Notes on "Perfect Competition, the Profit Criterion, and the Organization of Economic Activity" by Louis Makowski (1980)

Juan Pablo Xandri

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Main Idea: The author introduces a very general framework to study endogenous determination of commodity spaces in general equilibrium. The paper studies Walrasian equilibrium and given its shortcomings on stability of innovations, the concept of "Full Walrasian Equilibrium" is introduced, as one in which no firm has incentives to market new commodities. The author then investigates conditions on equilibria for them to be efficient, and then find conditions on fundamentals to guarantee efficiency (namely, convexity and smoothness of the aggregate production technology and trading possibilities spaces). Finally, the author studies how mergers between firms can help achieve efficiency in exploiting innovation complementarities

1 The Model

1.1 Brief Description

General equilibrium model, of the kind presented in Mas-Collel et al (chapters 15-17)

- I consumers with complete, transitive preferences over a "large" finite set of commodities L, with endowments over this commodities.

- F firms with technology given by production sets. Each firm (or type of firm) can only produce a set of personalized commodities (that is, commodities that only firm F can produce).

- The set of traded commodities will be a subset of L which firms produce in strictly positive amount.

1.2 Consumers

1.2.1 Preferences and Endowments

There is a finite set of consumers (or households) I. For simplicity, I will stand both for the set and the number of elements (that is, $\#I = I$). Also for simplicity, assume that $\#L = L$. Consumers have preferences $\succsim_i$ on net trades on consumption bundles of a set of goods L . To understand this idea, first we need some definitions:

- $\widetilde{X}^i \in \mathbb{R}_+^L$ is the consumption space for agent i

- Agent i receives endowment $\omega^i \in \mathbb{R}_+^L$. Some restrictions are made over the endowment in relation with the set of commodities firms produce. We will discuss it in the next section.

- Define the set of net trades as $X^i = \left\{ x \in \mathbb{R}^L :\ \text{exist } z \in \widetilde{X}^i \text{ such that } \underbrace{z}_{\text{consumption}} = \underbrace{\omega^i}_{\text{endowment}} + \underbrace{x}_{\text{net trade}} \right\}$

The author then assumes that preferences $\succsim_i$ are defined over X^i. In this way, we can get rid of endowments for all proofs and definitions. We assume that agents have non-satiated preferences over X^i : given $x^i \in X^i$ and $\varepsilon > 0$, there exist $y_\varepsilon^i \succ_i x^i$ with $\left\| y_\varepsilon^i - x^i \right\| < \varepsilon$.

Agents are also endowed with a fraction of each of the firms. Let π_f denote the profits of firm f and define $s(i, f) \in [0, 1]$ to be the fraction of the profits of firm f that are given to agent i. The only natural restriction these weights have to satisfy is that:

$$\sum_{i \in I} s(i, f) = 1 \text{ for all } f \in F$$

This paper is concerned with endogenous commodity spaces, which in this model can be interpreted as expanding the set of traded commodities. Given a subset $T \subseteq L$ of commodities and a net trade vector $x^i \in X^i$, we say that a **household i trades in** T if $x_l^i = 0$ for all commodities that are not in T (that is, $x_l^i = 0$ for all $l \notin T$). Based on this concept, define

$$X^i(T) = \left\{ x^i \in X^i : x_l^i = 0 \text{ for all } l \notin T \right\} \tag{1}$$

as the trading set of consumer i if the set of traded commodities is $T \subseteq L$, so the net trade over commodities $L - T$ is restricted to be zero.

1.2.2 Budget Set and the Demand Correspondence

Suppose that a subset $T \subseteq L$ are actually traded, and let $p \in \mathbb{R}^L_+$ the price vector for **all** commodities (not just defined for traded commodities, although prices for non-traded commodities can be anything). Define the budget set of agent i conditional on subset T being traded as

$$BS(p, T) = \left\{ x^i \in X^i(T) : \ px^i \leq \sum_{f \in F} s(i, f) \pi_f \right\} \tag{2}$$

with π_f the profits generated by firm f (to be explained below). Likewise, the demand correspondence given profits from firms is

$$X^*_i(p, T) = \left\{ x^i \in BS(p, T) : \ \nexists y \in BS(p, T) \ \text{with} \ y \succ_i x^i \right\}$$

1.3 Firms

There are F firms, each with a production possibility set $X^f \in \mathbb{R}^L$. The way the author introduces notation will allow him to treat symmetrically both firms and consumers (that's why we use the same notation for net trades and production possibility sets).

We interpret $x^f_l < 0$ as l being an output of production, and $x^f_l > 0$ as l being an input. This is not what we usually do (for example in Mas-Collel et al) but this allows us to treat symmetrically firms and households, in the sense that in equilibrium, the sum of net trades of households and net consumption of firms (which is minus net production of firms) has to sum up to zero. We will return to this aspect of the notation shortly.

Example 1 *Take an economy with only two commodities: labor (l) and food (c). A firm f has l as the only input of a production function of food, which is $Q(l) = \sqrt{l}$. Then the production possibility set of firm f can be written as $X^f = \left\{ (x_1, x_2) \in \mathbb{R}^2 : x_1 \geq 0 \ and \ -x_2 \leq \sqrt{x_1} \right\}$ where x_1 stands for labor and x_2 for food production.*

As we did in the case of consumers, assume that trading can only occur in a subset of commodities $T \subseteq L$, define

$$X^f(T) = \left\{ x^f \in X^f : x^f_l = 0 \ \text{for all} \ l \notin T \right\} \tag{3}$$

Note that this definition is identical to the analogous set for households, as in (1). This is a consequence of the goal of treating both firms and households symmetrically. Based on this notation, profits of firm f are simply

$$\pi_f = p(-x_f) \tag{4}$$

with $p \in \mathbb{R}_+^L$.

Each firm has a set of **personalized commodities**. A commodity $l \in L$ is a personalized commodity for firm f if it is a commodity that can be produced by f and such that no consumer or firm can have an excess supply of it (that is, all other firms and consumers can only demand it, not supply it). Moreover, each firm can only produce personalized commodities (there are no common commodities). We further assume that **households do not have a positive endowment of the firms personalized commodities**. The following definition states this idea more formally

Definition 2 (Personalized commodity) *A commodity $l \in L$ is personalized to firm f (and we write $l \in L(f)$) if and only if :*

1. *There exist $\widetilde{x}^f \in X^f$ such that $\widetilde{x}_l^f < 0$ (so firm f can produce commodity l)*

2. *For all $i \in I$ and all $x^i \in X^i$ we have that $x_l^i \geq 0$ (so consumers can only demand commodity l, and have no endowment of it)*

3. *For all $\widetilde{f} \in F$ such that $\widetilde{f} \neq f$ and all $x^{\widetilde{f}} \in X^{\widetilde{f}}$ we have that $x_l^{\widetilde{f}} \geq 0$ (so firms can only demand commodity l as an input, not produce it)*

In the definition, we denote $L(f)$ as **the set of personalized commodities to firm** f. This is an exogenous technological constraint on the nature of commodities, which need not be restrictive, since we can extend L to include perfect substitutes. To illustrate this idea, take the following example: suppose firms f_1 and f_2 can both produce commodity l_1. Then, we can formally create two distinct commodities l_1 and l_1', one personalized for each firm which enter as perfect substitutes in consumer preferences and does not affect the productive structure of the economy.

