Labor Economics, 14.661. Lecture 11: Partial Equilibrium Sequential Search

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Introduction

- Central question for labor and macro: what determines the level of employment and unemployment in the economy?
- Textbook answer: labor supply, labor demand, and unemployment as "leisure".
- Neither realistic nor a useful framework for analysis.
- Alternative: *labor market frictions*
- Related questions raised by the presence of frictions:
  - is the level of employment efficient/optimal?
  - how is the composition and quality of jobs determined, is it efficient?
  - distribution of earnings across workers.

Introduction (continued)

- Applied questions:
  - why was unemployment around 4-5% in the US economy until the 1970s?
  - why did the increase in the 70s and 80s, and then decline again in the late 90s?
  - why did European unemployment increase in the 1970s and remain persistently high?
  - why is the composition of employment so different across countries?
    - male versus female, young versus old, high versus low wages

Introduction (continued)

- Challenge: how should labor market frictions be modeled?
- Alternatives:
    - incentive problems, efficiency wages
    - wage rigidities, bargaining, non-market clearing prices
    - *search*
- Search and matching: costly process of workers finding the "right" jobs.
- *Theoretical interest:* how do markets function without the Walrasian auctioneer?
- *Empirically important,*
- But how to develop a tractable and rich model?

McCall Partial Equilibrium Search Model

- The simplest model of search frictions.
- Problem of an individual getting draws from a given wage distribution
- Decision: which jobs to accept and when to start work.
- Jobs sampled sequentially.
- Alternative: Stigler, fixed sample search (choose a sample of n jobs and then take the most attractive one).
- Sequential search typically more reasonable.
- Moreover, whenever sequential search is possible, is preferred to fixed sample search (why?).

Environment

- Risk neutral individual in discrete time.
- At time $t = 0$, this individual has preferences given by

$$\sum_{t=0}^{\infty} \beta^t c_t$$

- $c_t =$consumption.
- Start as unemployed, with consumption equal to b
- All jobs are identical except for their wages, and wages are given by an exogenous stationary distribution of

$$F(w)$$

  with finite (bounded) support $\mathbb{W}$.
- At every date, the individual samples a wage $w_t \in W$, and has to decide whether to take this or continue searching.
- Jobs are for life.
- Draws from $\mathbb{W}$ over time are independent and identically distributed.

Environment (continued)

- *Undirected search*, in the sense that the individual has no ability to seek or direct his search towards different parts of the wage distribution (or towards different types of jobs).
- Alternative: *directed search*.

Environment (continued)

- Suppose search without recall.
- If the worker accepts a job with wage w_t, he will be employed at that job forever, so the net present value of accepting a job of wage w_t is

$$\frac{w_t}{1-\beta}.$$

- Class of decision rules of the agent:

$$a_t : \mathbb{W} \rightarrow [0, 1]$$

as acceptance decision (acceptance probability)
- Let

$$A^t = \prod_{s=0}^{t} A_s.$$

Environment (continued)

- Then a strategy for the individual in this game is

$$p_t : A^{t-1} \times \mathbb{W} \rightarrow [0,1]$$

- Let $\mathcal{P}$ be the set of such functions (with the property that $p_t(\cdot)$ is defined only if $p_s(\cdot) = 0$ for all $s \leq t$).

- Then the maximization problem is

$$\max_{\{p_t\}_{t=0}^{\infty} \in \mathcal{P}^{\infty}} \mathbb{E} \sum_{t=0}^{\infty} \beta^t c_t$$

subject to $c_t = b$ if $t < s$ and $c_t = w_s$ if $t \geq s$ where $s = \inf\{n \in \mathbb{N} : a'_n = 1\}$.

