14.452 Economic Growth: Lectures 1 (second half), 2 and 3
The Solow Growth Model

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Solow Growth Model

- Develop a simple framework for the *proximate* causes and the mechanics of economic growth and cross-country income differences.
- Solow-Swan model named after Robert (Bob) Solow and Trevor Swan, or simply the *Solow model*
- Before Solow growth model, the most common approach to economic growth built on the Harrod-Domar model.
- Harrod-Domar mdel emphasized potential dysfunctional aspects of growth: e.g, how growth could go hand-in-hand with increasing unemployment.
- Solow model demonstrated why the Harrod-Domar model was not an attractive place to start.
- At the center of the Solow growth model is the *neoclassical* aggregate production function.

Households and Production I

- Closed economy, with a unique final good.
- Discrete time running to an infinite horizon, time is indexed by $t = 0, 1, 2, \ldots$.
- Economy is inhabited by a large number of households, and for now households will not be optimizing.
- This is the main difference between the Solow model and the *neoclassical growth model*.
- To fix ideas, assume all households are identical, so the economy admits *a representative household*.

Households and Production II

- Assume households save a constant exogenous fraction s of their disposable income
- Same assumption used in basic Keynesian models and in the Harrod-Domar model; at odds with reality.
- Assume all firms have access to the same production function: economy admits a **representative firm**, with a representative (or aggregate) production function.
- Aggregate production function for the unique final good is

$$Y(t) = F[K(t), L(t), A(t)] \tag{1}$$

- Assume capital is the same as the final good of the economy, but used in the production process of more goods.
- $A(t)$ is a *shifter* of the production function (1). Broad notion of technology.
- Major assumption: technology is **free**; it is publicly available as a non-excludable, non-rival good.

Key Assumption

Assumption 1 **(Continuity, Differentiability, Positive and Diminishing Marginal Products, and Constant Returns to Scale)** The production function $F : \mathbb{R}^3_+ \to \mathbb{R}_+$ is twice continuously differentiable in K and L, and satisfies

$$F_K(K, L, A) \equiv \frac{\partial F(\cdot)}{\partial K} > 0, \quad F_L(K, L, A) \equiv \frac{\partial F(\cdot)}{\partial L} > 0,$$

$$F_{KK}(K, L, A) \equiv \frac{\partial^2 F(\cdot)}{\partial K^2} < 0, \quad F_{LL}(K, L, A) \equiv \frac{\partial^2 F(\cdot)}{\partial L^2} < 0.$$

Moreover, F exhibits constant returns to scale in K and L.

- Assume F exhibits *constant returns to scale* in K and L. I.e., it is *linearly homogeneous* (homogeneous of degree 1) in these two variables.

Review

Definition Let K be an integer. The function $g : \mathbb{R}^{K+2} \to \mathbb{R}$ is homogeneous of degree m in $x \in \mathbb{R}$ and $y \in \mathbb{R}$ if and only if

$$g(\lambda x, \lambda y, z) = \lambda^m g(x, y, z) \text{ for all } \lambda \in \mathbb{R}_+ \text{ and } z \in \mathbb{R}^K.$$

Theorem **(Euler's Theorem)** Suppose that $g : \mathbb{R}^{K+2} \to \mathbb{R}$ is continuously differentiable in $x \in \mathbb{R}$ and $y \in \mathbb{R}$, with partial derivatives denoted by g_x and g_y and is homogeneous of degree m in x and y. Then

$$\begin{aligned} mg(x, y, z) &= g_x(x, y, z)\, x + g_y(x, y, z)\, y \\ \text{for all } x &\in \mathbb{R},\ y \in \mathbb{R} \text{ and } z \in \mathbb{R}^K. \end{aligned}$$

Moreover, $g_x(x, y, z)$ and $g_y(x, y, z)$ are themselves homogeneous of degree $m-1$ in x and y.

Market Structure, Endowments and Market Clearing I

- We will assume that markets are competitive, so ours will be a prototypical *competitive general equilibrium model*.
- Households own all of the labor, which they supply inelastically.
- Endowment of labor in the economy, $\bar{L}(t)$, and all of this will be supplied regardless of the price.
- The *labor market clearing* condition can then be expressed as:

$$L(t) = \bar{L}(t)$$

  for all t, where $L(t)$ denotes the demand for labor (and also the level of employment).
- More generally, should be written in complementary slackness form.
- In particular, let the *wage rate* at time t be $w(t)$, then the labor market clearing condition takes the form

$$L(t) \leq \bar{L}(t), w(t) \geq 0 \text{ and } (L(t) - \bar{L}(t))\, w(t) = 0$$

Market Structure, Endowments and Market Clearing II

- But Assumption 1 and competitive labor markets make sure that wages have to be strictly positive.
- Households also own the capital stock of the economy and rent it to firms.
- Denote the *rental price of capital* at time t be $R(t)$.
- Capital market clearing condition:

$$K^s(t) = K^d(t)$$

- Take households' initial holdings of capital, $K(0)$, as given
- $P(t)$ is the price of the final good at time t, normalize the price of the final good to 1 *in all periods*.
- Build on an insight by Kenneth Arrow (Arrow, 1964) that it is sufficient to price *securities* (assets) that transfer one unit of consumption from one date (or state of the world) to another.

Market Structure, Endowments and Market Clearing III

- Implies that we need to keep track of an *interest rate* across periods, $r(t)$, and this will enable us to normalize the price of the final good to 1 in every period.
- *General equilibrium economies*, where different commodities correspond to the same good at different dates.
- The same good at different dates (or in different states or localities) is a different commodity.
- Therefore, there will be *an infinite number of commodities*.
- Assume capital depreciates, with "exponential form," at the rate δ: out of 1 unit of capital this period, only $1-\delta$ is left for next period.
- Loss of part of the capital stock affects the interest rate (rate of return to savings) faced by the household.
- *Interest rate* faced by the household will be $r(t) = R(t) - \delta$.

Firm Optimization I

- Only need to consider the problem of a *representative firm*:

$$\max_{L(t)\geq 0, K(t)\geq 0} F[K(t), L(t), A(t)] - w(t)\,L(t) - R(t)\,K(t)\,.$$

- Since there are no irreversible investments or costs of adjustments, the production side can be represented as a static maximization problem.
- Equivalently, *cost minimization problem.*
- Features worth noting:

1. Problem is set up in terms of aggregate variables.
2. Nothing multiplying the F term, price of the final good has normalized to 1.
3. Already imposes competitive factor markets: firm is taking as given $w(t)$ and $R(t)$.
4. Concave problem, since F is concave.

Firm Optimization II

- Since F is differentiable, first-order necessary conditions imply:

$$w(t) = F_L[K(t), L(t), A(t)], \tag{2}$$

and

$$R(t) = F_K[K(t), L(t), A(t)]. \tag{3}$$

- Note also that in (2) and (3), we used $K(t)$ and $L(t)$, the amount of capital and labor used by firms.
- In fact, solving for $K(t)$ and $L(t)$, we can derive the capital and labor demands of firms in this economy at rental prices $R(t)$ and $w(t)$.
- Thus we could have used $K^d(t)$ instead of $K(t)$, but this additional notation is not necessary.

Firm Optimization III

Proposition Suppose Assumption 1 holds. Then in the equilibrium of the Solow growth model, firms make no profits, and in particular,

$$Y(t) = w(t) L(t) + R(t) K(t).$$

- **Proof:** Follows immediately from Euler Theorem for the case of $m = 1$, i.e., constant returns to scale.
- Thus firms make no profits, so ownership of firms does not need to be specified.