The reason for this assumption is to later study how firms can introduce new commodities into the market in a way that certain commodities can only be introduced by certain firms.

An important assumption: **firms can only produce personalized commodities**, so there are no "common" commodities among firms (but again, this is not a restrictive assumption)

Firms are assumed to maximize profits (this is not as trivial as it sounds in this setting, as we will see later). The profit maximizing correspondence for firms if only a subset $T \subseteq L$ of commodities is traded is defined as:

$$X_f^*(p, T) \equiv \left\{ x^f \in X^f(T) : p\left(-x^f\right) \geq p(-y) \text{ for all } y \in X^f(T) \right\} \tag{5}$$

1.4 Walrasian Equilibrium For Subsets of Commodities

Define $J = I \cup F$ the total set of agents in this economy (firms plus consumers). The way we introduced the notation allows us to treat symmetrically both firms and consumers. An **allocation** for this economy is a vector $x = \left(x^1, x^2, ..., x^I, x^{f_1}, x^{f_2}, ..., x^F \right) \in \mathbb{R}^{(I+F)L} \equiv \mathbb{R}^{JL}$ that specifies the net trades in consumption bundles of consumers and the production plans for firms. We can separate it into (x_I, x_F) between firms and consumers.

Before defining Walrasian equilibrium in this setting, we can redefine the demand correspondence and the profit maximizing correspondence in terms of a fixed allocation $\overline{x}$. The budget set and demand correspondence for household i given that a subset $T \subseteq L$ is traded are rewritten as (using (4) into (2))

$$BC\left(\overline{x}, p, T\right) \equiv \left\{ x^i \in X^i\left(T\right): \ px^i \leq \sum_{f \in F} s\left(i, f\right) p\left(-\overline{x}^f\right) \right\} \tag{6}$$

$$X_i^*\left(\overline{x}, p, T\right) \equiv \left\{ x^i \in BC\left(\overline{x}, p, T\right): \ \nexists y \in BC\left(\overline{x}, p, T\right) \text{ with } y \succ_i x^i \right\} \tag{7}$$

Also, to maintain symmetry of notation between firms and households, rewrite:

$$X_f^*\left(\overline{x}, p, T\right) \equiv X_f^*\left(p, T\right) \tag{8}$$

In our concept of equilibrium, we will always allow trade in commodities not produced by firms (that is, agents can exchange their endowments). Namely, let $\mathbf{T} \equiv L - \left\{ \bigcup_{f \in F} L\left(f\right) \right\}$ be the set of of commodities that cannot be produced by firms. In this paper, we will consider only subset of commodities such that for any T, $\mathbf{T} \subseteq T$. This implies that introduction of trading in "new" commodities is not "made" by consumers (which could trade their endowments) but only made by firms trying to market new products, interpreted as commodities not yet produced by any firm.

From now on, each time we take a subset of commodities $T \subseteq L$, we will assume $\mathbf{T} \subseteq T$ (so T is "big enough" to allow households to trade their endowments)

Given a subset of traded commodities $T \subseteq L$ such that $\mathbf{T} \subseteq T$ we want to introduce what we mean by $T-$ feasible allocations and $T-$ Walrasian equilibria.

Definition 3 ($T-$ feasibility) *An allocation x is $T - feasibility$ iff $x^j \in X^j\left(T\right)$ for all $j \in J$, and $\sum_{j \in J} x^j = 0$*

That is, an allocation is $T-feasible$ iff the aggregate net trades between consumers and firms are zero (so demand equals supply). Using this, together with utility and profit maximization of households and firms respectively, we can define a Walrasian equilibrium.

Definition 4 ($T-$Walrasian Equilibrium) *A pair (x,p) is a Walrasian equilibrium given T iff*

1. x is a $T-feasible$ allocation

2. For all $j \in J$, $x^j \in X_j^* (x, p, T)$

We write $(x,p) \in W(T)$, or simply $x \in W(T)$ whenever we need to refer both for allocations and prices, or only allocations, respectively.

2 Stability of Equilibria: Full Walrasian Equilibrium

So far, we defined the typical concept of equilibrium, given a subset of commodities T. However, some firm might be able to introduce new commodities to the economy, and hence affect the equilibrium (since the set of available commodities would be $T' \supset T$, which will have another equilibrium). Then, a proper equilibrium concept has to also have the property that no firm wants to alter the set of available commodities. This concept will be the **Full Walrasian Equilibrium,** to be defined shortly

Take a Walrasian equilibrium $(x,p) \in W(T)$, and take some $y^f \in X^f$ for some firm f. Define $\widehat{L}(y^f) \equiv \left\{ l \in L(f) : y_l^f < 0 \right\}$ as the **set of goods produced by firm f at production plan y^f** (remembering that the production of a given good by firms is given by a negative number, and demand of an intermediate good for firms is given by positive numbers). Given this definition, define $\widehat{T}(y^f)$ as the **set of commodities produced by all other firms, together with the commodities that firm f produces under plan y^f**. More formally:

$$\widehat{T}(y^f) \equiv \underbrace{(T - \{L(f)\})}_{\substack{\text{Traded commodities} \\ \text{produced by other firms} \\ \text{or from endowments}}} \cup \underbrace{\widehat{L}(y^f)}_{\substack{\text{Commodities produced} \\ \text{by } f \text{ at plan } y^f}} \tag{9}$$

Now, we can define a quasi-equilibrium:

Definition 5 (Quasi-Equilibrium) *Given a subset of commodities $T \subseteq L$, a pair $\left(\overline{y}, p^f\right)$ is a Quasi-Equilibrium for y^f (and we write $\left(\overline{y}, p^f\right) \in Q(y^f)$) if and only if the following conditions hold:*

1. $\overline{y}$ is $\widehat{T}(y^f) - feasible$

2. $\overline{y}^f = y^f$

3. For all $j \in J = I \cup F$ we have $y^j \in X_j^* \left(y, p^f, \widehat{T}\left(y^f\right) \right)$ *(according to the definitions in* (7) *and* (8) *)*

A quasi-equilibrium is then the equilibrium that would arise if instead of (x, p), firm f would deviate and produce y^f, moving the economy to this allocation with equilibrium prices p^f. However, not all quasi-equilibria will be considered feasible deviations from the original equilibrium (x, p). The author proposes two qualifications on quasi-equilibria to be potential deviations from equilibrium:

1. A deviating firm does not alter the set of commodities marketed by other firms.

2. A deviation must be both "unilaterally feasible" and "profitable" for the deviating firm

The first property is in line with a competitive, price-taking behavior of firms, in that no firm expects to change the behavior of other agents in the economy. This is what we mean by a deviation being **unilaterally feasible**. To illustrate this idea, imagine the following simple example: say there are two firms: a juice producer f_1 and a farmer f_2. The juice producer may want to introduce a new product: orange juice. However, to do so she has to buy oranges from the farmer. If the farmer is not producing oranges already, then the innovation is not unilaterally feasible

