Notes on Strategic Substitutes and Complements in Global Games

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1. A Private Values Global Game

A continuum of players choose action 0 or action 1.

Each player i has a payoff parameter x_i. The x_i are normally distributed in the population with mean θ and precision β; the mean is unknown to the players, and is itself normally distributed with mean y and precision α.

The payoff to action 1 is x_i. The payoff to action 0 is cl, where c is a positive or negative constant and l is the proportion of players choosing action 0.

We analyzed essentially this game in Morris and Shin (2002), in the special case where $c = 1$. Inspired by Guesnerie's (2004) discussion of eductive stability with incomplete information, we solve here for the case where c can be negative.

2. Summary of Results

We will show:

Proposition 1. *This game is dominance solvable (i.e., has an essentially unique strategy profile surviving iterated deletion of strictly dominated strategies) if and* *only if*

$$-\sqrt{2\pi\left(\frac{\alpha+\beta}{\beta\left(\alpha+2\beta\right)}\right)} \leq c \leq \sqrt{2\pi\left(\frac{\left(\alpha+\beta\right)\left(\alpha+2\beta\right)}{\alpha^{2}\beta}\right)}.$$

This follows from corollories 7 and 11 below. Notice that in the special case where $\alpha \to 0$, this condition becomes

$$-\sqrt{\frac{\pi}{\beta}} \leq c \leq \infty.$$

But as $\alpha \to \infty$, this condition becomes

$$-\sqrt{\frac{2\pi}{\beta}} \leq c \leq \sqrt{\frac{2\pi}{\beta}}.$$

This nicely illustrates an important point in Guesnerie (2004): adding aggregate uncertainty (decreasing α for a fixed β) tends to make dominance solvability easier to satisfy in the case of strategic complementarities, but harder to satisfy in the case of strategic substitutes.

Proposition 2. *If*

$$c < -\sqrt{2\pi\left(\frac{\left(\alpha+\beta\right)\left(\alpha+2\beta\right)}{\beta^{3}}\right)},$$

this game has no threshold equilibrium. If

$$-\sqrt{2\pi\left(\frac{\left(\alpha+\beta\right)\left(\alpha+2\beta\right)}{\beta^{3}}\right)} \leq c \leq \sqrt{2\pi\left(\frac{\left(\alpha+\beta\right)\left(\alpha+2\beta\right)}{\alpha^{2}\beta}\right)},$$

there is a unique threshold equilibrium. If

$$c > \sqrt{2\pi\left(\frac{\left(\alpha+\beta\right)\left(\alpha+2\beta\right)}{\alpha^{2}\beta}\right)},$$ *there are three threshold equilibria.*

This follows from corollary 5 and proposition 8 below. Interestingly, a similar observation to Guesnerie's holds for equilibrium as well: as $\alpha \to 0$, there is always exactly one threshold equilibrium under strategic substitutes, but there may be multiple equilibria under strategic complementarities. But as $\alpha \to \infty$, there is no threshold equilibrium for $c < -\sqrt{\frac{4\pi}{\beta}}$, but there is always a unique threshold equilibrium under strategic complementarities.

3. Key Expression

We introduce the key function to analyze this game. Observe that player i believes that any other player's private signal x_j is distributed normally with mean

$$\frac{\alpha y+\beta x_i}{\alpha+\beta}$$

and precision

$$\frac{\beta\left(\alpha+\beta\right)}{\alpha+2\beta}.$$

Thus the probability that any opponent observes a signal less than x^* is

$$\Phi\left(\sqrt{\frac{\beta\left(\alpha+\beta\right)}{\alpha+2\beta}}\left(x^*-\frac{\alpha y+\beta x_i}{\alpha+\beta}\right)\right).$$

Note that this is also the expected proportion observing a signal less than x^*. The $\widehat{x}$-threshold strategy is

$$s\left(x\right)=\begin{cases}1, \text{ if } x\geq\widehat{x}\\ 0, \text{ if } x<\widehat{x}\end{cases}.$$

