Online Appendix to Priority Design in Centralized Matching Markets by Çelebi and Flynn

B. Additional Technical Results

In this Appendix we collect technical results that are necessary for the proofs of our main results but that have been omitted from the main text for clarity and brevity.

B.1. Construction of the Allocation

In this section, we establish that we can characterize the allocation of students to schools under a coarsening Ξ as a conditional probability mass function $g_\Xi(c|\theta)$ that assigns each type θ a probability of being assigned to each school c. This allocation will serve as the object over which the planner has preferences and its construction and properties are important for the analysis.

Under a coarsening Ξ, a school prioritizes student i to student j if i is in a higher indifference class than j or i and j are in the same indifference class and i has higher tie-breaker. Hence the priorities of school c become lexicographic with order (p, τ), where $p = \Xi(s)$ for some score s. By Definition 1, any admissible coarsening map partitions students into a finite number of partitions or retains a strict priority structure. Moreover, there is no aggregate uncertainty in the economy: the same measure of students of each type is assigned to each school with probability one. This is stated rigorously in Lemma 1, which also establishes that the coarsened economy after tie-breaking inherits the full support condition that we assumed for the initial economy.

Lemma 1. *Let Ξ be a coarsening and $\tilde{F}_\Xi^{\tau^*}$ the distribution of students corresponding to a strict ordinal economy for a given realized distribution of tiebreakers.*

1. *$\tilde{F}_\Xi^{\tau^*} = \tilde{F}_\Xi^{\tau}$ almost surely where $\tilde{F}_\Xi^{\tau}$ is given by $\tilde{f}_\Xi^{\tau}(\tilde{\theta}_\Xi, \tau) = \tilde{f}_\Xi(\tilde{\theta}_\Xi)$ for all $\tilde{\theta}_\Xi \in \tilde{\Theta}_\Xi$, $\tau \in [0,1]$. Moreover, $\tilde{F}_\Xi^{\tau}$ has full support.*
2. *In the coarsened ordinal economy with tie-breakers $\tilde{\Omega}_\Xi^{\tau} = (\tilde{F}_\Xi^{\tau}, Q, \tilde{\Theta}_\Xi^{\tau})$, for any student of any induced type $\tilde{\theta}_\Xi = (\succ_{\tilde{\theta}_\Xi}, s_c^{\tilde{\theta}_\Xi}) \in \tilde{\Theta}_\Xi$, there is a unique mapping $\tilde{\mu}_{\tilde{\theta}_\Xi} : [0,1] \to \mathcal{C}$ that determines the assignment of that student as a function of her realized tie-breaker.*

Proof. Part 1: Each induced type, $\tilde{\theta}_\Xi = (\succ_{\tilde{\theta}_\Xi}, s_c^{\tilde{\theta}_\Xi})$, draws a random number from $U[0,1]$, by the law of large numbers, for any type $\tilde{\theta}_\Xi$, the distribution of realized tie-breakers $\tilde{F}_\Xi^{\tau^*}(\tau|\tilde{\theta}_\Xi)$ is

$U[0,1]$ almost surely, which gives a unique strict economy type space almost surely.[48] As the interim economy (with induced types) before tie-breaking $\tilde{\Theta}$ has full support by Assumption 1, and $\tilde{f}^{\tau}_{\Xi}(\tilde{\theta}_{\Xi},\tau) = \tilde{f}_{\Xi}(\tilde{\theta}_{\Xi})$, it follows that $(\tilde{F}^{\tau}_{\Xi}, Q, \tilde{\Theta}^{\tau}_{\Xi})$ has full support.

Part 2: As $\tilde{\Omega}^{\tau}_{\Xi}$ has full support, it has a unique stable matching (see Theorem 1 in Azevedo and Leshno (2016)), denoted by $\tilde{\mu}$. Thus for any type $\tilde{\theta}_{\Xi}$ its assigned school is determined directly as a function of its tie-breaking number τ, $\mu(\tilde{\theta}_{\Xi},\tau)$. Now define $\tilde{\mu}_{\tilde{\theta}_{\Xi}} : [0,1] \to \mathcal{C}$ so that $\tilde{\mu}_{\tilde{\theta}_{\Xi}}(\tau) = \mu(\tilde{\theta}_{\Xi},\tau)$. □

Using this fact, under any coarsening Ξ one can construct the *allocation* $g_{\Xi} : \Theta \times \mathcal{C} \to [0,1]$, with the probability that type θ is assigned to school c given by $g_{\Xi}(c|\theta)$. This is stated formally in Lemma 2.