Dynamic Programming Formulation

- Define the value of the agent when he has sampled a job of $w \in \mathbb{W}$:

$$v\left(w\right) = \max\left\{\frac{w}{1-\beta}, \beta v + b\right\}, \tag{1}$$

  where

$$v = \int_{\mathbb{W}} v\left(\omega\right) dF\left(\omega\right) \tag{2}$$

- v is the continuation value of not accepting a job.
- Integral in (2) as a Lebesgue integral, since $F\left(w\right)$ could be a mixture of discrete and continuous.
- Intuition.
- We are interested in finding both the value function $v\left(w\right)$ and the optimal policy of the individual.

Dynamic Programming Formulation (continued)

- Previous two equations:

$$v\left(w\right) = \max\left\{\frac{w}{1-\beta}, b + \beta \int_{\mathbb{W}} v\left(\omega\right) dF\left(\omega\right)\right\}. \qquad (3)$$

- Existence of optimal policies follows from standard theorems in dynamic programming.
- But, even more simply (3) implies that $v\left(w\right)$ must be piecewise linear with first a flat portion and then an increasing portion.
- *Optimal policy*: $v\left(w\right)$ is non-decreasing, therefore optimal policy will take a cutoff form.
  →*reservation wage R*
    - all wages above R will be accepted and those $w < R$ will be turned down.
- Implication of the reservation wage policy→no recall assumption of no consequence (why?).

Reservation Wage

- Reservation wage given by

$$\frac{R}{1-\beta} = b + \beta \int_{\mathbb{W}} v(\omega) \, dF(\omega). \tag{4}$$

- Intuition?
- Since $w < R$ are turned down, for all $w < R$

$$\begin{aligned} v(w) &= b + \beta \int_{\mathbb{W}} v(\omega) \, dF(\omega) \\ &= \frac{R}{1-\beta}, \end{aligned}$$

  and for all $w \geq R$,

$$v(w) = \frac{w}{1-\beta}$$

- Therefore,

$$\int_{\mathbb{W}} v(\omega) \, dF(\omega) = \frac{RF(R)}{1-\beta} + \int_{w \geq R} \frac{w}{1-\beta} dF(w).$$

Reservation Wage (continued)

- Combining this with (4), we have

$$\frac{R}{1-\beta} = b + \beta \left[\frac{RF(R)}{1-\beta} + \int_{w \geq R} \frac{w}{1-\beta} dF(w) \right]$$

- Rewriting

$$\int_{w<R} \frac{R}{1-\beta} dF(w) + \int_{w \geq R} \frac{R}{1-\beta} dF(w) = b + \beta \left[\int_{w<R} \frac{R}{1-\beta} dF(w) \right.$$

- Subtracting $\beta R \int_{w \geq R} dF(w) / (1-\beta) + \beta R \int_{w<R} dF(w) / (1-\beta)$ from both sides,

$$\int_{w<R} \frac{R}{1-\beta} dF(w) + \int_{w \geq R} \frac{R}{1-\beta} dF(w)$$

$$-\beta \int_{w \geq R} \frac{R}{1-\beta} dF(w) - \beta \int_{w<R} \frac{R}{1-\beta} dF(w)$$

$$= b + \beta \left[\int_{w \geq R} \frac{w-R}{1-\beta} dF(w) \right]$$

Reservation Wage (continued)

- Collecting terms, we obtain

$$R - b = \frac{\beta}{1 - \beta} \left[\int_{w \geq R} (w - R) \, dF(w) \right]. \tag{5}$$

- The left-hand side is the cost of foregoing the wage of R.
- The right hand side is the expected benefit of one more search.
- At the reservation wage, these two are equal.

Reservation Wage (continued)

- Let us define the right hand side of equation (5) as

$$g(R) \equiv \frac{\beta}{1-\beta} \left[\int_{w \geq R} (w - R) \, dF(w) \right],$$

- This is the expected benefit of one more search as a function of the reservation wage.

- Differentiating

$$\begin{aligned}
g'(R) &= -\frac{\beta}{1-\beta}(R-R) f(R) - \frac{\beta}{1-\beta} \left[\int_{w \geq R} dF(w) \right] \\
&= -\frac{\beta}{1-\beta} [1 - F(R)] < 0
\end{aligned}$$

- Therefore equation (5) has a unique solution.

