

# 14.452 Economic Growth: Lecture 5, Neoclassical Growth

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# Introduction

- Ramsey or Cass-Koopmans model: differs from the Solow model only because it explicitly models the consumer side and endogenizes savings.
- Beyond its use as a basic growth model, also a workhorse for many areas of macroeconomics.

# Preferences, Technology and Demographics I

- Infinite-horizon, continuous time.
- Representative household with instantaneous utility function

$$u(c(t)), \quad (1)$$

**Assumption**  $u(c)$  is strictly increasing, concave, twice continuously differentiable with derivatives  $u'$  and  $u''$ , and satisfies the following Inada type assumptions:

$$\lim_{c \rightarrow 0} u'(c) = \infty \text{ and } \lim_{c \rightarrow \infty} u'(c) = 0.$$

- Suppose representative household represents set of identical households (normalized to 1).
- Each household has an instantaneous utility function given by (1).
- $L(0) = 1$  and

$$L(t) = \exp(nt). \quad (2)$$

## Preferences, Technology and Demographics II

- All members of the household supply their labor inelastically.
- Objective function of each household at  $t = 0$ :

$$U(0) \equiv \int_0^{\infty} \exp(-(\rho - n)t) u(c(t)) dt, \quad (3)$$

where

- $c(t)$  = consumption per capita at  $t$ ,
- $\rho$  = subjective discount rate, and effective discount rate is  $\rho - n$ .
- Objective function (3) embeds:
  - Household is fully altruistic towards all of its future members, and makes allocations of consumption (among household members) cooperatively.
  - Strict concavity of  $u(\cdot)$
- Thus each household member will have an equal consumption

$$c(t) \equiv \frac{C(t)}{L(t)}$$

## Preferences, Technology and Demographics III

- Utility of  $u(c(t))$  per household member at time  $t$ , total of  $L(t) u(c(t)) = \exp(nt) u(c(t))$ .
- With discount at rate of  $\exp(-\rho t)$ , obtain (3).

ASSUMPTION 4'.

$$\rho > n.$$

- Ensures that in the model without growth, discounted utility is finite. Will strengthen it in model with growth.
- Start model without any technological progress.
- Factor and product markets are competitive.
- Production possibilities set of the economy is represented by

$$Y(t) = F[K(t), L(t)],$$

- Standard constant returns to scale and Inada assumptions still hold.

# Preferences, Technology and Demographics IV

- Per capita production function  $f(\cdot)$

$$\begin{aligned} y(t) &\equiv \frac{Y(t)}{L(t)} \\ &= F\left[\frac{K(t)}{L(t)}, 1\right] \\ &\equiv f(k(t)), \end{aligned}$$

where, as before,

$$k(t) \equiv \frac{K(t)}{L(t)}. \quad (4)$$

- Competitive factor markets then imply:

$$R(t) = F_K[K(t), L(t)] = f'(k(t)). \quad (5)$$

and

$$w(t) = F_L[K(t), L(t)] = f(k(t)) - k(t)f'(k(t)). \quad (6)$$

## Preferences, Technology and Demographics V

- Denote asset holdings of the representative household at time  $t$  by  $\mathcal{A}(t)$ . Then,

$$\dot{\mathcal{A}}(t) = r(t) \mathcal{A}(t) + w(t) L(t) - c(t) L(t)$$

- $r(t)$  is the risk-free market flow rate of return on assets, and  $w(t) L(t)$  is the flow of labor income earnings of the household.
- Defining per capita assets as

$$a(t) \equiv \frac{\mathcal{A}(t)}{L(t)},$$

we obtain:

$$\dot{a}(t) = (r(t) - n) a(t) + w(t) - c(t). \quad (7)$$

- Household assets can consist of capital stock,  $K(t)$ , which they rent to firms and government bonds,  $B(t)$ .

# Preferences, Technology and Demographics VI

- With uncertainty, households would have a portfolio choice between  $K(t)$  and riskless bonds.
- With incomplete markets, bonds allow households to smooth idiosyncratic shocks. But for now no need.
- Thus, market clearing  $\Rightarrow$

$$a(t) = k(t).$$

- No uncertainty depreciation rate of  $\delta$  implies

$$r(t) = R(t) - \delta. \tag{8}$$



# The Budget Constraint

- The differential equation

$$\dot{a}(t) = (r(t) - n) a(t) + w(t) - c(t)$$

is a flow constraint

- Not sufficient as a proper budget constraint unless we impose a lower bound on assets.
- Three options:
  - 1 Lower bound on assets such as  $a(t) \geq 0$  for all  $t$
  - 2 Natural debt limit.
  - 3 No Ponzi Game Condition.