Lemma 2. *For any coarsening Ξ the probability that any student of type $\theta \in \Theta$ is assigned to a school $c \in \mathcal{C}$ is well-defined and can be represented by a conditional probability mass function $g_{\Xi}(c|\theta)$, which is given by:*

$$g_{\Xi}(c|\theta) = \int_0^1 \mathbb{I}\left[\tilde{\mu}_{\tilde{\theta}_{\Xi}(\theta)}(\tau) = c\right] d\tau \tag{78}$$

where $\tilde{\theta}_{\Xi}(\theta)$ is the induced ordinal type of type $\theta \in \Theta$ under the coarsening Ξ, and $\tilde{\mu}_{\tilde{\theta}_{\Xi}}$ is the constructed mapping from Lemma 1 for the coarsened ordinal economy with tie-breakers $\tilde{\Omega}^{\tau}_{\Xi}$ corresponding to Ξ.

Proof. As a result of Lemma 1, in the coarsened ordinal economy with tie-breakers $\tilde{\Omega}^{\tau}_{\Xi}$ corresponding to Ξ, for any ordinal type $\tilde{\theta}_{\Xi} \in \tilde{\Theta}_{\Xi}$ we know that their assigned school is uniquely determined by their tie-breaker τ according to $\tilde{\mu}_{\tilde{\theta}_{\Xi}}(\tau)$. The probability that type $\tilde{\theta}_{\Xi}$ is matched to a school c is then:

$$\mu_{\Xi}(c|\tilde{\theta}_{\Xi}) = \int_0^1 \mathbb{I}\left[\tilde{\mu}_{\tilde{\theta}_{\Xi}}(\tau) = c\right] d\tau \tag{79}$$

Moreover, given a type $\theta = (u^{\theta}, s^{\theta})$, we can deduce their induced type under a coarsening Ξ, or $\tilde{\theta}_{\Xi}(\theta) = (\succ^{\tilde{\theta}_{\Xi}(\theta)}, s^{\tilde{\theta}_{\Xi}(\theta)})$ by taking the ordinal representation of their cardinal utility and

[48] This unique economy can be constructed by the following method: for any school, let the number of equivalence classes that the school has be k and enumerate them in increasing priority, so the i'th class has measure $m(i)$. Divide the interval $[0,1]$ to k ordered sub intervals (where the intervals are $[a_0,a_1), [a_1,a_2), \ldots, [a_{k-1},a_k]$ with $a_0 = 0$ and $a_k = 1$) with each having measure $m(i)$ and score distribution $U[a_i, a_{i+1}]$ (in other words, with density $m(i)/(a_i - a_{i-1})$). In the unique economy after tie-breaking, any student who is in priority class i has a uniform probability of having a priority of any number in the i'th sub interval. So long as the initial economy has full support, the economy after coarsening and tie-breakers also has full support.

coarsening their priority according to Ξ. Thus, for a type θ, the probability that they are assigned to school c is simply $\mu_\Xi(c|\tilde{\theta}_\Xi(\theta))$. Consequently, we can define:

$$g_\Xi(c|\theta) = \mu_\Xi(c|\tilde{\theta}_\Xi(\theta)) \tag{80}$$

or, as claimed in the Lemma:

$$g_\Xi(c|\theta) = \int_0^1 \mathbb{I}\left[\tilde{\mu}_{\tilde{\theta}_\Xi(\theta)}(\tau) = c\right] d\tau \tag{81}$$

If the coarsening map is the identity function, then there is no need for tie breaking and the economy has full support, so $\tilde{\mu}(\tilde{\theta}, \tau) = c$ is still unique. As a result, an analogous argument yields the result. □

This construction simply takes the function $\tilde{\mu}$ for any student from Lemma 1 and averages over the distribution of tie-breakers to compute the assignment probabilities.