Second Key Assumption

Assumption 2 **(Inada conditions)** F satisfies the Inada conditions

$$\begin{aligned}\lim_{K\to 0} F_K(\cdot) &= \infty \text{ and } \lim_{K\to\infty} F_K(\cdot) = 0 \text{ for all } L > 0 \text{ all } A \\ \lim_{L\to 0} F_L(\cdot) &= \infty \text{ and } \lim_{L\to\infty} F_L(\cdot) = 0 \text{ for all } K > 0 \text{ all } A.\end{aligned}$$

- Important in ensuring the existence of *interior equilibria.*
- It can be relaxed quite a bit, though useful to get us started.

Production Functions

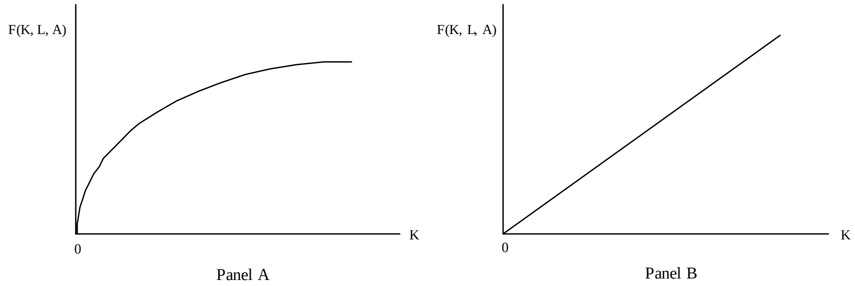


Figure: Production functions and the marginal product of capital. The example in Panel A satisfies the Inada conditions in Assumption 2, while the example in Panel B does not.

Fundamental Law of Motion of the Solow Model I

- Recall that K depreciates exponentially at the rate δ, so

$$K(t+1) = (1-\delta) K(t) + I(t), \tag{4}$$

where $I(t)$ is investment at time t.

- From national income accounting for a closed economy,

$$Y(t) = C(t) + I(t), \tag{5}$$

- *Behavioral rule* of the constant saving rate simplifies the structure of equilibrium considerably.
- Note not derived from the maximization of utility function: welfare comparisons have to be taken with a grain of salt.

Fundamental Law of Motion of the Solow Model II

- Since the economy is closed (and there is no government spending),

$$S(t) = I(t) = Y(t) - C(t).$$

- Individuals are assumed to save a constant fraction s of their income,

$$S(t) = sY(t), \tag{6}$$

$$C(t) = (1 - s)Y(t) \tag{7}$$

- Implies that the supply of capital resulting from households' behavior can be expressed as

$$K^s(t) = (1 - \delta)K(t) + S(t) = (1 - \delta)K(t) + sY(t).$$

Fundamental Law of Motion of the Solow Model III

- Setting supply and demand equal to each other, this implies $K^s(t) = K(t)$.
- We also have $L(t) = \bar{L}(t)$.
- Combining these market clearing conditions with (1) and (4), we obtain *the fundamental law of motion* the Solow growth model:

$$K(t+1) = sF[K(t), L(t), A(t)] + (1-\delta) K(t). \tag{8}$$

- Nonlinear *difference equation*.
- Equilibrium of the Solow growth model is described by this equation together with laws of motion for $L(t)$ (or $\bar{L}(t)$) and $A(t)$.

Definition of Equilibrium I

- Solow model is a mixture of an old-style Keynesian model and a modern dynamic macroeconomic model.
- Households do not optimize, but firms still maximize and factor markets clear.

Definition In the basic Solow model for a given sequence of $\{L(t), A(t)\}_{t=0}^{\infty}$ and an initial capital stock $K(0)$, an equilibrium path is a sequence of capital stocks, output levels, consumption levels, wages and rental rates $\{K(t), Y(t), C(t), w(t), R(t)\}_{t=0}^{\infty}$ such that $K(t)$ satisfies (8), $Y(t)$ is given by (1), $C(t)$ is given by (7), and $w(t)$ and $R(t)$ are given by (2) and (3).

- Note an equilibrium is defined as an entire path of allocations and prices: *not* a static object.

Equilibrium Without Population Growth and Technological Progress I

- Make some further assumptions, which will be relaxed later:
  1. There is no population growth; total population is constant at some level $L > 0$. Since individuals supply labor inelastically, $L(t) = L$.
  2. No technological progress, so that $A(t) = A$.
- Define the capital-labor ratio of the economy as

$$k(t) \equiv \frac{K(t)}{L}, \tag{9}$$

- Using the constant returns to scale assumption, we can express output (income) per capita, $y(t) \equiv Y(t)/L$, as

$$\begin{aligned} y(t) &= F\left[\frac{K(t)}{L}, 1, A\right] \\ &\equiv f(k(t)). \end{aligned} \tag{10}$$

Equilibrium Without Population Growth and Technological Progress II

- Note that $f(k)$ here depends on A, so I could have written $f(k, A)$; but A is constant and can be normalized to $A = 1$.
- From Euler Theorem,

$$\begin{aligned} R(t) &= f'(k(t)) > 0 \text{ and} \\ w(t) &= f(k(t)) - k(t) f'(k(t)) > 0. \end{aligned} \tag{11}$$

- Both are positive from Assumption 1.

Example: The Cobb-Douglas Production Function I

- Very special production function but widely used:

$$\begin{aligned} Y(t) &= F[K(t), L(t), A(t)] \\ &= AK(t)^{\alpha} L(t)^{1-\alpha}, \; 0 < \alpha < 1. \end{aligned}$$

- Satisfies Assumptions 1 and 2.
- Dividing both sides by $L(t)$,

$$y(t) = Ak(t)^{\alpha},$$

- From equation (11),

$$R(t) = \frac{\partial Ak(t)^{\alpha}}{\partial k(t)} = \alpha Ak(t)^{-(1-\alpha)}.$$

- From the Euler Theorem,

$$w(t) = y(t) - R(t)k(t) = (1-\alpha) Ak(t)^{\alpha}.$$

Example:The Cobb-Douglas Production Function II

- Alternatively, in terms of the original Cobb-Douglas production function,

$$\begin{aligned} R(t) &= \alpha A K(t)^{\alpha-1} L(t)^{1-\alpha} \\ &= \alpha A k(t)^{-(1-\alpha)}, \end{aligned}$$

and similarly, from (11),

$$\begin{aligned} w(t) &= (1-\alpha) A K(t)^{\alpha} L(t)^{-\alpha} \\ &= (1-\alpha) A k(t)^{\alpha}, \end{aligned}$$

verifying the Euler Theorem in this case.

Equilibrium Without Population Growth and Technological Progress I

- The per capita representation of the aggregate production function enables us to divide both sides of (8) by L to obtain:

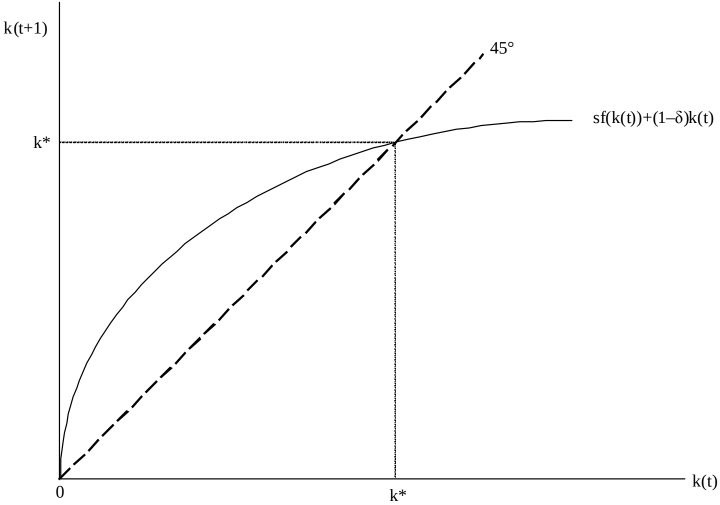
$$k(t+1) = sf(k(t)) + (1-\delta)k(t). \tag{12}$$

- Since it is derived from (8), it also can be referred to as the *equilibrium difference equation* of the Solow model
- The other equilibrium quantities can be obtained from the capital-labor ratio $k(t)$.