# The No Ponzi Game Condition

- Infinite-horizon no Ponzi game condition is:

$$\lim_{t \rightarrow \infty} a(t) \exp \left( - \int_0^t (r(s) - n) ds \right) \geq 0. \quad (9)$$

- Transversality condition ensures individual would never want to have positive wealth asymptotically, so no Ponzi game condition can be strengthened to (though not necessary in general):

$$\lim_{t \rightarrow \infty} a(t) \exp \left( - \int_0^t (r(s) - n) ds \right) = 0. \quad (10)$$

# Understanding the No Ponzi Game Condition

- Why?
- Write the single budget constraint of the form:

$$\int_0^T c(t) L(t) \exp\left(\int_t^T r(s) ds\right) dt + \mathcal{A}(T) \quad (11)$$

$$= \int_0^T w(t) L(t) \exp\left(\int_t^T r(s) ds\right) dt + \mathcal{A}(0) \exp\left(\int_0^T r(s) ds\right).$$

- Differentiating with respect to  $T$  and dividing  $L(t)$  gives (7).
- Now imagine that (11) applies to a finite-horizon economy .
- Flow budget constraint (7) by itself does not guarantee that  $\mathcal{A}(T) \geq 0$ .
- Thus in finite-horizon we would simply impose (11) as a boundary condition.
- The no Ponzi game condition is the infinite horizon equivalent of this (obtained by dividing by  $L(t)$  and multiplying both sides by  $\exp\left(-\int_0^T r(s) ds\right)$  and taking the limit as  $T \rightarrow \infty$ ).

## Definition of Equilibrium

**Definition** A competitive equilibrium of the Ramsey economy consists of paths  $[C(t), K(t), w(t), R(t)]_{t=0}^{\infty}$ , such that the representative household maximizes its utility given initial capital stock  $K(0)$  and the time path of prices  $[w(t), R(t)]_{t=0}^{\infty}$ , and all markets clear.

- Notice refers to the entire path of quantities and prices, not just steady-state equilibrium.

**Definition** A competitive equilibrium of the Ramsey economy consists of paths  $[c(t), k(t), w(t), R(t)]_{t=0}^{\infty}$ , such that the representative household maximizes (3) subject to (7) and (9) given initial capital-labor ratio  $k(0)$ , factor prices  $[w(t), R(t)]_{t=0}^{\infty}$  as in (5) and (6), and the rate of return on assets  $r(t)$  given by (8).

# Household Maximization I

- Maximize (3) subject to (7) and (10).
- First ignore (10) and set up the current-value Hamiltonian:

$$\hat{H}(a, c, \mu) = u(c(t)) + \mu(t) [w(t) + (r(t) - n)a(t) - c(t)],$$

- *Maximum Principle*  $\Rightarrow$  “candidate solution”

$$\hat{H}_c(a, c, \mu) = u'(c(t)) - \mu(t) = 0$$

$$\begin{aligned} \hat{H}_a(a, c, \mu) &= \mu(t)(r(t) - n) \\ &= -\dot{\mu}(t) + (\rho - n)\mu(t) \end{aligned}$$

$$\lim_{t \rightarrow \infty} [\exp(-(\rho - n)t) \mu(t) a(t)] = 0.$$

and the transition equation (7).

- Notice transversality condition is written in terms of the current-value costate variable.

## Household Maximization II

- For any  $\mu(t) > 0$ ,  $\hat{H}(a, c, \mu)$  is a concave function of  $(a, c)$  and strictly concave in  $c$ .
- The first necessary condition implies  $\mu(t) > 0$  for all  $t$ .
- Therefore, *Sufficient Conditions* imply that the candidate solution is an optimum (is it unique?)
- Rearrange the second condition:

$$\frac{\dot{\mu}(t)}{\mu(t)} = -(r(t) - \rho), \quad (12)$$

- First necessary condition implies,

$$u'(c(t)) = \mu(t). \quad (13)$$

# Household Maximization III

- Differentiate with respect to time and divide by  $\mu(t)$ ,

$$\frac{u''(c(t)) c(t) \dot{c}(t)}{u'(c(t)) c(t)} = \frac{\dot{\mu}(t)}{\mu(t)}.$$

- Substituting into (12), obtain another form of the consumer Euler equation:

$$\frac{\dot{c}(t)}{c(t)} = \frac{1}{\varepsilon_u(c(t))} (r(t) - \rho) \quad (14)$$

where

$$\varepsilon_u(c(t)) \equiv -\frac{u''(c(t)) c(t)}{u'(c(t))} \quad (15)$$

is the elasticity of the marginal utility  $u'(c(t))$ .

- Consumption will grow over time when the discount rate is less than the rate of return on assets.