B.2. Continuity of Allocations

To establish the existence of an optimum, we show that allocations are continuous in the cutoff vectors corresponding to coarsenings. We first use Theorem 1 to represent coarsenings as vectors $v \in \mathcal{V}$. We refer to $g_v : \Theta \times \mathcal{C} \to [0, 1]$ as the corresponding allocation. As highlighted in Footnote 25, the set of potential allocations $\mathcal{G}$ is a subset of the space $L^1(\Theta \times \mathcal{C})$ of functions measurable in the measure space $(\Theta \times \mathcal{C}, \Sigma, \check{F})$, where Σ is is the Borel σ-algebra generated by the product topology on $\Theta \times \mathcal{C}$ and $\check{F}$ extends the domain of the measure F from Θ to $\Theta \times \mathcal{C}$ by stacking $|\mathcal{C}|$ copies of F. We now prove that g_v is continuous in v in the sense that for any sequence $\{v_n\}_{n\in\mathbb{N}} \subset \mathcal{V}$ such that $v_n \to v$, we have that $g_{v_n} \to g_v$ in the L^1-norm.

Lemma 3. *g_v is continuous in v in the L^1-norm.*

Proof. As shown in Lemma 2, we may represent:

$$g_v(c|\theta) = \int_0^1 \mathbb{I}\left[\tilde{\mu}_{\tilde{\theta}_v(\theta)}(\tau) = c\right] d\tau \tag{82}$$

Fix any sequence $\{v_n\}_{n\in\mathbb{N}} \subset \mathcal{V}$ such that $v_n \to v$. If $g_{v_n}(c|\theta) \to g_v(c|\theta)$ for any such sequence for $\check{F}$-almost all (c, θ), then g_v is continuous in v in the L^1-norm. By the dominated convergence theorem, this is equivalent to showing the following for $\check{F}$-almost all (c, θ) (as

$L^1(\Theta \times \mathcal{C})$ is complete):

$$0 = \lim_{n\to\infty} \left[g_{(v_n)}(c|\theta) - g_v(c|\theta)\right] = \lim_{n\to\infty} \int_0^1 \left[\mathbb{I}\left[\tilde{\mu}_{\tilde{\theta}_{v_n}(\theta)}(\tau) = c\right] - \mathbb{I}\left[\tilde{\mu}_{\tilde{\theta}_v(\theta)}(\tau) = c\right]\right] d\tau \quad (83)$$

Now fix (c, θ), to show the above, once again by the dominated convergence theorem, it is sufficient to show that:

$$\lim_{n\to\infty} \left[\mathbb{I}\left[\tilde{\mu}_{\tilde{\theta}_{v_n}(\theta)}(\tau) = c\right] - \mathbb{I}\left[\tilde{\mu}_{\tilde{\theta}_v(\theta)}(\tau) = c\right]\right] = 0, \text{ for almost all } \tau \in [0, 1] \quad (84)$$

Now fix a $\tau \in [0, 1]$ and consider the sequence of economies with the priority structures induced by v_n. As $v_n \to v$, the priority structures converge pointwise. By Lemma 1, the full support assumption in the ex-ante economy implies that interim economy has full support for all n, hence the interim economy has a unique stable matching for all v_n. Thus, by Theorem 2.3 of Azevedo and Leshno (2016), we know that the stable matching is continuous in the induced economy with coarsening vector $v \in \mathcal{V}$ within the set of economies $\mathcal{O}$. Thus, for each $\tilde{\theta}_v(\theta)$ and almost all τ, there must exist an N such that for all $n > N$, we have that $\mathbb{I}\left[\tilde{\mu}_{\tilde{\theta}_{v_n}(\theta)}(\tau) = c\right] = \mathbb{I}\left[\tilde{\mu}_{\tilde{\theta}_v(\theta)}(\tau) = c\right]$ for almost all $\theta \in \Theta$ and $\tau \in [0, 1]$. Thus, we have shown that Equation 83 holds for $\check{F}$-almost all (c, θ) and completed the proof. □

C. Priority Design with Aggregate Uncertainty

In the main text we assumed that the mechanism designer knows the distribution of students' types. We start with a simple discrete example that shows this is necessary for the optimality of trinary coarsenings.