Definition A steady-state equilibrium without technological progress and population growth is an equilibrium path in which $k(t) = k^*$ for all t.

- The economy will tend to this steady state equilibrium over time (but never reach it in finite time).

Steady-State Capital-Labor Ratio



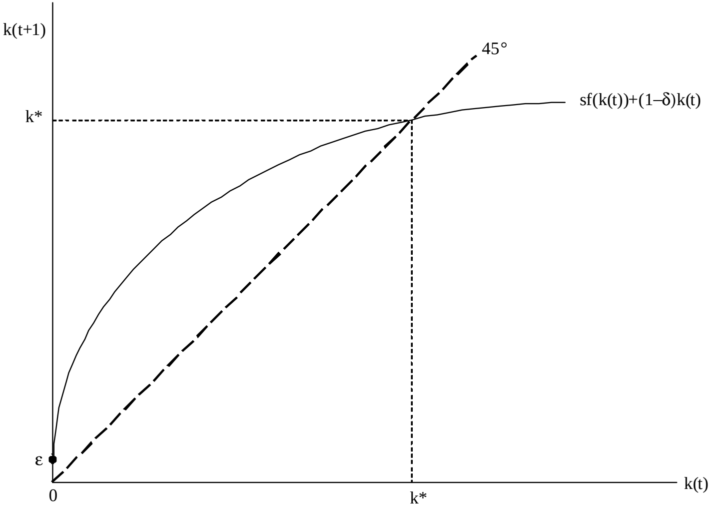
Equilibrium Without Population Growth and Technological Progress II

- Thick curve represents (12) and the dashed line corresponds to the 45° line.
- Their (positive) intersection gives the steady-state value of the capital-labor ratio k^*,

$$\frac{f(k^*)}{k^*} = \frac{\delta}{s}. \tag{13}$$

- There is another intersection at $k = 0$, because the figure assumes that $f(0) = 0$.
- Will ignore this intersection throughout:
  1. If capital is not essential, $f(0)$ will be positive and $k = 0$ will cease to be a steady state equilibrium
  2. This intersection, even when it exists, is an *unstable point*
  3. It has no economic interest for us.

Equilibrium Without Population Growth and Technological Progress III



Equilibrium Without Population Growth and Technological Progress IV

- Alternative visual representation of the steady state: intersection between δk and the function $sf(k)$. Useful because:
  1. Depicts the levels of consumption and investment in a single figure.
  2. Emphasizes the steady-state equilibrium sets investment, $sf(k)$, equal to the amount of capital that needs to be "replenished", δk.

Consumption and Investment in Steady State

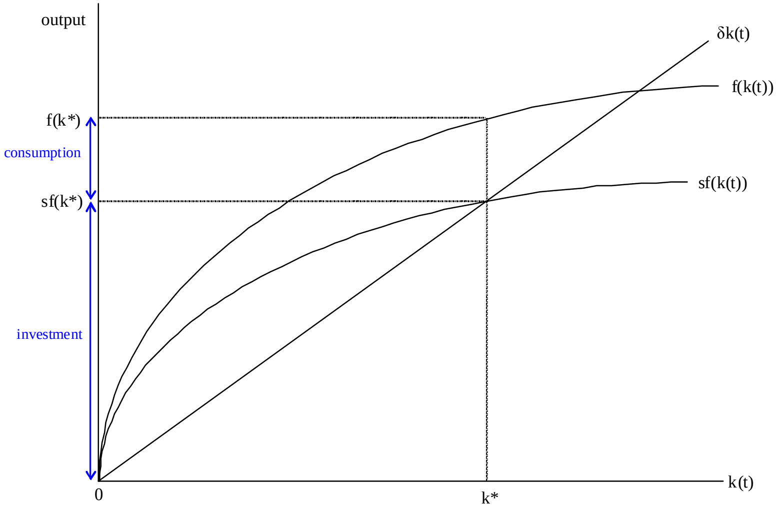


Figure: Investment and consumption in the steady-state equilibrium.

Equilibrium Without Population Growth and Technological Progress V

Proposition Consider the basic Solow growth model and suppose that Assumptions 1 and 2 hold. Then there exists a unique steady state equilibrium where the capital-labor ratio $k^* \in (0, \infty)$ is given by (13), per capita output is given by

$$y^* = f(k^*) \tag{14}$$

and per capita consumption is given by

$$c^* = (1 - s) f(k^*) . \tag{15}$$

Proof

- The preceding argument establishes that any k^* that satisfies (13) is a steady state.
- To establish existence, note that from Assumption 2 (and from L'Hospital's rule), $\lim_{k \to 0} f(k)/k = \infty$ and $\lim_{k \to \infty} f(k)/k = 0$.
- Moreover, $f(k)/k$ is continuous from Assumption 1, so by the Intermediate Value Theorem there exists k^* such that (13) is satisfied.
- To see uniqueness, differentiate $f(k)/k$ with respect to k, which gives

$$\frac{\partial [f(k)/k]}{\partial k} = \frac{f'(k)k - f(k)}{k^2} = -\frac{w}{k^2} < 0, \tag{16}$$

  where the last equality uses (11).
- Since $f(k)/k$ is everywhere (strictly) decreasing, there can only exist a unique value k^* that satisfies (13).
- Equations (14) and (15) then follow by definition.

Non-Existence and Non-Uniqueness

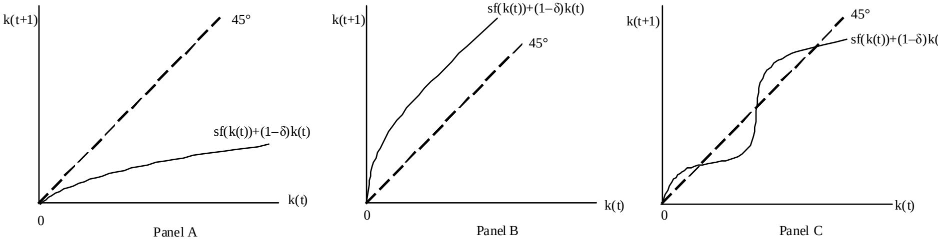


Figure: Examples of nonexistence and nonuniqueness of interior steady states when Assumptions 1 and 2 are not satisfied.

Equilibrium Without Population Growth and Technological Progress VI

- Comparative statics with respect to s, a and δ straightforward for k^* and y^*.
- But c^* will not be monotone in the saving rate (think, for example, of $s = 1$).
- In fact, there will exist a specific level of the saving rate, s_{gold}, referred to as the "golden rule" saving rate, which maximizes c^*.
- But cannot say whether the golden rule saving rate is "better" than some other saving rate.
- Write the steady state relationship between c^* and s and suppress the other parameters:

$$\begin{aligned} c^*(s) &= (1-s) f(k^*(s)), \\ &= f(k^*(s)) - \delta k^*(s), \end{aligned}$$

- The second equality exploits that in steady state $sf(k) = \delta k$.

Equilibrium Without Population Growth and Technological Progress X

- Differentiating with respect to s,

$$\frac{\partial c^*(s)}{\partial s} = \left[f'(k^*(s)) - \delta\right] \frac{\partial k^*}{\partial s}. \quad (17)$$

- s_{gold} is such that $\partial c^*(s_{gold}) / \partial s = 0$. The corresponding steady-state golden rule capital stock is defined as k^*_{gold}.