# Household Maximization IV

- Speed at which consumption will grow is related to the elasticity of marginal utility of consumption,  $\varepsilon_u(c(t))$ .
- Even more importantly,  $\varepsilon_u(c(t))$  is the inverse of the *intertemporal elasticity of substitution*:
  - regulates willingness to substitute consumption (or any other attribute that yields utility) over time.
  - Elasticity between dates  $t$  and  $s > t$  is defined as

$$\sigma_u(t, s) = -\frac{d \log(c(s)/c(t))}{d \log(u'(c(s))/u'(c(t)))}.$$

- As  $s \downarrow t$ ,

$$\sigma_u(t, s) \rightarrow \sigma_u(t) = -\frac{u'(c(t))}{u''(c(t))c(t)} = \frac{1}{\varepsilon_u(c(t))}. \quad (16)$$



# Household Maximization V

- Integrating (12),

$$\begin{aligned}\mu(t) &= \mu(0) \exp\left(-\int_0^t (r(s) - \rho) ds\right) \\ &= u'(c(0)) \exp\left(-\int_0^t (r(s) - \rho) ds\right),\end{aligned}$$

- Substituting into the transversality condition,

$$\begin{aligned}0 &= \lim_{t \rightarrow \infty} \left[ \exp(-(\rho - n)t) a(t) u'(c(0)) \exp\left(-\int_0^t (r(s) - \rho) ds\right) \right] \\ 0 &= \lim_{t \rightarrow \infty} \left[ a(t) \exp\left(-\int_0^t (r(s) - n) ds\right) \right].\end{aligned}$$

- Thus the “strong version” of the no-Ponzi condition, (10) has to hold.

# Household Maximization VI

- Since  $a(t) = k(t)$ , transversality condition is also equivalent to

$$\lim_{t \rightarrow \infty} \left[ \exp \left( - \int_0^t (r(s) - n) ds \right) k(t) \right] = 0$$

- Notice term  $\exp \left( - \int_0^t r(s) ds \right)$  is a present-value factor: converts a unit of income at  $t$  to a unit of income at 0.
- When  $r(s) = r$ , factor would be  $\exp(-rt)$ . More generally, define an average interest rate between dates 0 and  $t$  given by  $\frac{1}{t} \int_0^t r(s) ds$ .

# Equilibrium Prices

- Equilibrium prices given by (5) and (6).
- Thus market rate of return for consumers,  $r(t)$ , is given by (8), i.e.,

$$r(t) = f'(k(t)) - \delta.$$

- Substituting this into the consumer's problem, we have

$$\frac{\dot{c}(t)}{c(t)} = \frac{1}{\varepsilon_u(c(t))} (f'(k(t)) - \delta - \rho) \quad (17)$$

- Equation (??) similarly generalizes for the case of iso-elastic utility function.

# Optimal Growth I

- In an economy that admits a representative household, optimal growth involves maximization of utility of representative household subject to technology and feasibility constraints:

$$\max_{[k(t), c(t)]_{t=0}^{\infty}} \int_0^{\infty} \exp(-(\rho - n)t) u(c(t)) dt,$$

subject to

$$\dot{k}(t) = f(k(t)) - (n + \delta)k(t) - c(t),$$

and  $k(0) > 0$ .

- Versions of the First and Second Welfare Theorems for economies with a continuum of commodities: solution to this problem should be the same as the equilibrium growth problem.
- But straightforward to show the equivalence of the two problems.

## Optimal Growth II

- Again set up the current-value Hamiltonian:

$$\hat{H}(k, c, \mu) = u(c(t)) + \mu(t) [f(k(t)) - (n + \delta)k(t) - c(t)],$$

- Candidate solution from the *Maximum Principle*:

$$\hat{H}_c(k, c, \mu) = 0 = u'(c(t)) - \mu(t),$$

$$\begin{aligned} \hat{H}_k(k, c, \mu) &= -\dot{\mu}(t) + (\rho - n)\mu(t) \\ &= \mu(t) (f'(k(t)) - \delta - n), \end{aligned}$$

$$\lim_{t \rightarrow \infty} [\exp(-(\rho - n)t) \mu(t) k(t)] = 0.$$

- *Sufficiency Theorem*  $\Rightarrow$  unique solution ( $\hat{H}$  and thus the maximized Hamiltonian strictly concave in  $k$ ).

## Optimal Growth III

- Repeating the same steps as before, these imply

$$\frac{\dot{c}(t)}{c(t)} = \frac{1}{\varepsilon_u(c(t))} (f'(k(t)) - \delta - \rho),$$

which is identical to (17), and the transversality condition

$$\lim_{t \rightarrow \infty} \left[ k(t) \exp \left( - \int_0^t (f'(k(s)) - \delta - n) ds \right) \right] = 0,$$

which is, in turn, identical to (10).

- Thus the competitive equilibrium is a Pareto optimum and that the Pareto allocation can be decentralized as a competitive equilibrium.