Definition 6 (Feasible innovations) *Given a subset of traded commodities $T \subseteq L$, we say that a production plan y^f is a **feasible innovation** iff $y_l^f \leq 0$ for all $l \notin T - L(f)$. The **feasible innovations from commodity space T for firm f** is denoted as $Y^f(T)$, and can be defined as:*

$$Y^f(T) = \left\{ y^f \in X^f : \; y_l^f \leq 0 \text{ for all } \underbrace{l \notin T - L(f)}_{\substack{\text{potential innovations} \\ \text{of other firms}}} \right\} \tag{10}$$

From the previous definition we can see that a potential deviating firm f cannot demand as inputs (i.e. $y^f > 0$) commodities not already marketed by other firms (that is, goods not in $T - L(f)$). The author makes the following strong assumption:

$$\text{if } y^f \in Y^f(T) \implies Q\left(y^f\right) \neq \varnothing \tag{11}$$

That is, **for any feasible deviation of a firm from a given equilibrium, there is a quasi-equilibrium that would support it**. Now, given the Walrasian equilibrium (x, p) we can define profitable innovations as follows

Definition 7 (Profitable Innovation) *Given a Walrasian equilibrium $(x, p) \in W(T)$, with $T \subseteq L$, we say that there exists a **profitable innovation for firm f** iff there exist $y^f \in Y^f(T)$, an allocation $\overline{y} \in \mathbb{R}^{JL}$, and a price vector $p^f \in \mathbb{R}_+^L$ such that:*

1. $\left(\overline{y}, p^f\right) \in Q\left(y^f\right)$

2. $p^f\left(-y^f\right) > p\left(-x^f\right)$

So, a profitable innovation is simply a potential equilibrium that could arise if firm f introduces some new commodities, together with the fact that the firm would make greater profits in such an equilibrium. With the last definition, we are in position of defining the relevant concept of "stable" equilibrium in this economy, which is given by the Full Walrasian Equilibrium:

Definition 8 (Full Walrasian Equilibrium) *A Full Walrasian equilibrium is a set of commodities T, an allocation $x \in \mathbb{R}^{JL}$ and a price vector $p \in \mathbb{R}_+^L$ such that*

$(\mathbf{a}):$ $(x, p) \in W(T)$

$(\mathbf{b}):$ *For all $f \in F$, there are no profitable innovations.*

We write "(x, p, T) is a FWE"

So, A FWE is a Walrasian equilibrium in a stable set of commodities T, in the sense that no firm will unilaterally want to alter the commodities traded in equilibrium. This will be the equilibrium concept used throughout the paper.

3   Profit Criterion

In this section we will investigate whether profit maximization from the side of the firms is actually what shareholders will want the firm to do. While this is obvious in a classical setting with a fixed set of marketed commodities and competitive firms, is not as obvious in this setting, since the profit criterion also dictates the decision of whether or not to introduce new commodities. However, if firms are "perfect competitors", then the profit maximization objective of firms is the expected behavior of the firm.

Important note: This section could be totally separated from the paper and it would still be proper paper. This section is included in the paper only to justify why in equilibrium we focus on profit maximizing firms

The notion of a firm being a perfect competitor is conditioned on the equilibrium that is played. We say that a firm f is a **perfect competitor at equilibrium** $(x, p) \in W(T)$ if the following conditions hold:

$(\mathbf{a}):$ Firm f cannot influence prices of commodities not produced by other firms if chooses to deviate from equilibrium. More formally, for all feasible innovations $y^f \in Y^f(T)$ and all allocations and price vectors (y, p^f) such that $(y, p^f) \in Q(y^f)$, we have that

$$p_l^f = p_l \text{ for all } l \in T - L(f) \tag{12}$$

(i.e. goods produced by others). That is, any quasi-equilibrium for a feasible deviation must induce the same prices for the commodities traded by other firms and households.

(**b**) : Cannot influence the price of its own commodities. That is, for a given equilibrium, that firm can move the economy to (by changing the set of traded commodities) there is another allocation which is as good for both households and firms as the quasi-equilibrium proposed, but in which no one trades in firms f's personalized commodities.

More formally, for all agents (households and firms) $j \in I \cup F$ with $j \neq f$, all feasible innovations $y^f \in Y^f(T)$ and all quasi-equilibria (y, p^f) we have that there exist an allocation $y(f) \in \mathbb{R}^{JL}$ such that for all j,

$$y(f)^j \in X_j^* \left(y, p^f, \widehat{T}(y^f) \right) \text{ and } \overline{y}(f)_l^j = 0 \text{ for all } l \in L(f) \tag{13}$$

Condition (**b**) : can be interpreted as various firms producing perfect substitute goods, and hence the 0 value on the demand of goods by f mean that they buy all the units from a competitor, and still get the same utility (because of perfect substitutability).

Finally, we need to define the preferences of shareholders over the production plans of a given firm f, if it is thinking of making a deviation. We say shareholder i (that is, $s(i, f) > 0$) **prefers production plan** $y^f \in Y^f(T)$ **to the equilibrium production plan** x^f if and only if for all quasi-equilibrium allocations $(y, p^f) \in Q(y^f)$ agent i prefers $y^i \in X_i^* \left(y, p^f, \widehat{T}(y^f) \right)$ to her equilibrium allocation $x^i \in X_i^*(x, p, T)$. We write "$y^i \succ_i x^i$". We can define analogously $x^i \succ_i y^i$ and $x^i \sim_i y^i$

Once we have this definition of perfect competitor, we are in position to state the following theorem, which indicates under which assumptions profit maximization by firms is the desired behavior by shareholders. That is, any quasi-equilibrium induced by a profitable deviation by firm f is preferred to the equilibrium allocation of each shareholder i of firm f.

Theorem 9 (Shareholder Unanimity) [1] *Let $(x, p) \in W(T)$ and a firm f that is a perfect competitor in (x, p). Also, let $y^f \in Y^f(T)$ and $(y, p^f) \in Q(y^f)$. Then, the following hold:*

(**a**) : *For each $i \in I$ with $s(i, f) > 0$, then the following holds:*

$$
\begin{aligned}
(i) &\quad : \quad p^f\left(-y^f\right) > p\left(-x^f\right) \implies y^i \succ_i x^i \\
(ii) &\quad : \quad p^f\left(-y^f\right) < p\left(-x^f\right) \implies y^i \prec_i x^i \\
(iii) &\quad : \quad p^f\left(-y^f\right) = p\left(-x^f\right) \implies y^i \sim_i x^i
\end{aligned}
$$

(**b**) : *For each $i \in I$ such that $s(i, f) = 0$, we have $x^i \sim_i y^i$*

Although shareholder unanimity is an obvious result in typical Walrasian economies with a fixed set of commodities, it is not as obvious when firms can actually alter the equilibrium by changing the set

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[1]It is mentioned that this result is a simpler exposition of a more general result by Hart (1983), "On shareholder unanimity in large stock market economies", *Econometrica*

of traded commodities. According to this theorem, firms should act as profit maximizers when they are perfect competitors in a given equilibrium.