Proposition In the basic Solow growth model, the highest level of steady-state consumption is reached for s_{gold}, with the corresponding steady state capital level k^*_{gold} such that

$$f'\left(k^*_{gold}\right) = \delta. \quad (18)$$

The Golden Rule

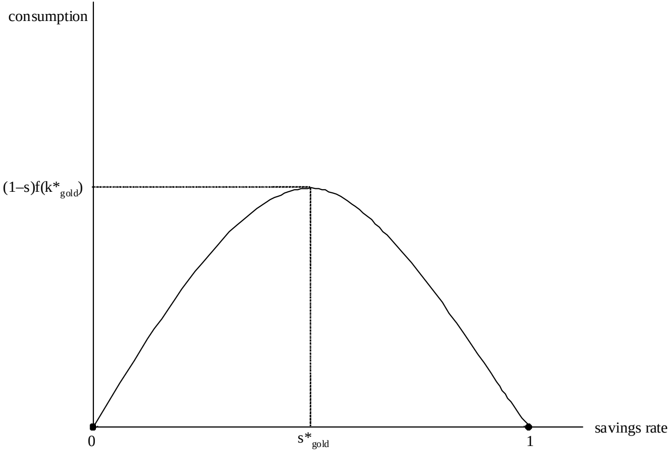


Figure: The "golden rule" level of savings rate, which maximizes steady-state consumption.

Dynamic Inefficiency

- When the economy is below k^*_{gold}, higher saving will increase consumption; when it is above k^*_{gold}, steady-state consumption can be increased by saving less.
- In the latter case, capital-labor ratio is too high so that individuals are investing too much and not consuming enough (*dynamic inefficiency)*.
- But no utility function, so statements about "inefficiency" have to be considered with caution.
- Such dynamic inefficiency will not arise once we endogenize consumption-saving decisions.

Summing up: the Discrete-Time Solow Model

- Per capita capital stock evolves according to

$$k(t+1) = sf(k(t)) + (1-\delta)k(t).$$

- The steady-state value of the capital-labor ratio k^* is given by

$$\frac{f(k^*)}{k^*} = \frac{\delta}{s}.$$

- Consumption is given by

$$C(t) = (1-s)Y(t)$$

- And factor prices are given by

$$\begin{aligned} R(t) &= f'(k(t)) > 0 \text{ and} \\ w(t) &= f(k(t)) - k(t)f'(k(t)) > 0. \end{aligned}$$

Steady State Equilibrium

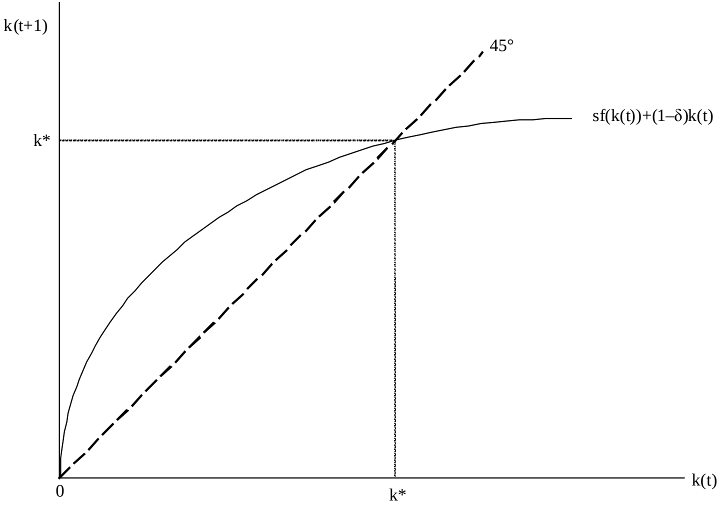


Figure: Steady-state capital-labor ratio in the Solow model.

Transitional Dynamics

- *Equilibrium path*: not simply steady state, but entire path of capital stock, output, consumption and factor prices.
  - In engineering and physical sciences, equilibrium is point of rest of dynamical system, thus *the steady state equilibrium*.
  - In economics, non-steady-state behavior also governed by optimizing behavior of households and firms and market clearing.
- Need to study the "transitional dynamics" of the equilibrium difference equation (12) starting from an arbitrary initial capital-labor ratio $k(0) > 0$.
- Key question: whether economy will tend to steady state and how it will behave along the transition path.

Transitional Dynamics: Review I

- Consider the nonlinear system of autonomous difference equations,

$$\mathbf{x}(t+1) = \mathbf{G}(\mathbf{x}(t)), \tag{19}$$

- $\mathbf{x}(t) \in \mathbb{R}^n$ and $\mathbf{G} : \mathbb{R}^n \to \mathbb{R}^n$.
- Let $\mathbf{x}^*$ be a fixed point of the mapping $\mathbf{G}(\cdot)$, i.e.,

$$\mathbf{x}^* = \mathbf{G}(\mathbf{x}^*).$$

- $\mathbf{x}^*$ is sometimes referred to as "an equilibrium point" of (19).
- We will refer to $\mathbf{x}^*$ as a stationary point or a *steady state* of (19).

Definition A steady state $\mathbf{x}^*$ is (locally) *asymptotically stable* if there exists an open set $B(\mathbf{x}^*) \ni \mathbf{x}^*$ such that for any solution $\{\mathbf{x}(t)\}_{t=0}^{\infty}$ to (19) with $\mathbf{x}(0) \in B(\mathbf{x}^*)$, we have $\mathbf{x}(t) \to \mathbf{x}^*$. Moreover, $\mathbf{x}^*$ is *globally asymptotically stable* if for all $\mathbf{x}(0) \in \mathbb{R}^n$, for any solution $\{\mathbf{x}(t)\}_{t=0}^{\infty}$, we have $\mathbf{x}(t) \to \mathbf{x}^*$.

Transitional Dynamics: Review II

Simple Result About Stability

- Let $x(t), a, b \in \mathbb{R}$, then the unique steady state of the linear difference equation $x(t+1) = ax(t) + b$ is globally asymptotically stable (in the sense that $x(t) \rightarrow x^* = b/(1-a)$) if $|a| < 1$.
- Suppose that $g : \mathbb{R} \rightarrow \mathbb{R}$ is differentiable at the steady state x^*, defined by $g(x^*) = x^*$. Then, the steady state of the nonlinear difference equation $x(t+1) = g(x(t))$, x^*, is locally asymptotically stable if $|g'(x^*)| < 1$. Moreover, if $|g'(x)| < 1$ for all $x \in \mathbb{R}$, then x^* is globally asymptotically stable.

Transitional Dynamics in the Discrete Time Solow Model

Proposition Suppose that Assumptions 1 and 2 hold, then the steady-state equilibrium of the Solow growth model described by the difference equation (12) is globally asymptotically stable, and starting from any $k(0) > 0$, $k(t)$ monotonically converges to k^*.

Proof of Proposition: Transitional Dyamics I

- Let $g(k) \equiv sf(k) + (1-\delta)k$. First observe that $g'(k) > 0$ for all k.
- Next, from (12),
$$k(t+1) = g(k(t)), \tag{20}$$
with a unique steady state at k^*.
- From (13), the steady-state capital k^* satisfies $\delta k^* = sf(k^*)$, or
$$k^* = g(k^*). \tag{21}$$
- Recall that $f(\cdot)$ is concave and differentiable from Assumption 1 and satisfies $f(0) \geq 0$ from Assumption 2.

Proof of Proposition: Transitional Dyamics II

- For any strictly concave differentiable function,

$$f(k) > f(0) + kf'(k) \geq kf'(k), \tag{22}$$

- The second inequality uses the fact that $f(0) \geq 0$.
- Since (22) implies that $\delta = sf(k^*)/k^* > sf'(k^*)$, we have $g'(k^*) = sf'(k^*) + 1 - \delta < 1$. Therefore,

$$g'(k^*) \in (0, 1).$$

- The Simple Result then establishes local asymptotic stability.