- Moreover, by the implicit function theorem,

$$\frac{dR}{db} = \frac{1}{1 - g'(R)} > 0.$$

Reservation Wage (continued)

- Suppose that the density of $F(R)$, denoted by $f(R)$, exists (was this necessary until now?).

- Then the second derivative of g also exists and is

$$g''(R) = \frac{\beta}{1-\beta} f(R) \geq 0.$$

- This implies the right hand side of equation (5) is also convex.

- What does this mean?

Unemployment with Sequential Search

- Suppose that there is now a continuum 1 of identical individuals sampling jobs from the same stationary distribution F.
- Once a job is created, it lasts until the worker dies, which happens with probability s.
- There is a mass of s workers born every period, so that population is constant
- New workers start out as unemployed.
- The death probability means that the effective discount factor of workers is equal to $\beta(1-s)$.
- Consequently, the value of having accepted a wage of w is:

$$v^a(w) = \frac{w}{1 - \beta(1-s)}.$$

Unemployment with Sequential Search (continued)

- With the same reasoning as before, the value of having a job offer at wage w at hand is

$$v(w) = \max\left\{v^a(w), b + \beta(1-s)v\right\}$$

with

$$v = \int_{\mathbb{W}} v(w) \, dF.$$

- Therefore, the reservation wages given by

$$R - b = \frac{\beta(1-s)}{1 - \beta(1-s)}\left[\int_{w \geq R} (w - R) \, dF(w)\right].$$

Law of Motion of Unemployment

- Let us start time t with U_t unemployed workers.
- There will be s new workers born into the unemployment pool.
- Out of the U_t unemployed workers, those who survive and do not find a job will remain unemployed.
- Therefore

$$U_{t+1} = s + (1 - s) F(R) U_t.$$

- Here $F(R)$ is the probability of not finding a job, so $(1 - s) F(R)$ is the joint probability of not finding a job and surviving.
- Simple first-order linear difference equation (only depending on the reservation wage R, which is itself independent of the level of unemployment, U_t).
- Since $(1 - s) F(R) < 1$, it is asymptotically stable, and will converge to a unique steady-state level of unemployment.

Flow Approached Unemployment

- This gives us the simplest version of the flow approach to unemployment.
- Subtracting U_t from both sides:

$$U_{t+1} - U_t = s\left(1 - U_t\right) - \left(1 - s\right)\left(1 - F(R)\right)U_t.$$

- If period length is arbitrary, this can be written as

$$U_{t+\Delta t} - U_t = s\left(1 - U_t\right)\Delta t - \left(1 - s\right)\left(1 - F(R)\right)U_t\Delta t + o\left(\Delta t\right).$$

- Dividing by Δt and taking limits as $\Delta t \rightarrow 0$, we obtain the continuous time version

$$\dot{U}_t = s\left(1 - U_t\right) - \left(1 - s\right)\left(1 - F(R)\right)U_t.$$

Flow Approached Unemployment (continued)

- The unique steady-state unemployment rate where $U_{t+1} = U_t$ (or $\dot{U}_t = 0$) given by

$$U = \frac{s}{s + (1-s)(1-F(R))}.$$

- Canonical formula of the flow approach.
- The steady-state unemployment rate is equal to the job destruction rate (here the rate at which workers die, s) divided by the job destruction rate plus the job creation rate (here in fact the rate at which workers leave unemployment, which is different from the job creation rate).
- Clearly, an increases in s will raise steady-state unemployment.
- Moreover, an increase in R, that is, a higher reservation wage, will also depress job creation and increase unemployment.

Aside on Riskiness and Mean Preserving Spreads

- Question: what is the effect of a more unequal (spread out) wage offer distribution on reservation wages, equilibrium wage distribution, and unemployment
    - why a difference between offer distribution and equilibrium distribution?
- Key concept *mean preserving spreads*.
- Loosely speaking, a mean preserving spread is a change in distribution that increases risk.