**Proposition** In the neoclassical growth model described above, with standard assumptions on the production function (assumptions 1-4'), the equilibrium is Pareto optimal and coincides with the optimal growth path maximizing the utility of the representative household.

# Steady-State Equilibrium I

- Steady-state equilibrium is defined as an equilibrium path in which capital-labor ratio, consumption and output are constant, thus:

$$\dot{c}(t) = 0.$$

- From (17), as long as  $f(k^*) > 0$ , *irrespective* of the exact utility function, we must have a capital-labor ratio  $k^*$  such that

$$f'(k^*) = \rho + \delta. \quad (18)$$

- Pins down the steady-state capital-labor ratio only as a function of the production function, the discount rate and the depreciation rate.
- Modified golden rule*: level of the capital stock that does not maximize steady-state consumption, because earlier consumption is preferred to later consumption.

# Steady-State Equilibrium II

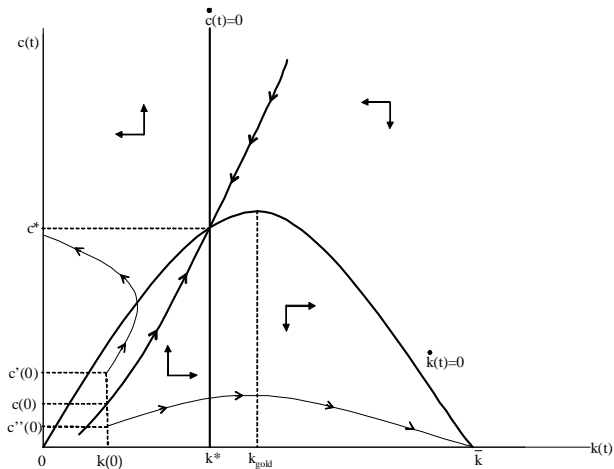


Figure: Steady state in the baseline neoclassical growth model



## Steady-State Equilibrium III

- Given  $k^*$ , steady-state consumption level:

$$c^* = f(k^*) - (n + \delta)k^*, \quad (19)$$

- Given Assumption 4', a steady state where the capital-labor ratio and thus output are constant necessarily satisfies the transversality condition.

**Proposition** In the neoclassical growth model described above, with Assumptions 1, 2, assumptions on utility above and Assumption 4', the steady-state equilibrium capital-labor ratio,  $k^*$ , is uniquely determined by (18) and is independent of the utility function. The steady-state consumption per capita,  $c^*$ , is given by (19).

- Comparative statics again straightforward.

## Steady-State Equilibrium IV

- Instead of the saving rate, it is now the discount factor that affects the rate of capital accumulation.
- Loosely, lower discount rate implies greater patience and thus greater savings.
- Without technological progress, the steady-state saving rate can be computed as

$$s^* = \frac{\delta k^*}{f(k^*)}. \quad (20)$$

- Rate of population growth has no impact on the steady state capital-labor ratio, which contrasts with the basic Solow model.
  - result depends on the way in which intertemporal discounting takes place.
- $k^*$  and thus  $c^*$  do *not* depend on the instantaneous utility function  $u(\cdot)$ .
  - form of the utility function only affects the transitional dynamics
  - not true when there is technological change.

# Transitional Dynamics I

- Equilibrium is determined by two differential equations:

$$\dot{k}(t) = f(k(t)) - (n + \delta)k(t) - c(t) \quad (21)$$

and

$$\frac{\dot{c}(t)}{c(t)} = \frac{1}{\varepsilon_u(c(t))} (f'(k(t)) - \delta - \rho). \quad (22)$$

- Moreover, we have an initial condition  $k(0) > 0$ , also a boundary condition at infinity,

$$\lim_{t \rightarrow \infty} \left[ k(t) \exp \left( - \int_0^t (f'(k(s)) - \delta - n) ds \right) \right] = 0.$$

# Transitional Dynamics II

- Appropriate notion of *saddle-path stability*:
  - consumption level (or equivalently  $\mu$ ) is the control variable, and  $c(0)$  (or  $\mu(0)$ ) is free: has to adjust to satisfy transversality condition
  - since  $c(0)$  or  $\mu(0)$  can jump to any value, need that there exists a one-dimensional manifold tending to the steady state (*stable arm*).
  - If there were more than one path equilibrium would be indeterminate.
- Economic forces are such that indeed there will be a one-dimensional manifold of stable solutions tending to the unique steady state.
- See Figure.