Proof of Proposition: Transitional Dyamics III

- To prove global stability, note that for all $k(t) \in (0, k^*)$,

$$\begin{aligned} k(t+1) - k^* &= g(k(t)) - g(k^*) \\ &= -\int_{k(t)}^{k^*} g'(k)\,dk, \\ &< 0 \end{aligned}$$

- First line follows by subtracting (21) from (20), second line uses the fundamental theorem of calculus, and third line follows from the observation that $g'(k) > 0$ for all k.

Proof of Proposition: Transitional Dyamics IV

- Next, (12) also implies

$$\begin{aligned}\frac{k(t+1)-k(t)}{k(t)} &= s\frac{f(k(t))}{k(t)}-\delta \\ &> s\frac{f(k^*)}{k^*}-\delta \\ &= 0.\end{aligned}$$

  Moreover, for any $k(t) \in (0, k^* - \varepsilon)$, this is uniformly so.
- Second line uses the fact that $f(k)/k$ is decreasing in k (from (22) above) andlast line uses the definition of k^*.
- These two arguments together establish that for all $k(t) \in (0, k^*)$, $k(t+1) \in (k(t), k^*)$.
- An identical argument implies that for all $k(t) > k^*$, $k(t+1) \in (k^*, k(t))$.
- Therefore, $\{k(t)\}_{t=0}^{\infty}$ monotonically converges to k^* and is globally stable.

Transitional Dynamics III

- Stability result can be seen diagrammatically in the Figure:
  - Starting from initial capital stock $k(0) < k^*$, economy grows towards k^*, *capital deepening* and growth of per capita income.
  - If economy were to start with $k'(0) > k^*$, reach the steady state by decumulating capital and contracting.
- As a consequence:

Proposition Suppose that Assumptions 1 and 2 hold, and $k(0) < k^*$, then $\{w(t)\}_{t=0}^{\infty}$ is an increasing sequence and $\{R(t)\}_{t=0}^{\infty}$ is a decreasing sequence. If $k(0) > k^*$, the opposite results apply.

- Thus far Solow growth model has a number of nice properties, but no growth, except when the economy starts with $k(0) < k^*$.

Transitional Dynamics in Figure

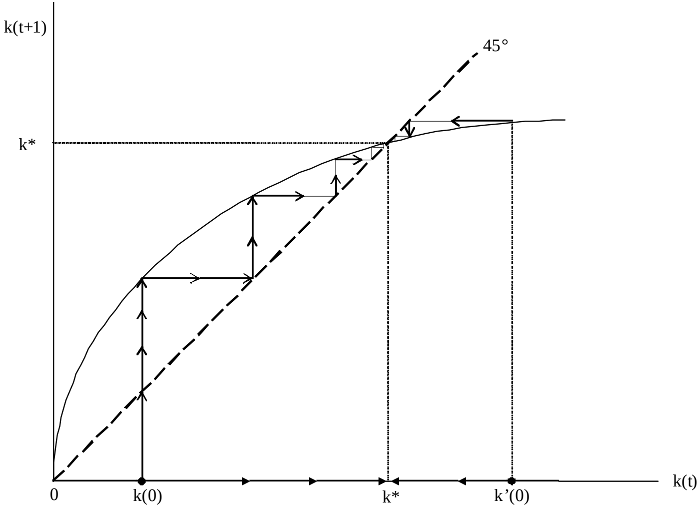


Figure: Transitional dynamics in the basic Solow model.

From Difference to Differential Equations I

- Start with a simple difference equation

$$x(t+1) - x(t) = g(x(t)). \tag{23}$$

- Now consider the following approximation for any $\Delta t \in [0, 1]$,

$$x(t+\Delta t) - x(t) \simeq \Delta t \cdot g(x(t)),$$

- When $\Delta t = 0$, this equation is just an identity. When $\Delta t = 1$, it gives (23).
- In-between it is a linear approximation, not too bad if $g(x) \simeq g(x(t))$ for all $x \in [x(t), x(t+1)]$

From Difference to Differential Equations II

- Divide both sides of this equation by Δt, and take limits

$$\lim_{\Delta t \to 0} \frac{x(t+\Delta t) - x(t)}{\Delta t} = \dot{x}(t) \simeq g(x(t)), \tag{24}$$

  where

$$\dot{x}(t) \equiv \frac{dx(t)}{dt}$$

- Equation (24) is a differential equation representing (23) for the case in which t and $t+1$ is "small".

The Fundamental Equation of the Solow Model in Continuous Time I

- Nothing has changed on the production side, so (11) still give the factor prices, now interpreted as instantaneous wage and rental rates.
- Savings are again

$$S(t) = sY(t),$$

- Consumption is given by (7) above.
- Introduce population growth,

$$L(t) = \exp(nt)\, L(0). \tag{25}$$

- Recall

$$k(t) \equiv \frac{K(t)}{L(t)},$$

The Fundamental Equation of the Solow Model in Continuous Time II

- Implies

$$\begin{aligned}\frac{\dot{k}(t)}{k(t)} &= \frac{\dot{K}(t)}{K(t)} - \frac{\dot{L}(t)}{L(t)}, \\ &= \frac{\dot{K}(t)}{K(t)} - n.\end{aligned}$$

- From the limiting argument leading to equation (24),

$$\dot{K}(t) = sF[K(t), L(t), A(t)] - \delta K(t).$$

- Using the definition of $k(t)$ and the constant returns to scale properties of the production function,

$$\frac{\dot{k}(t)}{k(t)} = s\frac{f(k(t))}{k(t)} - (n+\delta), \tag{26}$$

The Fundamental Equation of the Solow Model in Continuous Time III

Definition In the basic Solow model in continuous time with population growth at the rate n, no technological progress and an initial capital stock $K(0)$, an equilibrium path is a sequence of capital stocks, labor, output levels, consumption levels, wages and rental rates $[K(t), L(t), Y(t), C(t), w(t), R(t)]_{t=0}^{\infty}$ such that $L(t)$ satisfies (25), $k(t) \equiv K(t)/L(t)$ satisfies (26), $Y(t)$ is given by the aggregate production function, $C(t)$ is given by (7), and $w(t)$ and $R(t)$ are given by (11).

- As before, *steady-state* equilibrium involves $k(t)$ remaining constant at some level k^*.

Steady State With Population Growth

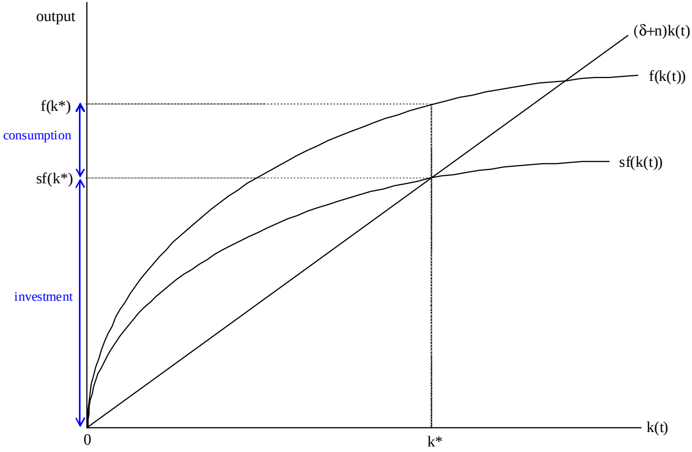


Figure: Investment and consumption in the steady-state equilibrium with population growth.