Concepts of Riskiness

- Let a family of distributions over some set $X \subset \mathbb{R}$ with generic element x be denoted by $F(x, r)$, where r is a shift variable, which changes the distribution function.
- An example will be $F(x, r)$ to stand for mean zero normal variables, with r parameterizing the variance of the distribution.
- Normal distribution is special in the sense that, the mean and the variance completely describe the distribution, so the notion of risk can be captured by the variance.
- This is generally not true.
- The notion of "riskier" is a more stringent notion than having a greater variance.

Mean Preserving Spreads and Stochastic Dominance

Definition

$F(x, r)$ is less risky than $F(x, r')$, written as $F(x, r) \succeq_R F(x, r')$, if $\int_X x dF(x, r) = \int_X x dF(x, r')$ and for all concave and increasing $u : \mathbb{R} \rightarrow \mathbb{R}$, we have

$$\int_X u(x) \, dF(x, r) \geq \int_X u(x) \, dF(x, r').$$

- A related definition is that of second-order stochastic dominance.

Definition

$F(x, r)$ second order stochastically dominates $F(x, r')$, written as $F(x, r) \succeq_{SD} F(x, r')$, if

$$\int_{-\infty}^{c} F(x, r) \, dx \leq \int_{-\infty}^{c} F(x, r') \, dx, \text{ for all } c \in X.$$

Second-Order Stochastic Dominance

- The definition of second-order stochastic dominance requires the distribution function of $F(x, r)$ to start lower and always keep a lower integral than that of $F(x, r')$.

- One easy case where this will be satisfied is when both distribution functions have the same mean and they intersect only once: "single crossing") with $F(x, r)$ cutting $F(x, r')$ from below.

- These definitions could also be stated with strict instead of weak inequalities.

- It can also be established that if $F(x, r)$ second-order stochastic the dominates $F(x, r')$ and $u(\cdot)$ is strictly increasing and concave, then

$$\int_X u(x) \, dF(x, r) \geq \int_X u(x) \, dF(x, r').$$

Riskiness and Mean Preserving Spreads

Theorem

(Blackwell, Rothschild and Stiglitz) *Suppose*
$\int_X x dF(x, r) = \int_X x dF(x, r')$. *Then* $F(x, r) \succeq_R F(x, r')$ *if and only if*
$F(x, r) \succeq_{SD} F(x, r')$.

Wage Dispersion and Search

- Start with equation (5), which is

$$R - b = \frac{\beta}{1-\beta} \left[\int_{w \geq R} (w - R) \, dF(w) \right].$$

- Rewrite this as

$$R - b = \frac{\beta}{1-\beta} \left[\int_{w \geq R} (w - R) \, dF(w) \right] + \frac{\beta}{1-\beta} \left[\int_{w \leq R} (w - R) \, dF \right.$$

$$- \frac{\beta}{1-\beta} \left[\int_{w \leq R} (w - R) \, dF(w) \right],$$

$$= \frac{\beta}{1-\beta} (Ew - R) - \frac{\beta}{1-\beta} \left[\int_{w \leq R} (w - R) \, dF(w) \right],$$

where

$$Ew = \int_{\mathbb{W}} w \, dF(w)$$

is the mean of the distribution.

Wage Dispersion and Search (continued)

- Rearranging the previous equation

$$R - b = \beta \left(Ew - b \right) - \beta \int_{w \leq R} \left(w - R \right) dF \left(w \right).$$

- Applying integration by parts to the integral on the right hand side, i.e., noting that

$$
\begin{aligned}
\int_{w \leq R} w dF \left(w \right) &= \int_0^R w dF \left(w \right) \\
&= \left. wF \left(w \right) \right|_0^R - \int_0^R F \left(w \right) dw \\
&= RF \left(R \right) - \int_0^R F \left(w \right) dw.
\end{aligned}
$$

- We obtain

$$R - b = \beta \left(Ew - b \right) + \beta \int_0^R F \left(w \right) dw. \tag{6}$$

Wage Dispersion and Search (continued)

- Now consider a shift from F to $\tilde{F}$ corresponding to a mean preserving spread.
- This implies that Ew is unchanged
- But by definition of a mean preserving spread (second-order stochastic dominance), the last integral increases.
- Therefore, the mean preserving spread induces a shift in the reservation wage from R to $\tilde{R} > R$.
- Intuition?
- Relation to the convexity of $v(w)$?