Proof. Take $(y, p^f) \in Q(y^f)$. Since firm f is a perfect competitor, because of (13) there exist an allocation $y(f)$ such that $y(f)_l^j = 0$ for all $l \in L(f)$ and all $j \in I \cup F - \{f\}$ and such that for any firm $\tilde{f} \neq f$:

$$p^f\left(-y^{\tilde{f}}\right) = p^f\left(-y(f)^{\tilde{f}}\right) \tag{14}$$

that is, $y(f)^{\tilde{f}}$ is also in the supply correspondence of firm $\tilde{f}$ iff it gives the same profits as $y^{\tilde{f}}$. In particular, since $(x, p) \in Q(x^f)$ (that is, the original Walrasian equilibrium is also a quasi-equilibrium with $p^f = p$ and $T(x^f) = T$) we can use again condition (13) with $y = x$ to ensure the existence of an allocation $x(f)$ such that $x(f)_l^j = 0$ for all $l \in L(f)$ and all $j \in I \cup F$, such that:

$$p\left(-x^{\tilde{f}}\right) = p\left(-x(f)^{\tilde{f}}\right) \tag{15}$$

Finally, because of condition (12) the prices firm $\tilde{f} \neq f$ faces for its personalized commodities in any quasi-equilibrium (y, p^f) must be the same as the ones in the original Walrasian equilibrium (x, p). Therefore, since the supply correspondence of each firm only depend on prices, and under both equilibria (the original equilibrium and the quasi-equilibrium) firm $\tilde{f}$ faces the same prices, we must have that $y^{\tilde{f}} \in X_{\tilde{f}}^*(x, p, T)$ and $x^{\tilde{f}} \in X_{\tilde{f}}^*(y, p^f, T(y^f))$ (since they are both optimal under the same objective function and the same feasible set). This further implies that both allocations must give the same profits for firm $\tilde{f}$, so $p^f\left(-y^{\tilde{f}}\right) = p\left(-x(f)^{\tilde{f}}\right)$. Putting this together with (14) and (15) we get:

$$p^f\left(-y(f)^{\tilde{f}}\right) = p^f\left(-y^{\tilde{f}}\right) = p\left(-x(f)^{\tilde{f}}\right) = p\left(-x^{\tilde{f}}\right) \tag{16}$$

From the point of view of consumers, for all $i \in I$ we must also have that $x(f)^i \in X_i^*(x, p, T)$ and $y(f)^i \in X_i^*\left(y, p^f, \widehat{T}(y^f)\right)$ because of (13). Since $(y, p^f) \in Q(y^f)$ and $(x, p) \in W(T)$ we also have that $x^i \in X_i^*(x, p, T)$ and $y^i \in X_i^*\left(y, p^f, \widehat{T}(y^f)\right)$ which implies that :

$$x(f)^i \sim_i x^i \text{ and } y(f)^i \sim_i y^i \text{ for all } i \in I \tag{17}$$

that is, both allocations are in the demand correspondence to allocations x and y respectively iff household i is indifferent between both.

Having proved the above result, let's first attempt to prove (**b**). For this, we need to show that if $s(i, f) = 0 \implies x^i \sim_i y^i$. For this, see that if $s(i, f) = 0$, then all firms $\tilde{f} \neq f$ make the exact same profits under allocations $y, y(f), x(f)$ and x (because of (16)) which implies that such an agent has the exactly same income under these allocations. Also, we know that $x(f)^i \in X_i^*\left(x(f), p, \widehat{T}(y^f)\right)$ and for a commodity l such that $x(f)_l^i > 0 \implies p_l = p_l^f$ (since $l \notin L(f)$). This paired with the equality of profits of all firms $\tilde{f} \neq f$ under allocations y and $x(f)$ implies that for household i, $x(f)^i$ is a feasible consumption bundle under prices p^f and allocation y (i.e. $x(f)^i \in BC\left(y, p^f, \widehat{T}(y^f)\right)$) which by the definition of demand correspondence further implies that

$$y^i \succsim_i x(f)^i \tag{18}$$

By the same reasoning comparing allocations x and $y(f)$, we can show that:

$$x^i \succsim_i y(f)^i \tag{19}$$

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Equations (17) ,(18) and (19), together with and the transitivity of $\succsim_i$ implies

$$y^i \underset{\text{by (18)}}{\succsim_i} x\left(f\right)^i \underset{\text{by (17)}}{\sim_i} x^i \underset{\text{by (19)}}{\succsim_i} y\left(f\right)^i \underset{\text{by (17)}}{\sim_i} y^i \implies x^i \sim y^i$$

Let's now prove (**a**). Take a household i such that $s\left(i,f\right) > 0$. The simplest case is (iii), when $p^f\left(-y^f\right) = p\left(-x^f\right)$, since the proof is identical to (**b**). Suppose by assumption (i) that $p^f\left(-y^f\right) > p\left(-x^f\right)$. Because $\succsim_i$ satisfies local non-satiation and $x\left(f\right)^i \in X_i^*\left(x\left(f\right),p,T\right)$, then $x\left(f\right)^i$ must hit the budget constraint on the set $BC\left(x,p,T\right)$ (i.e. the budget constraint is satisfied with equality): that is

$$px\left(f\right)^i = \sum_{\widetilde{f}\in F} s\left(i,\widetilde{f}\right) p\left(-x^{\widetilde{f}}\right) \tag{20}$$

Using (16) into (20) together with $p^f\left(-y^f\right) > p\left(-x^f\right)$ implies that

$$px\left(f\right)^i = s\left(i,f\right) p\left(-x^f\right) + \sum_{\widetilde{f}\neq f} s\left(i,\widetilde{f}\right) p\left(-x^{\widetilde{f}}\right) \underset{\text{by (16)}}{=} s\left(i,f\right) p\left(-x^f\right) + \sum_{\widetilde{f}\in F} s\left(i,\widetilde{f}\right) p^{\widetilde{f}}\left(-y^{\widetilde{f}}\right)$$

$$\underset{\substack{\text{because we assumed}}}{<} s\left(i,f\right) p^f\left(-y^f\right) + \sum_{\widetilde{f}\in F} s\left(i,\widetilde{f}\right) p^{\widetilde{f}}\left(-y^{\widetilde{f}}\right) = \text{Income in eq. } \left(y,p^f\right) \tag{21}$$

$$p^f\left(-y^f\right) > p\left(-x^f\right)$$

That is, income under the potential equilibrium $\left(y,p^f\right)$ for household i is strictly greater than the cost of buying the bundle $x\left(f\right)^i$, which by non-satiation must be equal to the cost of the bundle x^i from the original equilibrium.

Moreover, we had from before that $x\left(f\right)^i \in BC\left(y,p^f,\widehat{T}\left(y^f\right)\right)$ and also that $y^i \in X_i^*\left(y,p^f,\widehat{T}\left(y^f\right)\right)$ so $y^i \succsim_i x\left(f\right)^i$ (because of the definition of quasi-equilibrium). This last fact together with (21) and local non-satiation implies that y^i must be strictly preferred to $x\left(f\right)^i$, so:

$$y^i \succ_i x\left(f\right)^i \underset{\text{by (17)}}{\sim_i} x^i$$

proving the desired result. Finally, an analogous result holds when $p^f\left(-y^f\right) < p\left(-x^f\right)$ ∎

3.1   Perfect Competitors and Continuum of firms

How realistic is the assumption of firms being perfect competitors in a Walrasian economy? See that the definition given above does not only involve firms being price takers, but states that any firm is basically **inessential**. That is, the existence of a single firm does not affect prices, nor is essential for consumers, since condition (13) implies that for any allocation for which firm f produces something, all consumers and firms will have the same welfare/profits with another allocation that excludes firm f from the economy.