Steady State of the Solow Model in Continuous Time

- Equilibrium path (26) has a unique *steady state* at k^*, which is given by a slight modification of (13) above:

$$\frac{f(k^*)}{k^*} = \frac{n+\delta}{s}. \tag{27}$$

Proposition Consider the basic Solow growth model in continuous time and suppose that Assumptions 1 and 2 hold. Then there exists a unique steady state equilibrium where the capital-labor ratio is equal to $k^* \in (0, \infty)$ and is given by (27), per capita output is given by

$$y^* = f(k^*)$$

and per capita consumption is given by

$$c^* = (1-s) f(k^*).$$

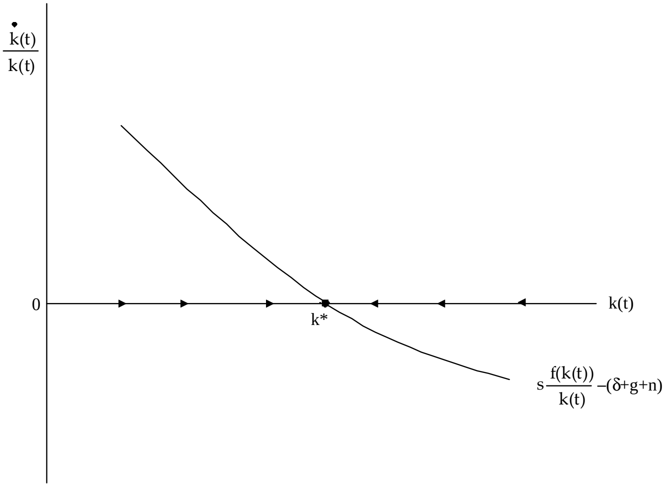
- Similar comparative statics to the discrete time model.

Transitional Dynamics in the Continuous Time Solow Model I

Simple Result about Stability In Continuous Time Model

- Let $g : \mathbb{R} \to \mathbb{R}$ be a differentiable function and suppose that there exists a unique x^* such that $g(x^*) = 0$. Moreover, suppose $g(x) < 0$ for all $x > x^*$ and $g(x) > 0$ for all $x < x^*$. Then the steady state of the nonlinear differential equation $\dot{x}(t) = g(x(t))$, x^*, is globally asymptotically stable, i.e., starting with any $x(0)$, $x(t) \to x^*$.

Simple Result in Figure



Transitional Dynamics in the Continuous Time Solow Model II

Proposition Suppose that Assumptions 1 and 2 hold, then the basic Solow growth model in continuous time with constant population growth and no technological change is globally asymptotically stable, and starting from any $k(0) > 0$, $k(t) \to k^*$.

- **Proof:** Follows immediately from the Theorem above by noting whenever $k < k^*$, $sf(k) - (n + \delta) k > 0$ and whenever $k > k^*$, $sf(k) - (n + \delta) k < 0$.
- Figure: plots the right-hand side of (26) and makes it clear that whenever $k < k^*$, $\dot{k} > 0$ and whenever $k > k^*$, $\dot{k} < 0$, so k monotonically converges to k^*.

A First Look at Sustained Growth I

- Cobb-Douglas already showed that when α is close to 1, adjustment to steady-state level can be very slow.
- Simplest model of sustained growth essentially takes $\alpha = 1$ in terms of the Cobb-Douglas production function above.
- Relax Assumptions 1 and 2 and suppose

$$F\left[K\left(t\right), L\left(t\right), A\left(t\right)\right] = AK\left(t\right), \tag{28}$$

  where $A > 0$ is a constant.
- So-called "AK" model, and in its simplest form output does not even depend on labor.
- Results we would like to highlight apply with more general constant returns to scale production functions,

$$F\left[K\left(t\right), L\left(t\right), A\left(t\right)\right] = AK\left(t\right) + BL\left(t\right), \tag{29}$$

A First Look at Sustained Growth II

- Assume population grows at n as before (cfr. equation (25)).
- Combining with the production function (28),

$$\frac{\dot{k}(t)}{k(t)} = sA - \delta - n.$$

- Therefore, if $sA - \delta - n > 0$, there will be sustained growth in the capital-labor ratio.
- From (28), this implies that there will be sustained growth in output per capita as well.

A First Look at Sustained Growth III

Proposition Consider the Solow growth model with the production function (28) and suppose that $sA - \delta - n > 0$. Then in equilibrium, there is sustained growth of output per capita at the rate $sA - \delta - n$. In particular, starting with a capital-labor ratio $k(0) > 0$, the economy has

$$k(t) = \exp((sA - \delta - n)t)\, k(0)$$

and

$$y(t) = \exp((sA - \delta - n)t)\, Ak(0).$$

- Note no transitional dynamics.

Sustained Growth in Figure

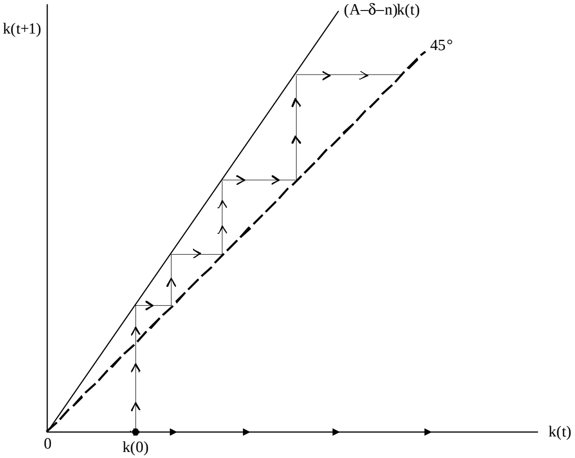


Figure: Sustained growth with the linear AK technology with $sA - \delta - n > 0$.

A First Look at Sustained Growth IV

- Unattractive features:
  1. Knife-edge case, requires the production function to be ultimately linear in the capital stock.
  2. Implies that as time goes by the share of national income accruing to capital will increase towards 1.
  3. Technological progress seems to be a major (perhaps the most major) factor in understanding the process of economic growth.

Balanced Growth I

- Production function $F[K(t), L(t), A(t)]$ is too general.
- May not have *balanced growth*, i.e. a path of the economy consistent with the *Kaldor facts* (Kaldor, 1963).
- Kaldor facts:
  - while output per capita increases, the capital-output ratio, the interest rate, and the distribution of income between capital and labor remain roughly constant.

Historical Factor Shares

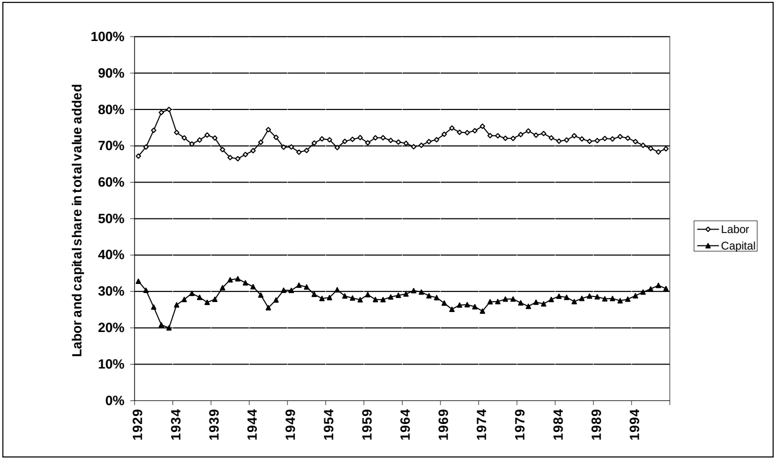


Figure: Capital and Labor Share in the U.S. GDP.

Balanced Growth II

- Note capital share in national income is about $1/3$, while the labor share is about $2/3$.
- Ignoring land, not a major factor of production.
- But in poor countries land is a major factor of production.
- This pattern often makes economists choose $AK^{1/3}L^{2/3}$.
- Main advantage from our point of view is that balanced growth is the same as a steady-state in transformed variables
  - i.e., we will again have $\dot{k} = 0$, but the definition of k will change.
- But important to bear in mind that growth has many non-balanced features.
  - e.g., the share of different sectors changes systematically.