Now suppose that all player follow the x^*-threshold strategy. Then the expected payoff to action 1 is x_i; the expected payoff to action 0 is

$$c\Phi\left(\sqrt{\frac{\beta\left(\alpha+\beta\right)}{\alpha+2\beta}}\left(x^*-\frac{\alpha y+\beta x_i}{\alpha+\beta}\right)\right).$$

Thus the expected gain to choosing action 1 is

$$u\left(x_i,x^*,y\right)=x_i-c\Phi\left(\sqrt{\frac{\beta\left(\alpha+\beta\right)}{\alpha+2\beta}}\left(x^*-\frac{\alpha y+\beta x_i}{\alpha+\beta}\right)\right).$$

Proposition 3. *There is a unique value of x solving $u\left(x,x,y\right)=0$ (for all y) if and only if*

$$c\leq\sqrt{2\pi\left(\frac{\left(\alpha+\beta\right)\left(\alpha+2\beta\right)}{\beta\alpha^2}\right)}.$$

PROOF.

$$u(x,x,y) = x - c\Phi\left(\sqrt{\frac{\beta(\alpha+\beta)}{\alpha+2\beta}}\left(\frac{\alpha}{\alpha+\beta}\right)(x-y)\right).$$

$$\begin{aligned}
\frac{d}{dx}u(x,x,y) &= 1 - c\sqrt{\frac{\beta(\alpha+\beta)}{\alpha+2\beta}}\left(\frac{\alpha}{\alpha+\beta}\right)\phi\left(\sqrt{\frac{\beta(\alpha+\beta)}{\alpha+2\beta}}\left(\frac{\alpha}{\alpha+\beta}\right)(x-y)\right) \\
&= 1 - c\sqrt{\frac{\beta\alpha^2}{(\alpha+\beta)(\alpha+2\beta)}}\phi\left(\sqrt{\frac{\beta(\alpha+\beta)}{\alpha+2\beta}}\left(\frac{\alpha}{\alpha+\beta}\right)(x-y)\right)
\end{aligned}$$

The expression on the right hand side is minimized when $x = y$. Since $\phi(0) = \frac{1}{\sqrt{2\pi}}$, we have

$$\begin{aligned}
\left.\frac{d}{dx}u(x,x,y)\right|_{y=x} &= 1 - c\sqrt{\frac{\beta\alpha^2}{(\alpha+\beta)(\alpha+2\beta)}}\phi(0) \\
&= 1 - c\sqrt{\frac{1}{2\pi}\left(\frac{\beta\alpha^2}{(\alpha+\beta)(\alpha+2\beta)}\right)}.
\end{aligned}$$

This establishes the sufficiency of

$$c \leq \sqrt{2\pi\left(\frac{(\alpha+\beta)(\alpha+2\beta)}{\beta\alpha^2}\right)}$$

for uniqueness (for any y). Now suppose that

$$c > \sqrt{2\pi\left(\frac{(\alpha+\beta)(\alpha+2\beta)}{\beta\alpha^2}\right)}.$$

Now if $y = x = \frac{c}{2}$, we have $u(x,x,y) = 0$ and

$$\left.\frac{d}{dx}u(x,x,y)\right|_{y=x=\frac{c}{2}} = 1 - c\sqrt{\frac{1}{2\pi}\left(\frac{\beta\alpha^2}{(\alpha+\beta)(\alpha+2\beta)}\right)} < 0,$$

so there are other solutions to $u(x,x,y) = 0$.

4. Strategic Complementarities $(c \geq 0)$

Assume throughout this section that $c \geq 0$. The results in this section are minor variants of our results elsewhere (e.g., in Morris and Shin (2002)), and exploit the fact that $u(x_i, x^*, y)$ is increasing in x_i.

4.1. Equilibrium

Proposition 4. *There is a $\widehat{x}$-threshold equilibrium if and only if $u(\widehat{x}, \widehat{x}, y) = 0$.*

Corollary 5. *There is a unique threshold equilibrium (for all y) if and only if*

$$c \leq \sqrt{2\pi \left(\frac{(\alpha + \beta)(\alpha + 2\beta)}{\beta \alpha^2} \right)}.$$

4.2. Dominance Solvability

Let $\underline{x}(y)$ and $\overline{x}(y)$ be the smallest and largest solutions to the equation $u(x, x, y) = 0$.