Such a property is found naturally in models in which instead of thinking of the set of firms as single firms, we can think of them as **types of firms**, such that for each type of firm $f \in F$ there exist a **continuum of firms, each with zero mass**. Each type of firm is identified by the production possibilities set, and if we think as the set of personalized commodities as just relabellings of the commodities produced by such a type (as in Example 1) then we can use the basic framework of this paper to analyze its properties. We will actually investigate this property in Pesendorfer (1995) in the next lectures.

4 Optimality of Full Walrasian Equilibrium

In this section we will investigate conditions under which we can achieve optimality under this equilibrium concept. We will first find that if there is only one firm (or type of firms), then equilibria will be Pareto optima. Secondly, we will find conditions ($(C1)$ and $(C2)$ below) on the resulting FWE to guarantee its Pareto optimality. Finally, we study conditions on the fundamentals of the setup such that these conditions are satisfied.

4.1 Individual Improvement of Welfare

Take (x, p, T) a FWE, and take any $T' \supset T$. A FWE will be Pareto optimal if there does not exist a $T' - feasible$ allocation y that dominates x^*, for any $L \supset T' \supset T$. The following theorem states that a FWE cannot be consistent with the existence of profitable deviations by any individual firms $f \in F$.

Theorem 10 *Take $(x, p) \in W(T)$ for some $T \subseteq T' \subseteq L$, and take a $T' - feasible$ allocation y that Pareto-dominates x. Then, if there exist some perfect competitor firm f^* such that $T' - T \subseteq L(f^*)$, then there are profits to innovation by f^* (which means that (x, p, T) is NOT a FWE)*

Proof. Proof is by contradiction: assume that y is a $T'-$ feasible allocation that Pareto Dominates x, that firm f^* is a perfect competitor at (x, p) and such that $T' - T \subset L(f^*)$, but such that there are no profits to innovation by f^*. Then, we will show that y is not a feasible allocation, achieving then a contradiction, proving the desired result.

The fact that y is $T'-$ feasible and $T' - T \subset L(f^*)$ implies that $y^{f^*} \in Y^{f^*}(T)$, as defined in (10). Assumption (11) implies that there exists a quasi-equilibrium $\left(\overline{y}, p^{f^*}\right) \in Q\left(y^{f^*}\right)$ (that is, with $\overline{y}^{f^*} = y^{f^*}$), since this assumptions says that for any potential deviation, there exist some quasi-equilibrium that supports it.

The assumption that $T' - T \subset L(f^*)$ also implies that $\widehat{T}\left(y^{f^*}\right) = T'$, with $\widehat{T}\left(y^{f^*}\right)$ as defined in (9). The non-existence of profits to innovation means that

$$p^{f^*}\left(-\overline{y}^{f^*}\right) = p^{f^*}\left(-y^{f^*}\right) \le p\left(-x^{f^*}\right) \tag{22}$$

Using Theorem 9 on Shareholder unanimity, since f^* is a perfect competitor firm in the equilibrium (x, p), implies that

$$x^i \succsim_i \overline{y}^i \text{ for all } i \in I \tag{23}$$

Now, since y Pareto-dominates x we know that

$$y^i \succsim_i x^i \text{ for all } i \in I \text{ and } \exists i^* \in I : y^{i^*} \succ_{i^*} x^{i^*} \tag{24}$$

Using (23) and (24) together with transitivity of $\succsim_i$ implies that

$$y^i \succsim_i \overline{y}^i \text{ for all } i \in I \text{ and } y^{i^*} \succ_{i^*} \overline{y}^{i^*} \tag{25}$$

Since $\left(\overline{y}, p^{f^*}\right) \in Q\left(y^{f^*}\right)$ we know that $\overline{y}^i \in X_i^*\left(\overline{y}, p^{f^*}, \widehat{T}\left(y^{f^*}\right)\right) = X_i^*\left(\overline{y}, p^{f^*}, T'\right)$, where as we defined before, $\widehat{T}\left(y^f\right)$ is the set of commodities actually traded when we move from the set of commodities T to the one implied by the production plan y^f.

This fact, together with local non-satiation and (25) implies that

$$p^{f^*} y^i \geq p^{f^*}\overline{y}^i \text{ and } p^{f^*} y^{i^*} > p^{f^*}\overline{y}^{i^*} \Longrightarrow p^{f^*}\sum_{i \in I} y^i > p^{f^*}\sum_{i \in I}\overline{y}^i \tag{26}$$

For all firms $f \neq f^*$ firm profit optimization implies that

$$p^{f^*}\left(-\overline{y}^f\right) \geq p^{f^*}\left(-y^f\right) \Longleftrightarrow p^{f^*} y^f \geq p^{f^*}\overline{y}^f \tag{27}$$

Moreover, since $p^{f^*}\overline{y}^{f^*} \equiv p^{f^*} y^{f^*}$ (because $y^{f^*} = \overline{y}^{f^*}$), we can sum up over f and get

$$p^{f^*}\sum_{f \in F} y^f \geq p^{f^*}\sum_{f \in F}\overline{y}^f \tag{28}$$

Summing (26) and (28) we get

$$0 = p^{f^*}\underbrace{\left(\sum_{f \in F} y^f + \sum_{i \in I} y^i\right)}_{=0 \text{ because } y \text{ is } T'-\text{feasible by def. 3}} > p^{f^*}\underbrace{\left(\sum_{f \in F}\overline{y}^f + \sum_{i \in I}\overline{y}^i\right)}_{=0 \text{ because } \overline{y} \text{ is } T'-\text{feasible by def. 3}} = 0 \tag{29}$$

clearly a contradiction, which implies that either y is not feasible. Therefore, there must exist a profitable innovation (i.e. $p^{f^*}\left(-y^{f^*}\right) > p\left(-x^{f^*}\right)$) as we wanted to show $\blacksquare$

A very useful corollary from the previous theorem is that if there is only one type of firm, in the sense that production set are identical (up to the relabellings of perfect substitute goods) then any FWE is Pareto efficient. To state this result more formally, let's define what be mean by two firms having similar technologies

Definition 11 (Similar technologies) [2] *We say that firms f and f' have similar technologies iff there exists a permutation function $h : L \to L$ (that is, a renaming of commodities of commodities) such that:*

1. *h is one to one*

2. *$l \in L\left(f\right) \Longleftrightarrow h\left(l\right) \in L\left(f'\right)$ (so h gives a function from personalized commodities of f to personalized commodities of f')*

3. *If $y^f = \left(y_1^f, y_2^f, ..., y_L^f\right) \in Y^f \Longrightarrow \left(y_{h(1)}^f, y_{h(2)}^f, ..., y_{h(L)}^f\right) \in Y^{f'}$ (so the production sets are identical up to relabellings)*

4. *For any $l \in L\left(f\right)$ we have that l and $l' \equiv h\left(l\right) \in L\left(f'\right)$ are perfect substitutes for all agents (both for consumers as consumption goods and for firms as inputs)*

Intuitively, two firms have similar technologies if they have identical production sets. However, in this setting where we assume the existence of personalized commodities, we have to take into account that perfect substitute goods.