# Transitional Dynamics III

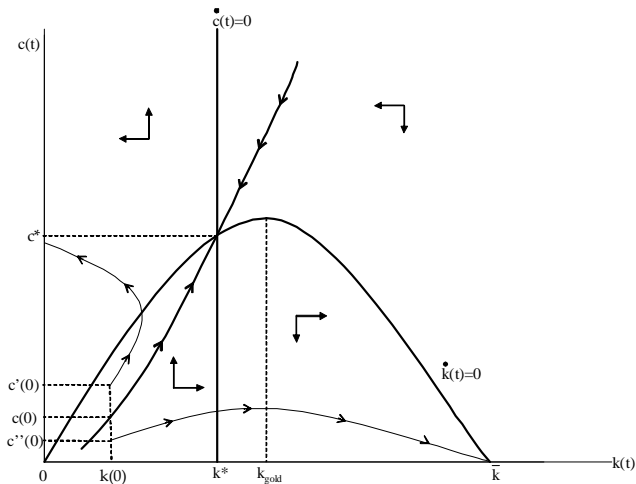


Figure: Transitional dynamics in the baseline neoclassical growth model

# Transitional Dynamics: Sufficiency

- Why is the stable arm unique?
- Three different (complementary) lines of analysis
  - 1 Sufficiency Theorem
  - 2 Global Stability Analysis
  - 3 Local Stability Analysis
- *Sufficiency Theorem*: solution starting in  $c(0)$  and limiting to the steady state satisfies the necessary and sufficient conditions, and thus unique solution to household problem and unique equilibrium.

**Proposition** In the neoclassical growth model described above, with Assumptions 1, 2, assumptions on utility above and Assumption 4', there exists a unique equilibrium path starting from any  $k(0) > 0$  and converging to the unique steady-state  $(k^*, c^*)$  with  $k^*$  given by (18). Moreover, if  $k(0) < k^*$ , then  $k(t) \uparrow k^*$  and  $c(t) \uparrow c^*$ , whereas if  $k(0) > k^*$ , then  $k(t) \downarrow k^*$  and  $c(t) \downarrow c^*$ .

# Global Stability Analysis

- Alternative argument:
  - if  $c(0)$  started below it, say  $c''(0)$ , consumption would reach zero, thus capital would accumulate continuously until the maximum level of capital (reached with zero consumption)  $\bar{k} > k_{gold}$ . This would violate the transversality condition. Can be established that transversality condition necessary in this case, thus such paths can be ruled out.
  - if  $c(0)$  started above this stable arm, say at  $c'(0)$ , the capital stock would reach 0 in finite time, while consumption would remain positive. But this would violate feasibility (a little care is necessary with this argument, since necessary conditions do not apply at the boundary).

# Local Stability Analysis I

- Linearize the set of differential equations, and looking at their eigenvalues.
- Recall the two differential equations:

$$\dot{k}(t) = f(k(t)) - (n + \delta)k(t) - c(t)$$

and

$$\frac{\dot{c}(t)}{c(t)} = \frac{1}{\varepsilon_u(c(t))} (f'(k(t)) - \delta - \rho).$$

- Linearizing these equations around the steady state  $(k^*, c^*)$ , we have (suppressing time dependence)

$$\begin{aligned} \dot{k} &= \text{constant} + (f'(k^*) - n - \delta)(k - k^*) - c \\ \dot{c} &= \text{constant} + \frac{c^* f''(k^*)}{\varepsilon_u(c^*)} (k - k^*). \end{aligned}$$



## Local Stability Analysis II

- From (18),  $f'(k^*) - \delta = \rho$ , so the eigenvalues of this two-equation system are given by the values of  $\zeta$  that solve the following quadratic form:

$$\det \begin{pmatrix} \rho - n - \zeta & -1 \\ \frac{c^* f''(k^*)}{\varepsilon_u(c^*)} & 0 - \zeta \end{pmatrix} = 0.$$

- Since  $c^* f''(k^*) / \varepsilon_u(c^*) < 0$ , there are two real eigenvalues, one negative and one positive.
- Thus local analysis also leads to the same conclusion, but can only establish local stability.

# Neoclassical Growth Model in Discrete Time

- Economically, nothing is different in discrete time.
- Mathematically, a few details need to be sorted out (in particular, different tools from 14.451 to be used).
- Sometimes discrete time will be more convenient to work with, and sometimes continuous time.
- See recitation for details of the discrete time model.

# Technological Change and the Neoclassical Model

- Extend the production function to:

$$Y(t) = F[K(t), A(t)L(t)], \quad (23)$$

where

$$A(t) = \exp(gt) A(0).$$

- A consequence of Uzawa Theorem.: (23) imposes purely labor-augmenting—Harrod-neutral—technological change.
- Continue to adopt all usual assumptions, and Assumption 4' will be strengthened further in order to ensure finite discounted utility in the presence of sustained economic growth.