Types of Neutral Technological Progress I

- For some constant returns to scale function $\tilde{F}$:
  - *Hicks-neutral* technological progress:

$$\tilde{F}\left[K(t), L(t), A(t)\right] = A(t) F\left[K(t), L(t)\right],$$

    - Relabeling of the isoquants (without any change in their shape) of the function $\tilde{F}\left[K(t), L(t), A(t)\right]$ in the L-K space.
  - *Solow-neutral* technological progress,

$$\tilde{F}\left[K(t), L(t), A(t)\right] = F\left[A(t) K(t), L(t)\right].$$

    - Capital-augmenting progress: isoquants shifting with technological progress in a way that they have constant slope at a given labor-output ratio.
  - *Harrod-neutral* technological progress,

$$\tilde{F}\left[K(t), L(t), A(t)\right] = F\left[K(t), A(t) L(t)\right].$$

    - Increases output as if the economy had more labor: slope of the isoquants are constant along rays with constant capital-output ratio.

Isoquants with Neutral Technological Progress

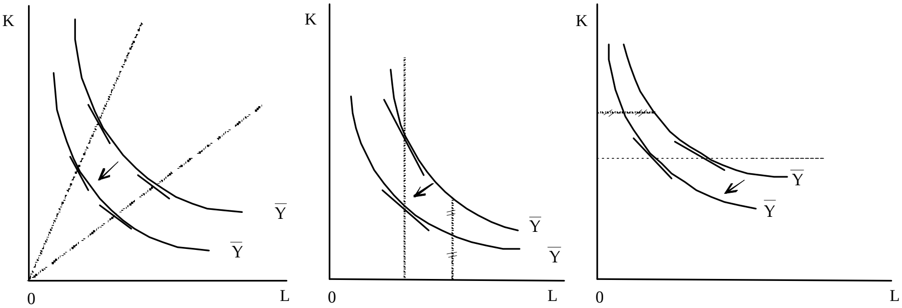


Figure: Hicks-neutral, Solow-neutral and Harrod-neutral shifts in isoquants.

Types of Neutral Technological Progress II

- Could also have a vector valued index of technology $\mathbf{A}(t) = (A_H(t), A_K(t), A_L(t))$ and a production function

$$\tilde{F}[K(t), L(t), \mathbf{A}(t)] = A_H(t) F[A_K(t) K(t), A_L(t) L(t)],$$

- Nests the constant elasticity of substitution production function introduced in the Example above.
- But even this is a restriction on the form of technological progress, $A(t)$ could modify the entire production function.
- Balanced growth necessitates that all technological progress be labor augmenting or Harrod-neutral.

Uzawa's Theorem I

- Focus on continuous time models.
- Key elements of balanced growth: constancy of factor shares and of the capital-output ratio, $K(t)/Y(t)$.
- By factor shares, we mean

$$\alpha_L(t) \equiv \frac{w(t)L(t)}{Y(t)} \text{ and } \alpha_K(t) \equiv \frac{R(t)K(t)}{Y(t)}.$$

- By Assumption 1 and Euler Theorem $\alpha_L(t) + \alpha_K(t) = 1$.

Uzawa's Theorem II

Theorem

(Uzawa I) *Suppose* $L(t) = \exp(nt) L(0)$,

$$Y(t) = \tilde{F}(K(t), L(t), \tilde{A}(t)),$$

$\dot{K}(t) = Y(t) - C(t) - \delta K(t)$, *and* $\tilde{F}$ *is CRS in* K *and* L. *Suppose for* $\tau < \infty$, $\dot{Y}(t) / Y(t) = g_Y > 0$, $\dot{K}(t) / K(t) = g_K > 0$ *and* $\dot{C}(t) / C(t) = g_C > 0$. *Then,*

1. $g_Y = g_K = g_C$; *and*
2. *for any* $t \geq \tau$, $\tilde{F}$ *can be represented as*

$$Y(t) = F(K(t), A(t) L(t)),$$

where $A(t) \in \mathbb{R}_+$, $F : \mathbb{R}_+^2 \to \mathbb{R}_+$ *is homogeneous of degree 1, and*

$$\dot{A}(t) / A(t) = g = g_Y - n.$$

Implications of Uzawa's Theorem

Corollary Under the assumptions of Uzawa Theorem, after time τ technological progress can be represented as Harrod neutral (purely labor augmenting).

- Remarkable feature: stated and proved without any reference to equilibrium behavior or market clearing.
- Also, contrary to Uzawa's original theorem, not stated for a balanced growth path but only for an asymptotic path with constant rates of output, capital and consumption growth.
- **But**, not as general as it seems;
  - the theorem gives only one representation.

Stronger Theorem

Theorem

(Uzawa's Theorem II) *Suppose that all of the hypothesis in Uzawa's Theorem are satisfied, so that* $\tilde{F} : \mathbb{R}^2_+ \times \mathcal{A} \to \mathbb{R}_+$ *has a representation of the form* $F(K(t), A(t)L(t))$ *with* $A(t) \in \mathbb{R}_+$ *and* $\dot{A}(t)/A(t) = g = g_Y - n$*. In addition, suppose that factor markets are competitive and that for all* $t \geq T$*, the rental rate satisfies* $R(t) = R^*$ *(or equivalently,* $\alpha_K(t) = \alpha_K^*$*). Then, denoting the partial derivatives of* $\tilde{F}$ *and* F *with respect to their first two arguments by* $\tilde{F}_K$*,* $\tilde{F}_L$*,* F_K *and* F_L*, we have*

$$\begin{aligned} \tilde{F}_K\left(K(t), L(t), \tilde{A}(t)\right) &= F_K\left(K(t), A(t)L(t)\right) \text{ and} \\ \tilde{F}_L\left(K(t), L(t), \tilde{A}(t)\right) &= A(t)F_L\left(K(t), A(t)L(t)\right). \end{aligned} \tag{30}$$

Moreover, if (30) holds and factor markets are competitive, then $R(t) = R^*$ *(and* $\alpha_K(t) = \alpha_K^*$*) for all* $t \geq T$*.*

Intuition

- Suppose the labor-augmenting representation of the aggregate production function applies.
- Then note that with competitive factor markets, as $t \geq \tau$,

$$\begin{aligned} \alpha_K(t) &\equiv \frac{R(t) K(t)}{Y(t)} \\ &= \frac{K(t)}{Y(t)} \frac{\partial F[K(t), A(t) L(t)]}{\partial K(t)} \\ &= \alpha_K^*, \end{aligned}$$

- Second line uses the definition of the rental rate of capital in a competitive market
- Third line uses that $g_Y = g_K$ and $g_K = g + n$ from Uzawa Theorem and that F exhibits constant returns to scale so its derivative is homogeneous of degree 0.

Intuition for the Uzawa's Theorems

- We assumed the economy features capital accumulation in the sense that $g_K > 0$.
- From the aggregate resource constraint, this is only possible if output and capital grow at the same rate.
- Either this growth rate is equal to n and there is no technological change (i.e., proposition applies with $g = 0$), or the economy exhibits growth of per capita income and capital-labor ratio.
- The latter case creates an asymmetry between capital and labor: capital is accumulating faster than labor.
- Constancy of growth requires technological change to make up for this asymmetry
- But this intuition does not provide a reason for why technology should take labor-augmenting (Harrod-neutral) form.
- But if technology did not take this form, an asymptotic path with constant growth rates would not be possible.

Interpretation

- Distressing result:
  - Balanced growth is only possible under a very stringent assumption.
  - Provides no reason why technological change should take this form.
- But when technology is endogenous, intuition above also works to make technology endogenously more labor-augmenting than capital augmenting.
- Not only requires labor augmenting asymptotically, i.e., along the balanced growth path.
- This is the pattern that certain classes of endogenous-technology models will generate.