Proposition 12 [3] *Take (x,p) a FWE with subset of traded commodities $T \subset L$. If either there is only 1 firm which is a perfect competitor, or all firms are perfect competitors at (x,p) and have similar technologies, then x is a Pareto optimal allocation*

Proof. Let's prove the first statement: let f_0 be the only firm in the economy. The proof will be by contradiction: assume that x is not Pareto optimal and then show that (x,p) was not a FWE, achieving a contradiction.

If there is only one firm (call it f_0) then by Theorem 10 we know that any introduction of new commodities to move the set of traded commodities from T to $T' \supset T$ can only be done by the only firm in the economy. Therefore $T' - T \subset L(f)$, so if x is Pareto dominated by a $T' - feasible$ allocation, it must be profitable for f_0 to implement it by changing the set of traded commodities to T', which implies that (x,p) was not a FWE to begin with.

The second statement is basically the same as the first, since in essence, there is only one technology with different labels. The idea of the proof is the following: take any $T'-$ feasible allocation y, with $T' \supset T$ and such that y Pareto dominates x. Without loss of generality, let $T' - T = \{l_1, l_2, ..., l_k\}$. In principle, $T' - T$ may not be inside the set of personalized commodities of any firm (i.e. $\nexists f : T' - T \subset L(f)$).

Because all firms have similar technologies, there exist a firm f^*, another commodity set $\widetilde{T} \supset T$ and a $\widetilde{T}-$ feasible allocation $\widetilde{y}$ such that $\widetilde{y}$ Pareto dominates x and such that $\widetilde{T} - T \subset L(f^*)$. Such an allocation can be constructed by getting the relabelling of each commodity in $T' - T$ into $\widetilde{T} - T$, as $\widetilde{T} - T = \left\{l_{h(1)}, l_{h(2)}, ..., l_{h(k)}\right\}$ with h the relabelling function from $T' - T$ to L, and then defining $\widetilde{y}$ as

$$\widetilde{y}_l = y_l \text{ for all } l \in T \text{ and } \widetilde{y}_l = y_{h^{-1}(l)} \text{ for all } l \in \widetilde{T} \tag{30}$$

That is, since all firms have similar technologies, even if $T' - T$ is not in the set of personalized commodities of any firm, there exist some relabelling of them for which there is a firm (actually, **ANY** firm) that has those relabeled commodities in its set of personalized commodities. Then, we can use the previous theorem to conclude that then firm f^* has a profitable deviation, which implies that (x,p) was not a FWE to begin with, achieving a contradiction. ∎

The idea is that if there is only one firm that could move the set of traded commodities from T to T' and hence improve welfare of consumers, it would do it solely on the grounds of profit maximization. Problems will occur when the change in the commodity space depends on multiple firms doing this change, as we will study in the next sections.

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[3] This Proposition is not included in the original paper

4.2 Conditions to achieve efficiency of FWE

Let (x, p, T) be a FWE. We will aim to get conditions on the nature of the FWE itself to get efficiency of the outcome. Let's introduce some notation first:

- $L^* = L - T$ is the set of commodities that are potential innovations from T (remember that $\mathbf{T} \subseteq T$, so these commodities can only be introduced by firms).

- $F^* = \{f \in F : L(f) \cap L^* \neq \varnothing\}$ is the set of firms that are potential innovators at T

- $J^* = J - F^* = (I \cup F) - F^*$ is the set of agents that are not innovators (all consumers and the rest of the firms)

- $P^* = \{\overline{p} \in \mathbb{R}_+^L : \overline{p}_l = p_l$ for all $l \in T$, and such that for all $j \in J^*$, $x^j \in X_j^*(x, \overline{p}, L)\}$ is the set of prices that could extend the defined prices over all commodities L, such that the decisions of all agents in the economy who are not innovators, remain optimal if we expand it to some alternative price vector.

What conditions do we need on (x, p, T) such that there are no favorable pecuniary externalities in the innovation process? This will be given by the following conditions **(C1)** and **(C2)**.

Condition **(C1)** is a requirement that basically states that potential innovations for each firm $f \in F^*$ do not depend on innovations of other firms. That is, any innovation is unilaterally feasible. This requirement fails to hold in situations in which a potential innovation y_1 of firm f_1 requires as an input in production a commodity that is not traded at T but could be introduced by firm $f_2 \neq f_1$. Below we will illustrate this point with some examples.

Condition 13 (C1) *For each $f \in F^*$, $X^f = Y^f(T)$. (that is, for all potential innovators, the production possibility set is exactly the set of unilaterally feasible deviations)*

Condition **(C2)** is very natural: innovation does not change the existing prices for the worse: they are uniformly bounded from below for existing commodities by the same prices for all possible innovations. Also, this acts as a common lower bound on the prices that innovators will expect in equilibrium.

Condition 14 (C2) *There exist $\widetilde{p} \in P^*$ with the following property: for all $f \in F^*$, all $y^f \in Y^f(T)$ and all p^f such that $(y, p^f) \in Q(y^f)$, we get that $p_l^f \geq \widetilde{p}_l$ for all $l \in \widehat{L}(y^f)$*

We show that these two conditions imply the following theorem

Theorem 15 *If (x, p, T) is a FWE and conditions **(C1)** and **(C2)** are satisfied, then there exist $\widetilde{p} \in \mathbb{R}^L$ such that $(x, \widetilde{p}) \in W(L)$, and therefore is Pareto-optimal*

Proof. Take $\widetilde{p}$ to be the price vector defined in **(C2)**. From the definition of P^*. we have that $x^j \in X_j^* (x, \widetilde{p}, L)$ for all $j \in J^*$ (that is, x^i is in the demand correspondence of all households and non-innovating firms at prices $\widetilde{p}$, allocation x and set of traded commodities L). Therefore, to show that $(x, \widetilde{p}) \in W(L)$, we only need to show that the allocation is profit maximizing for all $f \in F^*$: that is, for all $f \in F^*$, $\widetilde{p}(-x^f) \geq \widetilde{p}(-y^f)$ for all $y^f \in X^f$.

From **(C1)**, we have that $y^f \in X^f \equiv Y^f(T)$, and because of condition (11), given this is a feasible deviation, we know that there exist a quasi-equilibrium $(y, p^f) \in Q(y^f)$ for which y^f is the innovation plan. Because (x, p, T) is a FWE, no firm has a profitable deviation, so we must have that:

$$p(-x^f) \geq p^f(-y^f) \tag{31}$$

We know that p is equal to $\widetilde{p}$ in T because of the definition of $\widetilde{p}$ of condition **(C2)**; that is, $\widetilde{p} \in P^*$, which implies that $\widetilde{p}_l = p_l$ for all $l \in T$. Using this, we have that $p(-x^f) = \widetilde{p}(-x^f)$. This together with (31) implies that:

$$\widetilde{p}(-x^f) \geq p^f(-y^f) \tag{32}$$

Using again **(C2)** we know that for any quasi-equilibrium price $p^f \in Q(y^f)$ we have that $p_l^f \geq \widetilde{p}_l$ for all $l \in \widehat{L}(y^f)$ (that is, $\widetilde{p}$ has a smaller price for every commodity which is introduced by f at the innovation plan y^f). Then, since $y_l^f = 0$ for all $l \notin \widehat{L}(y^f)$ (by definition of $\widehat{L}(y^f)$) we have that:

$$p^f(-y^f) \geq \widetilde{p}(-y^f) \tag{33}$$

Putting (32) and (33) together we get:

$$\widetilde{p}(-x^f) \geq p^f(-y^f) \geq \widetilde{p}(-y^f) \Longrightarrow \widetilde{p}(-x^f) \geq \widetilde{p}(-y^f) \tag{34}$$

Since y^f was an arbitrary production plan on X^f, we have shown that $x^f \in X_f^*(x, \widetilde{p}, L)$ for all $f \in F^*$. Therefore, $(x, \widetilde{p}) \in W(L)$.