Paradoxes of Search

- The search framework is attractive especially when we want to think of a world without a Walrasian auctioneer, or alternatively a world with "frictions".
- Search theory holds the promise of potentially answering these questions, and providing us with a framework for analysis.
- But...

The Rothschild Critique

- The key ingredient of the McCall model is non-degenerate wage distribution $F(w)$.
- Where does this come from?
- Presumably somebody is offering every wage in the support of this distribution.
- *Wage posting* by firms.
- The basis of the Rothschild critique is that it is difficult to rationalize the distribution function $F(w)$ as resulting from profit-maximizing choices of firms.

The Rothschild Critique (continued)

- Imagine that the economy consists of a mass 1 of identical workers similar to our searching agent.
- On the other side, there are N firms that can productively employ workers. Imagine that firm j has access to a technology such that it can employ l_j workers to produce

$$y_j = x_j l_j$$

units of output (with its price normalized to one as the numeraire, so that w is the real wage).

- Suppose that each firm can only attract workers by posting a single vacancy.
- Moreover, to simplify the discussion, suppose that firms post a vacancy at the beginning of the game at $t = 0$, and then do not change the wage from then on. (why is this useful?)

The Rothschild Critique (continued)

- Suppose that the distribution of x in the population of firms is given by $G(x)$ with support $X \subset \mathbb{R}_+$.
- Also assume that there is some cost $\gamma > 0$ of posting a vacancy at the beginning, and finally, that $N >> 1$ (i.e., $N = \int_{-\infty}^{\infty} dG(x) >> 1$) and each worker samples one firm from the distribution of posting firms.
- As before, suppose that once a worker accepts a job, this is permanent, and he will be employed at this job forever.
- Moreover let us set $b = 0$, so that there is no unemployment benefits.
- Finally, to keep the environment entirely stationary, assume that once a worker accepts a job, a new worker is born, and starts search.
- Will these firms offer a non-degenerate wage distribution $F(w)$?

Equilibrium Wage Distribution?

- The answer is no.
- Denote whether the firm is posting a vacancy or not by

$$p : X \rightarrow \{0, 1\},$$

  and the wage on for by

$$h : X \rightarrow \mathbb{R}_+.$$

- Intuitively, $h(x)$ should be indecreasing (higher wages are more attractive to high productivity firms).
- Let us suppose that this is so (not necessary).
- Then, the along-the-equilibrium path wage distribution is

$$F(w) = \frac{\int_{-\infty}^{h^{-1}(w)} p(x) \, dG(x)}{\int_{-\infty}^{\infty} p(x) \, dG(x)}.$$

- Intuition?

Equilibrium Wage Distribution? (continued)

- In addition, the strategies of workers can be represented by a function

$$a : \mathbb{R}_+ \longrightarrow [0, 1]$$

  denoting the probability that the worker will accept any wage in the "potential support" of the wage distribution, with 1 standing for acceptance.

- This is general enough to nest non-symmetric or mixed strategies.

- The natural equilibrium concept is subgame perfect Nash equilibrium, whereby the strategies of firms (p, h) and those of workers, a, are best responses to each other in all subgames.

Equilibrium Wage Distribution? (continued)

- Previous analysis: all workers will use a reservation wage, so

$$a\left(w\right) \quad = \quad 1 \text{ if } w \geq R$$
$$= \quad 0 \text{ otherwise}$$

- Since all workers are identical and the equation above determining the reservation wage, (5), has a unique solution, all workers will all be using the same reservation rule, accepting all wages $w \geq R$ and turning down those $w < R$.