Proposition 6. *A strategy s survives iterated deletion of strictly dominated strategies if and only if $x < \underline{x}(y) \Rightarrow s(x) = 0$ and $x > \overline{x}(y) \Rightarrow s(x) = 1$.*

Corollary 7. *The game is dominance solvable (for all y) if and only if*

$$c \leq \sqrt{2\pi \left(\frac{(\alpha + \beta)(\alpha + 2\beta)}{\beta \alpha^2} \right)}.$$

5. Strategic Substitutes $(c < 0)$

Assume throughout this section that $c < 0$. In this case, we have that $u(x_i, x^*, y)$ is strictly increasing in x^*. Note that we do not necessarily have u increasing in x_i.

5.1. Equilibrium

Proposition 8. *If*

$$c < -\sqrt{2\pi\left(\frac{(\alpha+\beta)(\alpha+2\beta)}{\beta^3}\right)},$$

then this game has no threshold equilibrium (for some y). If

$$-\sqrt{2\pi\left(\frac{(\alpha+\beta)(\alpha+2\beta)}{\beta^3}\right)} \leq c \leq 0,$$

then, for all y, there is a unique threshold equilibrium with cutoff equal to the unique solution to $u(x,x,y)=0$.

PROOF. A necessary condition for an $\widehat{x}$ threshold equilibrium is clearly that $u(\widehat{x},\widehat{x},y)=0$. If $c \leq 0$, then (by Proposition 3) there is at most one threshold equilibrium. Now a sufficient condition for this to be an equilibrium would be that

$$\frac{d}{dx_i}u(x_i,x^*,y) \geq 0.$$

Now

$$\begin{aligned}
\frac{d}{dx_i}u(x_i,x^*,y) &= 1+c\sqrt{\frac{\beta(\alpha+\beta)}{\alpha+2\beta}}\left(\frac{\beta}{\alpha+\beta}\right)\phi\left(\sqrt{\frac{\beta(\alpha+\beta)}{\alpha+2\beta}}\left(x^*-\frac{\alpha y+\beta x_i}{\alpha+\beta}\right)\right) \\
&\geq 1+c\sqrt{\frac{1}{2\pi}\left(\frac{\beta^3}{(\alpha+\beta)(\alpha+2\beta)}\right)}.
\end{aligned}$$

So if

$$c \geq -\sqrt{2\pi\left(\frac{(\alpha+\beta)(\alpha+2\beta)}{\beta^3}\right)},$$

we have existence. But suppose this condition fails, and we have

$$c < -\sqrt{2\pi\left(\frac{(\alpha+\beta)(\alpha+2\beta)}{\beta^3}\right)}.$$

Suppose that $y=\frac{c}{2}$. The unique value of x solving $u(x,x,y)=0$ is then $\frac{c}{2}$. But

$$\frac{d}{dx_i}u\left(\frac{c}{2},\frac{c}{2},\frac{c}{2}\right) = 1+c\sqrt{\frac{1}{2\pi}\left(\frac{\beta^3}{(\alpha+\beta)(\alpha+2\beta)}\right)} < 0.$$

This contradicts the existence of the threshold equilibrium.

5.2. Dominance Solvability

Now consider the values of x solving the equation $u(x, x^*, y) = 0$. Any solution must lie in the compact interval $[-1, 0]$. So by continuity of u, there exist $\overline{x}(x^*, y)$ and $\underline{x}(x^*, y)$, the largest and smallest solutions to the equation $u(x, x^*, y) = 0$. Since $\lim\limits_{x\to-\infty} u(x, x^*, y) = -\infty$ and $\lim\limits_{x\to\infty} u(x, x^*, y) = \infty$, observe (by continuity) that $u(x, x^*, y) > 0$ for all $x > \overline{x}(x^*, y)$ and $u(x, x^*, y) < 0$ for all $x < \underline{x}(x^*, y)$.