## Technological Change II

- Define

$$\begin{aligned}\hat{y}(t) &\equiv \frac{Y(t)}{A(t)L(t)} \\ &= F\left[\frac{K(t)}{A(t)L(t)}, 1\right] \\ &\equiv f(k(t)),\end{aligned}$$

where

$$k(t) \equiv \frac{K(t)}{A(t)L(t)}. \quad (24)$$

- Also need to impose a further assumption on preferences in order to ensure balanced growth.

## Technological Change III

- Define balanced growth as a pattern of growth consistent with the *Kaldor facts* of constant capital-output ratio and capital share in national income.
- These two observations together also imply that the rental rate of return on capital,  $R(t)$ , has to be constant, which, from (8), implies that  $r(t)$  has to be constant.
- Again refer to an equilibrium path that satisfies these conditions as a balanced growth path (BGP).
- Balanced growth also requires that consumption and output grow at a constant rate. Euler equation

$$\frac{\dot{c}(t)}{c(t)} = \frac{1}{\varepsilon_u(c(t))} (r(t) - \rho).$$

## Technological Change IV

- If  $r(t) \rightarrow r^*$ , then  $\dot{c}(t)/c(t) \rightarrow g_c$  is only possible if  $\varepsilon_u(c(t)) \rightarrow \varepsilon_u$ , i.e., if the elasticity of marginal utility of consumption is asymptotically constant.
- Thus balanced growth is only consistent with utility functions that have asymptotically constant elasticity of marginal utility of consumption.

**Proposition** Balanced growth in the neoclassical model requires that asymptotically (as  $t \rightarrow \infty$ ) all technological change is purely labor augmenting and the elasticity of intertemporal substitution,  $\varepsilon_u(c(t))$ , tends to a constant  $\varepsilon_u$ .

## Example: CRRA Utility I

- Recall the Arrow-Pratt coefficient of relative risk aversion for a twice-continuously differentiable concave utility function  $U(c)$  is

$$\mathcal{R} = -\frac{U''(c)c}{U'(c)}.$$

- Constant relative risk aversion (CRRA) utility function satisfies the property that  $\mathcal{R}$  is constant.
- Integrating both sides of the previous equation, setting  $\mathcal{R}$  to a constant, implies that the family of CRRA utility functions is given by

$$U(c) = \begin{cases} \frac{c^{1-\theta}-1}{1-\theta} & \text{if } \theta \neq 1 \text{ and } \theta \geq 0 \\ \ln c & \text{if } \theta = 1 \end{cases},$$

with the coefficient of relative risk aversion given by  $\theta$ .

- Details: see recitation.

# Technological Change V

- Given the restriction that balanced growth is only possible with a constant elasticity of intertemporal substitution, start with

$$u(c(t)) = \begin{cases} \frac{c(t)^{1-\theta} - 1}{1-\theta} & \text{if } \theta \neq 1 \text{ and } \theta \geq 0 \\ \ln c(t) & \text{if } \theta = 1 \end{cases},$$

- Elasticity of marginal utility of consumption,  $\varepsilon_u$ , is given by  $\theta$ .
- When  $\theta = 0$ , these represent linear preferences, when  $\theta = 1$ , we have log preferences, and as  $\theta \rightarrow \infty$ , infinitely risk-averse, and infinitely unwilling to substitute consumption over time.
- Assume that the economy admits a representative household with CRRA preferences

$$\int_0^{\infty} \exp(-(\rho - n)t) \frac{\tilde{c}(t)^{1-\theta} - 1}{1-\theta} dt, \quad (25)$$



## Technological Change VI

- $\tilde{c}(t) \equiv C(t) / L(t)$  is per capita consumption.
- Refer to this model, with labor-augmenting technological change and CRRA preference as given by (25) as the *canonical model*
- Euler equation takes the simpler form:

$$\frac{\dot{\tilde{c}}(t)}{\tilde{c}(t)} = \frac{1}{\theta} (r(t) - \rho). \quad (26)$$

- Steady-state equilibrium first: since with technological progress there will be growth in per capita income,  $\tilde{c}(t)$  will grow.

# Technological Change VII

- Instead define

$$\begin{aligned}c(t) &\equiv \frac{C(t)}{A(t)L(t)} \\ &\equiv \frac{\tilde{c}(t)}{A(t)}.\end{aligned}$$

- This normalized consumption level will remain constant along the BGP:

$$\begin{aligned}\frac{\dot{c}(t)}{c(t)} &\equiv \frac{\dot{\tilde{c}}(t)}{\tilde{c}(t)} - g \\ &= \frac{1}{\theta} (r(t) - \rho - \theta g).\end{aligned}$$

# Technological Change VIII

- For the accumulation of capital stock:

$$\dot{k}(t) = f(k(t)) - c(t) - (n + g + \delta)k(t),$$

where  $k(t) \equiv K(t) / A(t)L(t)$ .