Example 3. *There are six students, $\Theta = \{\theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \theta_6\}$ and one school with capacity 2, $\mathcal{C} = \{c\}$ and $Q_c = 2$. The scores of the students are given by their index number, i.e. $s_c^{\theta_i} = i$. The designer prefers to admit θ_6, θ_4 and to a lesser extent, θ_2 and has the following utility function:*

$$Z(\mu) = \sum_{i \in \{4,6\}} \mathbb{I}\{\theta_i \in \mu(c)\} + \frac{1}{2}\mathbb{I}\{\theta_2 \in \mu(c)\} \tag{85}$$

There are two states denoted by γ and γ' and both states have strictly positive probability. The ordinal preferences of student θ_i in state $j \in \{\gamma, \gamma'\}$ is denoted by $\succ_j^i$. The preferences are given by:

- $\succ_j^i = c, \theta_i$ *for* $i \in \{1, 2, 3, 6\}$ *and* $j = \{\gamma, \gamma'\}$.
- $\succ_\gamma^i = c, \theta_i$ *and* $\succ_{\gamma'}^i = \theta_i, c$ *for* $i \in \{4, 5\}$.

Under state γ, the unique optimal trinary coarsening is given by:

$$\Xi_\gamma(i) = \begin{cases} 3, i = 6 \\ 2, i \in \{4, 5\} \\ 1, i \in \{1, 2, 3\} \end{cases} \tag{86}$$

This coarsening admits 6 with probability 1 and maximizes the probability of admission for 4. Note that any other trinary coarsening will result in a strictly lower utility for the designer. Under state γ', Ξ_γ is not optimal as it admits 6 with probability 1 but admits 2 with probability 1/3. As Ξ_γ is the unique trinary coarsening that is optimal under γ, no trinary coarsening can attain the optimum under both realizations of uncertainty. Moreover, the following is an optimal trinary coarsening under γ':

$$\Xi_{\gamma'}(i) = \begin{cases} 3, i = 6 \\ 2, i \in \{2, 3, 4, 5\} \\ 1, i \in \{1\} \end{cases} \tag{87}$$

$\Xi\gamma'$ admits 6 with probability 1 and admits 2 with probability 1/2, and thus performs strictly better than Ξ_γ. However, there exists a coarsening with 4 indifference classes that

attains the optimum under both states:

$$\Xi^*(i) = \begin{cases} 4, i = 6 \\ 3, i \in \{4,5\} \\ 2, i \in \{2,3\} \\ 1, i \in \{1\} \end{cases} \tag{88}$$

Ξ^* replicates Ξ_γ under γ and $\Xi_{\gamma'}$ under γ', and thus attains the optimal allocation in both cases. This example shows that trinary coarsenings are not without loss of optimality under uncertainty.

We now depart from that assumption by considering a mechanism designer who believes finitely many distributions are possible and has a prior belief over those probability distributions. Formally, let $\mathcal{F}$ denote the set of finitely many possible distributions with a particular element f. Let $p(f)$ denote the probability measure over this finite set and g_f denote the allocation under f. In this environment, we introduce a new assumption (in place of Assumption 3) on the objective function of the planner that they are a subjective expected utility maximizer with respect to their prior as to the market they face:

Assumption 4. *Under uncertainty, the social planner has subjective V.N-M utility Z with a Bernoulli utility function $z : \mathcal{G} \to \mathbb{R}$ that is continuous in g_f.*

Notice that in any realization of the student distribution, the same coarsening rule must be applied. Hence schools must use the same priority structure across realizations of different market structures. This draws a parallel between uncertainty and our analysis of homogeneous coarsening in Online Appendix D: under homogeneity multiple schools must use the same coarsening rule; under uncertainty, the same rule must be used across multiple states.