- Workers' strategies are therefore again characterized by a reservation wage R.

Equilibrium Wage Distribution? (continued)

- Now take a firm with productivity x offering a wage $w' > R$.
- Its net present value of profits from this period's matches is

$$\pi\left(p = 1, w' > R, x\right) = -\gamma + \frac{1}{n}\frac{(x - w')}{1 - \beta}$$

where

$$n = \int_{-\infty}^{\infty} p\left(x\right) dG\left(x\right).$$

- This firm can deviate and cut its wage to some value in the interval $[R, w')$.
- All workers will still accept this job since its wage is above the reservation wage, and the firm will increase its profits to

$$\pi\left(p = 1, w \in [R, w'), x\right) = -\gamma + \frac{1}{n}\frac{x - w}{1 - \beta} > \pi\left(p = 1, w', x\right)$$

- Conclusion: there should not be any wages strictly above R.

Equilibrium Wage Distribution? (continued)

- Next consider a firm offering a wage $\tilde{w} < R$.
- This wage will be rejected by all workers, and the firm would lose the cost of posting a vacancy, i.e.,

$$\pi\left(p = 1, w < R, x\right) = -\gamma,$$

and this firm can deviate to $p = 0$ and make zero profits.

- Therefore, in equilibrium when workers use the reservation wage rule of accepting only wages greater than R, all firms will offer the same wage R, and there is no distribution and no search.

Theorem

(**Rothschild Paradox**) *When all workers are homogeneous and engage in undirected search, all equilibrium distributions will have a mass point at their reservation wage R.*

The Diamond Paradox

- In fact, the paradox is even deeper.

Theorem

(Diamond Paradox) *For all $\beta < 1$, the unique equilibrium in the above economy is $R = 0$.*

- Sketch: suppose $R > 0$, and $\beta < 1$.
- The optimal acceptance decision for to worker is

$$
\begin{aligned}
a(w) &= 1 \text{ if } w \geq R \\
&= 0 \text{ otherwise}
\end{aligned}
$$

- Therefore, all firms offering $w = R$ is an equilibrium
- But also...

The Diamond Paradox (continued)

Lemma

There exists $\varepsilon > 0$ such that when "almost all" firms are offering $w = R$, it is optimal for each worker to use the following acceptance strategy:

$$
\begin{aligned}
a(w) &= 1 \text{ if } w \geq R - \varepsilon \\
&= 0 \text{ otherwise}
\end{aligned}
$$

The Diamond Paradox (continued)

Sketch of the proof:

- If the worker accepts the wage of $R - \varepsilon$,

$$u^{accept} = \frac{R - \varepsilon}{1 - \beta}$$

- If he rejects and waits until next period, then since "almost all" firms are offering R,

$$u^{reject} = \frac{\beta R}{1 - \beta}$$

- For all $\beta < 1$, there exists $\varepsilon > 0$ such that

$$u^{accept} > u^{reject}.$$

The Diamond Paradox (continued)

- Implication: starting from an allocation where all firms offer R, any firm can deviate and offer a wage of $R - \varepsilon$ and increase its profits.
- This proves that no wage $R > 0$ can be the equilibrium, proving the proposition.
- Is the same true for Nash equilibria?

Solutions to the Diamond Paradox

- How do we resolve this paradox?

1. By assumption: assume that $F(w)$ is not the distribution of wages, but the distribution of "fruits" exogenously offered by "trees". This is clearly unsatisfactory, both from the modeling point of view, and from the point of view of asking policy questions from the model (e.g., how does unemployment insurance affect the equilibrium? The answer will depend also on how the equilibrium wage distribution changes).

2. Introduce other dimensions of heterogeneity.

3. Modify the wage determination assumptions→bargaining rather than *wage posting*: the most common and tractable alternative (though is it the most realistic?)