Implications for Modeling of Growth

- Does not require $Y(t) = F[K(t), A(t)L(t)]$, but only that it has a representation of the form $Y(t) = F[K(t), A(t)L(t)]$.
- Allows one important exception. If,

$$Y(t) = [A_K(t)K(t)]^{\alpha}[A_L(t)L(t)]^{1-\alpha},$$

  then both $A_K(t)$ and $A_L(t)$ could grow asymptotically, while maintaining balanced growth.
- Because we can define $A(t) = [A_K(t)]^{\alpha/(1-\alpha)} A_L(t)$ and the production function can be represented as

$$Y(t) = [K(t)]^{\alpha}[A(t)L(t)]^{1-\alpha}.$$

- Differences between labor-augmenting and capital-augmenting (and other forms) of technological progress matter when the elasticity of substitution between capital and labor is not equal to 1.

Further Intuition

- Suppose the production function takes the special form $F\left[A_K(t)K(t), A_L(t)L(t)\right]$.
- The stronger theorem implies that factor shares will be constant.
- Given constant returns to scale, this can only be the case when $A_K(t)K(t)$ and $A_L(t)L(t)$ grow at the same rate.
- The fact that the capital-output ratio is constant in steady state (or the fact that capital accumulates) implies that $K(t)$ must grow at the same rate as $A_L(t)L(t)$.
- Thus balanced growth can only be possible if $A_K(t)$ is asymptotically constant.

The Solow Growth Model with Technological Progress: Continuous Time I

- From Uzawa Theorem, production function must admit representation of the form

$$Y(t) = F[K(t), A(t)L(t)],$$

- Moreover, suppose

$$\frac{\dot{A}(t)}{A(t)} = g, \tag{31}$$

$$\frac{\dot{L}(t)}{L(t)} = n.$$

- Again using the constant saving rate

$$\dot{K}(t) = sF[K(t), A(t)L(t)] - \delta K(t). \tag{32}$$

The Solow Growth Model with Technological Progress: Continuous Time II

- Now define $k(t)$ as the *effective capital-labor* ratio, i.e.,

$$k(t) \equiv \frac{K(t)}{A(t)L(t)}. \tag{33}$$

- Slight but useful abuse of notation.
- Differentiating this expression with respect to time,

$$\frac{\dot{k}(t)}{k(t)} = \frac{\dot{K}(t)}{K(t)} - g - n. \tag{34}$$

- Output per unit of effective labor can be written as

$$\begin{aligned} \hat{y}(t) &\equiv \frac{Y(t)}{A(t)L(t)} = F\left[\frac{K(t)}{A(t)L(t)}, 1\right] \\ &\equiv f(k(t)). \end{aligned}$$

The Solow Growth Model with Technological Progress: Continuous Time III

- Income per capita is $y(t) \equiv Y(t)/L(t)$, i.e.,

$$\begin{aligned} y(t) &= A(t)\,\hat{y}(t) \\ &= A(t)\,f(k(t)). \end{aligned} \tag{35}$$

- Clearly if $\hat{y}(t)$ is constant, income per capita, $y(t)$, will grow over time, since $A(t)$ is growing.
- Thus should not look for "steady states" where income per capita is constant, but for *balanced growth paths*, where income per capita grows at a constant rate.
- Some transformed variables such as $\hat{y}(t)$ or $k(t)$ in (34) remain constant.
- Thus balanced growth paths can be thought of as steady states of a transformed model.

The Solow Growth Model with Technological Progress: Continuous Time IV

- Hence use the terms "steady state" and balanced growth path interchangeably.
- Substituting for $\dot{K}(t)$ from (32) into (34):

$$\frac{\dot{k}(t)}{k(t)} = \frac{sF[K(t), A(t)L(t)]}{K(t)} - (\delta + g + n).$$

- Now using (33),

$$\frac{\dot{k}(t)}{k(t)} = \frac{sf(k(t))}{k(t)} - (\delta + g + n), \tag{36}$$

- Only difference is the presence of g: k is no longer the capital-labor ratio but the *effective* capital-labor ratio.

The Solow Growth Model with Technological Progress: Continuous Time V

Proposition Consider the basic Solow growth model in continuous time, with Harrod-neutral technological progress at the rate g and population growth at the rate n. Suppose that Assumptions 1 and 2 hold, and define the effective capital-labor ratio as in (33). Then there exists a unique steady state (balanced growth path) equilibrium where the effective capital-labor ratio is equal to $k^* \in (0, \infty)$ and is given by

$$\frac{f(k^*)}{k^*} = \frac{\delta + g + n}{s}. \tag{37}$$

Per capita output and consumption grow at the rate g.

The Solow Growth Model with Technological Progress: Continuous Time VI

- Equation (37), emphasizes that now total savings, $sf(k)$, are used for replenishing the capital stock for three distinct reasons:
    1. depreciation at the rate δ.
    2. population growth at the rate n, which reduces capital per worker.
    3. Harrod-neutral technological progress at the rate g.
- Now replenishment of effective capital-labor ratio requires investments to be equal to $(\delta + g + n)\, k$.

The Solow Growth Model with Technological Progress: Continuous Time VII

Proposition Suppose that Assumptions 1 and 2 hold, then the Solow growth model with Harrod-neutral technological progress and population growth in continuous time is asymptotically stable, i.e., starting from any $k(0) > 0$, the effective capital-labor ratio converges to a steady-state value k^* $(k(t) \rightarrow k^*)$.

- Now model generates growth in output per capita, but entirely *exogenously*.

Comparative Dynamics I

- Comparative dynamics: dynamic response of an economy to a change in its parameters or to shocks.
- Different from comparative statics in Propositions above in that we are interested in the entire path of adjustment of the economy following the shock or changing parameter.
- For brevity we will focus on the continuous time economy.
- Recall

$$\dot{k}(t)/k(t) = sf(k(t))/k(t) - (\delta + g + n)$$

Comparative Dynamics in Figure

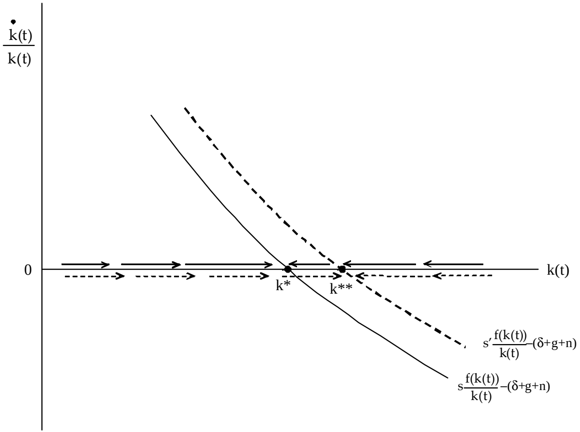


Figure: Dynamics following an increase in the savings rate from s to s'. The solid arrows show the dynamics for the initial steady state, while the dashed arrows

Comparative Dynamics II

- One-time, unanticipated, permanent increase in the saving rate from s to s'.
  - Shifts curve to the right as shown by the dotted line, with a new intersection with the horizontal axis, k^{**}.
  - Arrows on the horizontal axis show how the effective capital-labor ratio adjusts gradually to k^{**}.
  - Immediately, the capital stock remains unchanged (since it is a *state* variable).
  - After this point, it follows the dashed arrows on the horizontal axis.
- s changes in unanticipated manner at $t = t'$, but will be reversed back to its original value at some known future date $t = t'' > t'$.
  - Starting at t', the economy follows the rightwards arrows until t'.
  - After t'', the original steady state of the differential equation applies and leftwards arrows become effective.
  - From t'' onwards, economy gradually returns back to its original balanced growth equilibrium, k^*.

Conclusions

- Simple and tractable framework, which allows us to discuss capital accumulation and the implications of technological progress.
- Solow model shows us that if there is no technological progress, and as long as we are not in the AK world, there will be no sustained growth.
- Generate per capita output growth, but only exogenously: technological progress is a blackbox.
- Capital accumulation: determined by the saving rate, the depreciation rate and the rate of population growth. All are exogenous.
- Need to dig deeper and understand what lies in these black boxes.