all these facts, *monotonicity* still seems to be a natural condition, even under the presense of mistakes. But when should we impose it, and in what sense is it natural?

Note that all the violations of monotonicity in this section happen under the assumption of rational anticipation of mistakes. If we drop this assumption, that is, when the decision maker does not know how she makes the future mistakes, *monotonicity* becomes a natural heuristic for the decision maker's current-stage decisions. For a decision maker who is agnostic about her future choices, it is plausible for her to think that if a decision problem has strictly better prizes than another decision problem, then the first decision problem is likely to yield higher payoff (see Ke (2015b)).

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