First, and analogously to the previous analysis, we provide a result on the maximum number of partitions needed to achieve an optimal coarsening under uncertainty. This is stated as Proposition 6:

Proposition 6. *There exists an optimal coarsening Ξ with at most $2|\mathcal{F}|$ cutoffs at every school.*

Proof. Fix an arbitrary coarsening Ξ. Divide the schools to two subsets: $c \in \tilde{\mathcal{C}}$ if Ξ_c is finite and $c \in \hat{\mathcal{C}}$ if Ξ_c is the identity function. Take $c \in \tilde{\mathcal{C}}$, the coarsening Ξ takes any student with score $s_c^{\tilde{\theta}}$ to an equivalence class, hence $\Xi_c(s_c^{\tilde{\theta}}) \in \{P_1^c, P_2^c, \ldots, P_n^c\}$ with $P_1^c < P_2^c, \ldots, < P_n^c$. By Theorem 1, for any $f_i \in \mathcal{F}$, there can be at most one lottery region for school c and denote that region by $P_{f_i}^c$ and enumerate f_i in a way that $P_{f_i}^c$ is increasing in i. Let $P_n < P_{n+1} <$

$\ldots < P_m$ denote the indifference classes that are between two consecutive lottery regions $P^c_{f_i}$ and $P^c_{f_{i+1}}$. Define Ξ' by merging $P_n < P_{n+1} < \ldots < P_m$.[49] Note that the allocation under Ξ and Ξ' are the same under any f_i.[50] Repeating this for all schools $c \in \tilde{\mathcal{C}}$ and lottery regions results in a coarsening rule Ξ' that induces same allocation as the arbitrary coarsening Ξ. Moreover, there is only 1 partition between any two lottery regions for a schools in $\tilde{\mathcal{C}}$, hence there are at most $2|\mathcal{F}|$ cut-offs and $2|\mathcal{F}|+1$ partitions per schools in $\tilde{\mathcal{C}}$ under Ξ'.

Next, take $c \in \hat{\mathcal{C}}$, so Ξ_c is the identity function. As the economy has full support, there is a unique stable matching under any $f_i \in \mathcal{F}$. Let μ_i denote this stable matching. For any school c and distribution f_i, let t^i_c denote the threshold that the students must clear in order to gain admission to that school under f_i. Formally,

$$t^i_c = \min\{s^{\tilde{\theta}}_c : \mu_i(\tilde{\theta}) = c\} \tag{89}$$

Let $T_c = \cup_{i \leq |\mathcal{F}|}\{t^i_c\}$. Next, define Ξ' in the following way:

$$\Xi'_c(s^{\tilde{\theta}}_c) = \max\{t^i_c \in T_c : s^{\tilde{\theta}}_c \geq t^i_c\} \tag{90}$$

Note that from the stability of μ_i under the distribution f_i and the definition of matching (in particular, property (iv) from Footnote 23)) there cannot be a student k such that $c' \succ_k \mu_i(k)$ and $s^k_{c'} \geq t^i_{c'}$. To see that μ_i is stable under Ξ', assume that student pair l, j (with scores $s^l_{c'}$ and $s^j_{c'}$) blocks it in school c'. Then we have $c' \succ_l \mu_i(l)$, $\mu_i(j) = c'$ and $\Xi'_{c'}(l) \geq \Xi'_{c'}(j)$. As $\mu_i(j) = c'$, we have $\Xi'_{c'}(j) \geq t^i_c$. As $\Xi'_{c'}(l) \geq \Xi'_{c'}(j)$ and $\Xi'_{c'}(j) \geq t^i_c$, we have that $s^l_{c'} \geq t^i_{c'}$, which contradicts the stability of μ_i under Ξ as $c' \succ_l \mu_i(l)$. By the same argument as in the first case, it follows that $g_\Xi = g_{\Xi'}$. Then non-emptiness follows from the same argument in Theorem 2 under Assumption 4. □

This result generalizes Theorems 1 and 2 to the case with aggregate uncertainty. Instead of an optimum being attainable with at most 2 cutoffs at each school, now up to $2|\mathcal{F}|$ cutoffs are required: 2 for each state of the world. This demonstrates how the presence of aggregate uncertainty can substantially complicate the problem of priority design.