- Transversality condition, in turn, can be expressed as

$$\lim_{t \rightarrow \infty} \left\{ k(t) \exp \left( - \int_0^t [f'(k(s)) - g - \delta - n] ds \right) \right\} = 0. \quad (27)$$

- In addition, equilibrium  $r(t)$  is still given by (8), so

$$r(t) = f'(k(t)) - \delta$$

# Technological Change IX

- Since in steady state  $c(t)$  must remain constant:

$$r(t) = \rho + \theta g$$

or

$$f'(k^*) = \rho + \delta + \theta g, \quad (28)$$

- Pins down the steady-state value of the normalized capital ratio  $k^*$  uniquely.
- Normalized consumption level is then given by

$$c^* = f(k^*) - (n + g + \delta) k^*, \quad (29)$$

- Per capita consumption grows at the rate  $g$ .

# Technological Change X

- Because there is growth, to make sure that the transversality condition is in fact satisfied substitute (28) into (27):

$$\lim_{t \rightarrow \infty} \left\{ k(t) \exp \left( - \int_0^t [\rho - (1 - \theta)g - n] ds \right) \right\} = 0,$$

- Can only hold if  $\rho - (1 - \theta)g - n > 0$ , or alternatively :

ASSUMPTION 4:

$$\rho - n > (1 - \theta)g.$$

- Remarks:
  - Strengthens Assumption 4' when  $\theta < 1$ .
  - Alternatively, recall in steady state  $r = \rho + \theta g$  and the growth rate of output is  $g + n$ .
  - Therefore, equivalent to requiring that  $r > g + n$ .

# Technological Change XI

**Proposition** Consider the neoclassical growth model with labor augmenting technological progress at the rate  $g$  and preferences given by (25). Suppose that Assumptions 1, 2, assumptions on utility above hold and  $\rho - n > (1 - \theta)g$ . Then there exists a unique balanced growth path with a normalized capital to effective labor ratio of  $k^*$ , given by (28), and output per capita and consumption per capita grow at the rate  $g$ .

- Steady-state capital-labor ratio no longer independent of preferences, depends on  $\theta$ .
  - Positive growth in output per capita, and thus in consumption per capita.
  - With upward-sloping consumption profile, willingness to substitute consumption today for consumption tomorrow determines accumulation and thus equilibrium effective capital-labor ratio.

# Transitional Dynamics with Technological Change

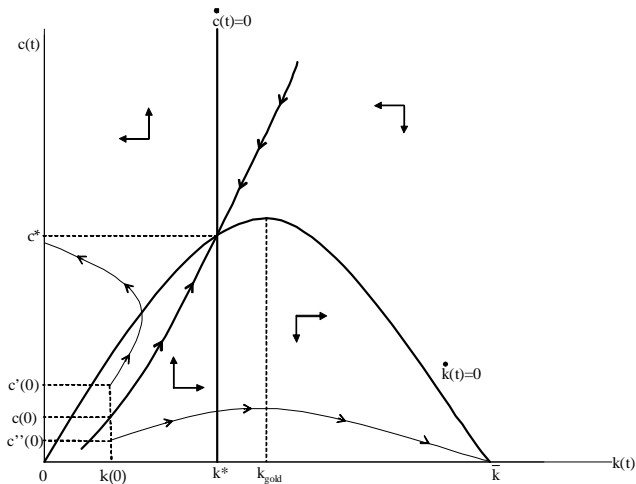


Figure: Transitional dynamics in the neoclassical growth model with technological change

## Technological Change XII

- Steady-state effective capital-labor ratio,  $k^*$ , is determined endogenously, but steady-state growth rate of the economy is given exogenously and equal to  $g$ .

**Proposition** Consider the neoclassical growth model with labor augmenting technological progress at the rate  $g$  and preferences given by (25). Suppose that Assumptions 1, 2, assumptions on utility above hold and  $\rho - n > (1 - \theta)g$ . Then there exists a unique equilibrium path of normalized capital and consumption,  $(k(t), c(t))$  converging to the unique steady-state  $(k^*, c^*)$  with  $k^*$  given by (28). Moreover, if  $k(0) < k^*$ , then  $k(t) \uparrow k^*$  and  $c(t) \uparrow c^*$ , whereas if  $k(0) > k^*$ , then  $k(t) \downarrow k^*$  and  $c(t) \downarrow c^*$ .



## Example: CRRA and Cobb-Douglas

- One solvable case: CRRA (or even better log) preferences and Cobb-Douglas production function, given by  $F(K, AL) = K^\alpha (AL)^{1-\alpha}$ , so that

$$f(k) = k^\alpha.$$

- See recitation.