Since all commodities are being produced, we are in the classical setup of the first welfare theorem, since preferences are assumed to be locally non-satiated and the set of traded commodities is fixed, so it must be the case that x is a Pareto optimal allocation, as we wanted to show. ∎

See that the conditions for Pareto Efficiency of this concept of equilibrium are substantially stricter than the ones needed for the classical First Welfare Theorem. As we will see in the examples and in the next section, condition **(C2)** is related to a notion of convexity and smoothness of both consumption and production sets, a requirement not needed for the first welfare theorem.

4.3 Some Examples of Inefficiencies of FWE

- **Complementarities in intermediate goods:** Two good economy, and two firms. There is only one consumption good produced by firm 1, call it l_1 and the other good, l_2 is just used as an input to lower costs of producing good l_1. Firm f_1 produce good l_1, and f_2 produce good l_2. Moreover, suppose that the cost of producing l_1 without l_2 is greater than the reservation price of consumers

to that good, but the cost of producing it with l_2 is below it. Here we can see a potential failure of efficiency: firm f_1 doesn't produce and neither does f_2, although it would be efficient to do so

- **Non-Smoothness and complementarities between goods:** Two good economy, two firms producing two different consumption goods. Only one consumer, with Leontief preferences between them (perfect complements). Here, we can get stuck at no production (bad coordination). This arises because of non-differentiability: the partial derivatives of the utility function represent the marginal reservation price of each good, and if differentiable, then the change in utility can be approximated with unilateral changes in each good (in this setting made by each firm, with the partial derivative being the reservation price). If it is not differentiable, then total change in utility from going from $(0,0)$ to say $(\varepsilon, \varepsilon)$ with $\varepsilon > 0$ and small, cannot be represented as the change in utility from a change in the consumption of each good individually, with each firm charging the reservation price.

- **Non-Convexities in Consumption Sets:** An economy with 4 goods: housing in two different locations (L_1 and L_2) and developing of the area (as golf courses, cinemas) in each area (c_1 and c_2). If agents can only live in one place (there is a fixed cost of moving from L_1 to L_2, and hence the non-convexity) then even if relocation to L_2 together with c_2 is more valuable than L_1 with c_1, if L_2 alone is worse than L_1, we can see that we can get stuck in an inefficient outcome. It is worth noting that the classical first welfare theorem on efficiency does not rely on any convexity of preferences or consumption sets.

4.4   Smoothness and Convexity

Here, we find some conditions on the fundamentals of the economy, rather than properties of equilibria, to guarantee efficiency of FWE. More specifically, we will try to find conditions under which **(C2)** holds. Let (x, p, T) be given. Some definitions:

Given $E \subseteq \mathbb{R}^L$ and $z \in \mathbb{R}^L$, we define the **support of** E **at** z (and we write $S_z(E)$) as

$$S_z(E) = \left\{ q \in \mathbb{R}^L : q'z \leq q'x \text{ for all } x \in E \right\}$$

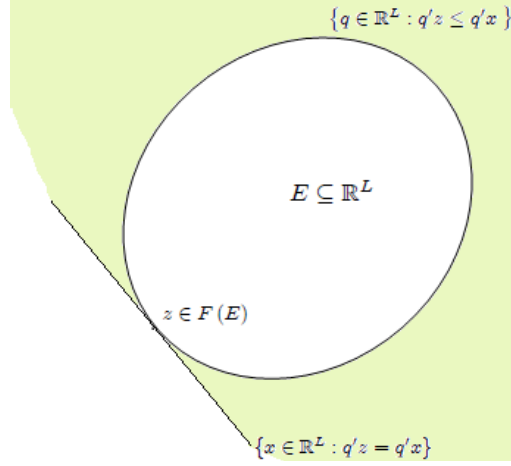the support will be the set-theoretic analog of the tangent of a curve, as we can see in Figure 1:

Figure 1: Vector $q \in S_z(E)$

We have that $S_z(E)$ is a non-empty convex cone.

- We say that E **is a smooth, convex set** if for all $z \in F(E)$ (frontier of E) there exist some $\widetilde{q} \in \mathbb{R}^L$ such that $S_z(E) \subseteq \{q : q = \lambda \widetilde{q} \text{ with } \lambda \in \mathbb{R}\}$. That is, there is a unique support to E at z (so, the frontier is "differentiable"). See that the example in Figure 1 is a smooth, convex set, since the only hyperplane that's in the support at $z \in F(E)$ is $\{x \in \mathbb{R}^L : q'z = a\}$

- $B_i(x^i) \equiv \{x \in X^i : x \succ_i x^i\}$ is the set of strictly preferred consumption bundles for agent i given x^i

- $B(x) \equiv \sum_i B^i(x^i) + \sum_{f \in F - F^*} X^f$ is the aggregate "better than x set" that is achievable without innovators (that is, without any firm producing new commodities)

- $B(T') \equiv B(x) \cap \{z \in \mathbb{R}^L : z_l = 0 \text{ for all } l \text{ not in } T'\}$ is the restriction of the above set to a subset of commodities T'

- $N(l) \equiv \{z \in \mathbb{R}^L : z_l \geq 0\}$ is the $l-$dimension positive orthant

- $L(F^*) \equiv \bigcup_{f \in F^*} L(f)$ is the set of all goods that can be produced by innovating firms

- $relint(E)$ is the relative interior of a set E

Now, we are in conditions on stating the assumptions that will guarantee **(C2)**:

(A1) : There exist some smooth, convex set $B^* \subseteq \mathbb{R}^L$ such that $B(x) = B^* \cap \left(\bigcap_{l \in L(F^*)} N(l) \right)$. That is, the "better than" set can be though of a smooth convex set on the commodity space that has all potential innovators NOT innovating. See Figure 2 to see graphically the sets B^* and $B(x)$

(**A2**) : For all $T' \supset T$, $relint\left(B\left(T'\right)\right) \neq \varnothing$

(**A3**) : For any $(x,p) \in W\left(T\right)$, $\sum_{j \in J^*} x_l^j > 0$ for all $j \in T \cap L\left(F^*\right)$

(**A4**) : For any $q \in \mathbb{R}^L$ and $i \in I$, if $qx^i \leq qz$ for all $z \in B^i\left(x^i\right)$, then if $q_l = p_l$ for all $l \in T$, then there exist $\widetilde{z} \in B^i\left(x^i\right)$ such that $qx^i < q\widetilde{z}$