Now observe that if $\overline{x}^* \geq \underline{x}^*$, then $u(x, \overline{x}^*, y) > u(x, \underline{x}^*, y) > 0$ for all $x > \overline{x}(\underline{x}^*, y)$ and $u(x, \underline{x}^*, y) < u(x, \overline{x}^*, y) < 0$ for all $x < \underline{x}(\overline{x}^*, y)$, implying that $\overline{x}(\overline{x}^*, y) \leq \overline{x}(\underline{x}^*, y)$ and $\underline{x}(\overline{x}^*, y) \leq \underline{x}(\underline{x}^*, y)$. Thus both $\overline{x}(x^*, y)$ and $\underline{x}(x^*, y)$ are decreasing in x^*.

Now define $\underline{z}^k$ and $\overline{z}^k$ inductively by $\underline{z}^0 = -\infty$, $\overline{z}^0 = \infty$, $\underline{z}^{k+1} = \underline{x}\left(\overline{z}^k, y\right)$ and $\overline{z}^{k+1} = \overline{x}\left(\underline{z}^k, y\right)$. Since $\overline{z}^0 > \overline{z}^1 > \underline{z}^1 > \underline{z}^0$, we have that by induction that $\underline{z}^k$ is an increasing sequence and $\overline{z}^k$ is a decreasing sequence with $\underline{z}^k \leq \overline{z}^k$ for all k. Let $\underline{z}^*(y) = \lim\limits_{k\to\infty} \underline{z}^k$ and $\overline{z}^*(y) = \lim\limits_{k\to\infty} \overline{z}^k$.

We will show (by induction) that strategy s survives k rounds of deletion of strictly dominated strategies if and only if $x < \underline{z}^k \Rightarrow s(x) = 0$ and $x > \overline{z}^k \Rightarrow s(x) = 1$. Vacuously true for $k = 0$. Suppose it is true for k.

Now the payoff gain to choosing action 1 for a player observing x if his opponent is following a strategy surviving k rounds is at most $u\left(x, \overline{z}^k, y\right)$. So if $x < \underline{x}\left(\overline{z}^k, y\right) = \underline{z}^{k+1}$, then action 1 cannot be a best response. Thus any strategy surviving $k+1$ rounds has $x < \underline{z}^{k+1} \Rightarrow s(x) = 0$.

Also the payoff gain to choosing action 1 for a player observing x if his opponent is following a strategy surviving k rounds is at least $u\left(x, \underline{z}^k, y\right)$. So if $x < \overline{x}\left(\underline{z}^k, y\right) = \overline{z}^{k+1}$, then action 0 cannot be a best response. Thus any strategy surviving $k+1$ rounds has $x > \overline{z}^{k+1} \Rightarrow s(x) = 1$.

Finally, observe that if $x \in \left[\underline{z}^{k+1}, \overline{z}^{k+1}\right]$

Thus we have:

Proposition 9. *A strategy s survives iterated deletion of strictly dominated strategies if and only if $x < \underline{z}^*(y) \Rightarrow s(x) = 0$ and $x > \overline{z}^*(y) \Rightarrow s(x) = 1$.*

Observe that by construction $\underline{z}^*(y)$ and $\overline{z}^*(y)$ are the unique pair of numbers satisfying the following properties:

1. $\overline{z}^*(y) \geq \underline{z}^*(y)$;

2. $u\left(\overline{z}^*(y), \underline{z}^*(y), y\right) = 0$;

3. $u\left(\underline{z}^*(y), \overline{z}^*(y), y\right) = 0$;

4. if $(\overline{z}, \underline{z})$ satisfy (i) $\overline{z} \geq \underline{z}$, (ii) $u(\overline{z}, \underline{z}, y) = 0$, and (iii) $u(\underline{z}, \overline{z}, y) = 0$, then $\overline{z}^*(y) \geq \overline{z} \geq \underline{z} \geq \underline{z}^*(y)$.