# Comparative Dynamics I

- Comparative statics: changes in steady state in response to changes in parameters.
- Comparative dynamics look at how the entire equilibrium path of variables changes in response to a change in policy or parameters.
- Look at the effect of a change in tax on capital (or discount rate  $\rho$ )
- Consider the neoclassical growth in steady state  $(k^*, c^*)$ .
- Tax declines to  $\tau' < \tau$ .
- From Propositions above, after the change there exists a unique steady state equilibrium that is saddle path stable.
- Let this steady state be denoted by  $(k^{**}, c^{**})$ .
- Since  $\tau' < \tau$ ,  $k^{**} > k^*$  while the equilibrium growth rate will remain unchanged.

## Comparative Dynamics II

- Figure: drawn assuming change is unanticipated and occurs at some date  $T$ .
- At  $T$ , curve corresponding to  $\dot{c}/c = 0$  shifts to the right and laws of motion represented by the phase diagram change.
- Following the decline  $c^*$  is above the stable arm of the new dynamical system: consumption must drop immediately
- Then consumption slowly increases along the stable arm
- Overall level of normalized consumption will necessarily increase, since the intersection between the curve for  $\dot{c}/c = 0$  and for  $\dot{k}/k = 0$  will necessarily be to the left side of  $k_{gold}$ .

# Comparative Dynamics III

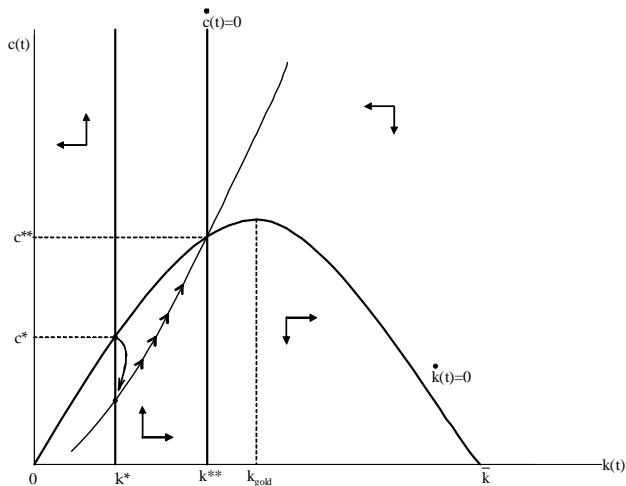


Figure: The dynamic response of capital and consumption to a decline in capital

# The Role of Policy I

- Growth of per capita consumption and output per worker (per capita) are determined exogenously.
- But level of income, depends on  $1/\theta$ ,  $\rho$ ,  $\delta$ ,  $n$ , and naturally the form of  $f(\cdot)$ .
- Proximate causes of differences in income per capita: here explain those differences only in terms of preference and technology parameters.
- Link between proximate and potential fundamental causes:
  - e.g. intertemporal elasticity of substitution and the discount rate can be as related to cultural or geographic factors.
- But an explanation for cross-country and over-time differences in economic growth based on differences or changes in preferences is unlikely to be satisfactory.
- More appealing: link incentives to accumulate physical capital (and human capital and technology) to the institutional environment.

## The Role of Policy II

- Simple way: through differences in policies.
- Introduce linear tax policy: returns on capital net of depreciation are taxed at the rate  $\tau$  and the proceeds of this are redistributed back to the consumers.
- Capital accumulation equation remains as above:

$$\dot{k}(t) = f(k(t)) - c(t) - (n + g + \delta)k(t),$$

- But interest rate faced by households changes to:

$$r(t) = (1 - \tau) (f'(k(t)) - \delta),$$

## The Role of Policy III

- Growth rate of normalized consumption is then obtained from the consumer Euler equation, (26):

$$\begin{aligned}\frac{\dot{c}(t)}{c(t)} &= \frac{1}{\theta} (r(t) - \rho - \theta g). \\ &= \frac{1}{\theta} ((1 - \tau) (f'(k(t)) - \delta) - \rho - \theta g).\end{aligned}$$

- Identical argument to that before implies

$$f'(k^*) = \delta + \frac{\rho + \theta g}{1 - \tau}. \quad (30)$$

- Higher  $\tau$ , since  $f'(\cdot)$  is decreasing, reduces  $k^*$ .
- Higher taxes on capital have the effect of depressing capital accumulation and reducing income per capita.
- But have not so far offered a reason why some countries may tax capital at a higher rate than others.

# Conclusions

- Major contribution: open the black box of capital accumulation by specifying the preferences of consumers.
- Also by specifying individual preferences we can explicitly compare equilibrium and optimal growth.
- Paves the way for further analysis of capital accumulation, human capital and endogenous technological progress.
- Did our study of the neoclassical growth model generate new insights about the sources of cross-country income differences and economic growth relative to the Solow growth model? Largely no.
- This model, by itself, does not enable us to answer questions about the fundamental causes of economic growth.
- But it clarifies the nature of the economic decisions so that we are in a better position to ask such questions.