(**A5**) : For any $f \in F^*, i \in I, y^f \in Y^f\left(T\right), \left(y,p^f\right) \in Q\left(y^f\right)$ such that $s\left(i,f\right) > 0$, there exist some other agent $i^* \in I$ with $s\left(i^*,f\right) = 0$ with identical preferences and trading space $X^{i^*} \equiv X^i$ such that $\sum_{f' \in F} s\left(i^*,f'\right) p^f y^{f'} = \sum_{f' \in F} s\left(i,f'\right) p^f y^{f'}$

Assumption (**A1**) is the differentiability and convexity assumption: namely it means that at the current commodity space available, we could extend the aggregate "better than" set smoothly, which mean that reservation prices are uniquely determined and well defined for the goods that are not provided in the equilibrium (the unique part comes from the unique support). (**A3**) simply says that innovators already market some goods in the equilibrium. Condition (A4) states that if for all consumption bundles z that are preferred for agent i to the one implied by the equilibrium allocation, such that it there is some "generalized price" q for all commodities which coincides with the equilibrium price p, and such that it any preferred consumption bundle costs at least the same under this fictional price than the equilibrium allocation, then it must be the case that there is a preferred consumption bundle $\widetilde{z}$ that costs strictly more. (**A5**) is made to eliminate small income effects through profits of firms.

Theorem 16 *Assume* **(C1)** *and* (**A1**) $-$ (**A5**) *are satisfied. Take a Walrasian equilibrium* $(x,p) \in W\left(T\right)$. *Then, there exist* $\widetilde{p} \in P^* \cap S_x\left(B\left(x\right)\right)$ *that satisfies* **(C2)**. *Then, if* (x,p,T) *is a FWE, then it is a Pareto Optimum*

Proof. (Omitted - A bit technical). The main idea comes from assumption (**A1**), which simply states that reservation prices for unmarketed commodities are well and uniquely defined, which means that firms could market them if they wanted to. The uniquely defined "extended price" would be $\widetilde{p}$ as in **(C2)**. In Figure 2, imagine that only commodity l_2 is traded in an FWE, resulting in allocation $\overline{x} = (0,\overline{x}_2)$. The dotted area is the better than set $B\left(\overline{x}\right)$, in which there is no production of commodity l_1. The gray area is the smooth convex set B^*, and see that as assumed in (**A1**) we have $B\left(\overline{x}\right) = B^* \cap \left(\bigcap_{l \in L\left(F^*\right)} N\left(l\right)\right)$. The extended price vector comes from the slope of the supporting hyperplane of B^* at $\overline{x}$, as seen in the figure $\blacksquare$

l_2

$B(\overline{x}) = B^* \cap \{N(l_1) \cap N(l_2)\}$

$\overline{x} = (0, \overline{x}_2)$

B^*

l_1

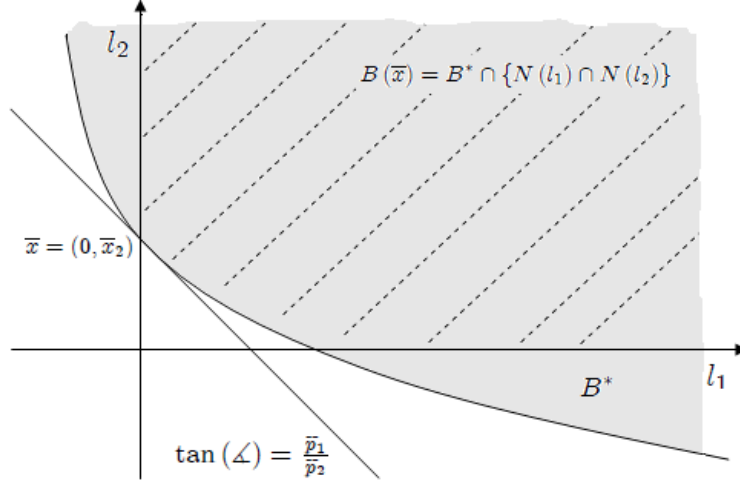$\tan(\angle) = \frac{\overline{p}_1}{\overline{p}_2}$

Figure 2: Idea of proof, Theorem 16

5   Coalitions of Firms to achieve efficiency

Theorem 13 stated that if profitable innovations possibilities were available to one firm, then in equilibrium they would be made. This raises the possibility of mergers of firms to take this profitable opportunities, and this could in principle, restore efficiency. This is what this section is about.

Take an equilibrium $(x, p) \in W(T)$. A **merger** is a coalition of firms in $F' \subseteq F$, which we will call $\widetilde{f}(F')$, with production possibilities set $X^{\widetilde{f}(F')} = \sum_{f \in F'} X^f$. At this equilibrium, the ownership shares for this integrated firm will be given by

$$s\left(i, \widetilde{f}(F')\right) = \frac{\sum_{f \in F'} s(i, f) \, px^f}{\sum_{f \in F'} px^f} \tag{35}$$

See that these shares are just the proportion of profits that agent i had on all the firms which are now forming the merger. Also note that shares on this corporation depend on the equilibrium that is being played. Some further definitions:

- $F(F') = (F - F') \cup f(F')$ is the new set of firms, that group firms in F' in a single firm $f(F')$, and the rest are treated identically as before

- $\mathcal{F}^*$ is the set of all subsets of F

- $\mathcal{F} \subseteq \mathcal{F}^*$ is the set of all possible coalitions between firms (that need not be all possible mergers, i.e. in principle, not all coalitions are not allowed)

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- $\mathcal{E}(F')$ stands for the economy with the same set of consumers, and set of firms $F(F')$ and shares defined as in (35)

- Given allocation $x \in \mathbb{R}^L, x(F')$ is an allocation in $\mathcal{E}(F')$ such that $x(F)^j = x^j$ for all $j \notin F'$, and $x(F)^{f(F')} = \sum_{f \in F'} x^f$

Now, we are in conditions to extend the concept of FWE with coalitions

Definition 17 (Full Walrasian Equilibrium with coalitions) *A triple (x, p, T) is a Full Walrasian Equilibrium with coalitions* [4] *such that for all $F' \in \mathcal{F}$ we have that $(x(F'), p, T)$ is a Full Walrasian equilibrium in economy $\mathcal{E}(F')$*

From this definitions and the theorems already proven, we can draw some corollaries:

Corollary 18 *Suppose that $(x, p) \in W(T)$ and some $T' - feasible$ allocation y dominates x. Then if $(T' - T) \subseteq \bigcup_{f \in F'} L(f)$ for some coalition $F' \in \mathcal{F}$, then there are profits to innovation by $\widetilde{f}(F')$. Moreover, if $\mathcal{F} = \mathcal{F}^*$, then any Full Walrasian Equilibrium with coalitions is Pareto optimum*

Proof. The first part comes directly as an application of Theorem 10, using as a single firm the coalition F', which then has feasible and profitable innovations. The second part is shown from Proposition 15, since if there exist any potential innovation that is not individually feasible, then firms can form a coalition and take advantages from this innovation. ∎

The important point of this section is to note that here, mergers would emerge NOT to exploit economies of scale, but rather to exploit complementarities on innovations.

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[4] In original paper, not diferentiated from FWE