Proposition 10. *$\underline{z}^*(y) = \overline{z}^*(y)$ for all y if and only if*

$$c \geq -\sqrt{\frac{\beta}{(\alpha+\beta)(\alpha+2\beta)}}\sqrt{2\pi}$$

PROOF. Do there exists $\overline{z}, \underline{z}$ and y such that (1) $\overline{z} > \underline{z}$; (2) $u(\overline{z}, \underline{z}, y) = 0$; and (3) $u(\underline{z}, \overline{z}, y) = 0$? Thus we require

$$\begin{aligned} \overline{z} &> \underline{z} \\ \overline{z} &= c\Phi\left(\sqrt{\frac{\beta(\alpha+\beta)}{\alpha+2\beta}}\left(\underline{z} - \frac{\alpha y + \beta\overline{z}}{\alpha+\beta}\right)\right) \\ \underline{z} &= c\Phi\left(\sqrt{\frac{\beta(\alpha+\beta)}{\alpha+2\beta}}\left(\overline{z} - \frac{\alpha y + \beta\underline{z}}{\alpha+\beta}\right)\right) \end{aligned}$$

Now carrying out the change of variables

$$\begin{aligned} w &= \sqrt{\frac{\beta(\alpha+\beta)}{\alpha+2\beta}}\left(\underline{z} - \frac{\alpha y + \beta\overline{z}}{\alpha+\beta}\right) \\ \text{and } \Delta &= \sqrt{\frac{\beta(\alpha+2\beta)}{\alpha+\beta}}\left(\overline{z} - \underline{z}\right), \end{aligned}$$

the first equation becomes

$$\Delta > 0$$

and, subtracting the third equation from the second equation, we have

$$\sqrt{\frac{\alpha+\beta}{\beta(\alpha+2\beta)}}\Delta = -c\left[\Phi(w+\Delta) - \Phi(w)\right].$$

But if there exists $\Delta > 0$ and w satisfying the observe equation, then we can clearly choose $\overline{z}, \underline{z}$ and y so that (1), (2) and (3) above are satisfied. Now observe that

$$\Phi(w+\Delta) - \Phi(w) < \frac{\Delta}{\sqrt{2\pi}}.$$

So a necessary condition to solve the above equation is that

$$\sqrt{\frac{\alpha+\beta}{\beta\left(\alpha+2\beta\right)}}\sqrt{2\pi}<\left|c\right|.$$

But if this condition holds, then setting $w=0$,

$$\sqrt{\frac{\alpha+\beta}{\beta\left(\alpha+2\beta\right)}}\Delta<-c\left[\Phi\left(w+\Delta\right)-\Phi\left(w\right)\right]$$

for sufficiently small $\Delta>0$,

$$\sqrt{\frac{\alpha+\beta}{\beta\left(\alpha+2\beta\right)}}\Delta>-c\left[\Phi\left(w+\Delta\right)-\Phi\left(w\right)\right]$$

for sufficently large $\Delta>0$, so by continuity there exists $\Delta>0$ solving

$$\sqrt{\frac{\alpha+\beta}{\beta\left(\alpha+2\beta\right)}}\Delta=-c\left[\Phi\left(w+\Delta\right)-\Phi\left(w\right)\right]$$

Thus we have:

Corollary 11. *The game is dominance solvable (for all y) if and only if*

$$c\geq-\sqrt{2\pi\left(\frac{\alpha+\beta}{\beta\left(\alpha+2\beta\right)}\right)}.$$

6. Common Values

The above analysis all concerned a "private value global game". Consider exactly the same game, except that the payoff to action 1 is θ instead of x_i. This is the game first studied by Carlsson and van Damme and in much of the applied literature. The corresponding dominance solvability condition for this game can (by similar methods) be shown to be:

$$-\sqrt{2\pi\left(\frac{\beta}{\left(\alpha+\beta\right)\left(\alpha+2\beta\right)}\right)}\leq c\leq\sqrt{2\pi\left(\frac{\beta\left(\alpha+2\beta\right)}{\alpha^{2}\left(\alpha+\beta\right)}\right)}.$$

But now as $\alpha\rightarrow0$, we have

$$-\sqrt{\frac{\pi}{\beta}}\leq c\leq\infty;$$

but as $\alpha\rightarrow\infty$, the condition is never satisfied for any $c\neq0$.

References

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