To understand the normative implications of aggregate uncertainty, we characterize when the presence of uncertainty induces a welfare loss to the planner. We denote the lottery region cut-offs of school c under distribution f by $[\underline{P^c_f}, \overline{P^c_f}]$. We can define a no-overlap condition in the case with uncertainty:

Definition 3. *For any problem, the set of distributions* $\{f_1, f_2, \ldots, f_n\} = \mathcal{F}$ *satisfies no*

[49] This can be done by setting $\Xi'(P_n) = \ldots = \Xi'(P_m) = \Xi(P_n)$.

[50] The argument for this is exactly same as the one made in the proof of Theorem 1.

overlap across distributions if for any f_i, there exists coarsenings:

$$\Xi_i = [\underline{P_i^1}, \ldots, \underline{P_i^{|\mathcal{C}|}} | \overline{P_i^1}, \ldots, \overline{P_i^{|\mathcal{C}|}}] \tag{91}$$

such that Ξ_i is optimal under f_i and for all c, k and j, $[\underline{P_k^c}, \overline{P_k^c}] \cap [\underline{P_j^c}, \overline{P_j^c}] = \emptyset$ or $[\underline{P_k^c}, \overline{P_k^c}] = [\underline{P_j^c}, \overline{P_j^c}]$.

We can now prove that there is a loss from uncertainty if and only if the no-overlap condition is not satisfied. This is stated as Proposition 7:

Proposition 7. *There is no welfare loss from uncertainty if and only if $\mathcal{F}$ satisfies no overlap across distributions.*

Proof. Assume that the problem satisfies no overlap across distributions and fix optimal trinary coarsenings Ξ_{f_i} that satisfy no overlap across distributions. Let $|\mathcal{F}| = n$. For any school c, Ξ_{f_i} has two cut-offs v_{f_i}: denote them by $L^c_{f_i}$ and $U^c_{f_i}$ with $L^c_{f_i} < U^c_{f_i}$. As the trinary coarsening satisfies no overlap across distributions, we can enumerate them as $L^c_{f_1} < U^c_{f_1} < \ldots < L^c_{f_n} < U^c_{f_n}$. Define a new set of cut-offs $w^c = L^c_{f_1}, U^c_{f_1}, \ldots, L^c_{f_n}, U^c_{f_n}$. Define a new coarsening v' by setting the cut-offs as w^c for all c. Notice that for any realization of f_i, the coarsenings v_{f_i} and v' induce the same allocation.[51] Hence the coarsening v' induces the same allocation as v_{f_i} for any f_i. As each v_{f_i} maximizes the mechanism designer's utility under f_i, the maximum utility can be replicated by using v' for any f_i and there is no loss from uncertainty.

For the converse result, fix an arbitrary coarsening w and suppose that the problem does not satisfy the no overlap across distributions condition. Consider the set of coarsenings that uses $v_f = w$ for all $f \in \mathcal{F}$. Notice that this set trivially has no overlap. Clearly, w and $\{v_f\}$ attain the same utility value for the mechanism designer. As the problem does not satisfy the no overlap across distributions assumption, there must be a set of coarsenings $\{v'_f\}$ that attains a strictly higher expected utility than $\{v_f\}$ and w. As w is arbitrary, there is welfare loss from uncertainty regardless of its choice. □

As no overlap is a very strenuous condition, the above result clarifies that generally uncertainty will lead to welfare losses for the planner. It moreover makes clear that this loss in welfare stems from the inability of the planner to preserve the lottery classes from the optimal *ex post* designs in each state.

[51] To see why this is true, notice that v_{f_i} can be obtained from v' by merging indifference classes that are above and below each lottery class for any school, using the construction in the proof of Theorem 1.

D. Priority Design with Homogeneous Coarsening

In many applications, the assumption that the policymaker can set a school-specific coarsening may be unrealistic. For example, in cases where the underlying score represents exam scores or some measure of student achievement, it is possible that the schools are constrained to use the same grading criteria. Hence, in this section, we examine optimal coarsenings when the coarsening rule is constrained to be the same across all schools. Formally, we find the optimal coarsening in the set $\mathcal{H}$ of homogeneous coarsenings:

Definition 4. *A homogeneous coarsening rule Ξ is a coarsening rule such that*

$$\Xi(s) = (\Xi_1(s_1), \ldots, \Xi_1(s_n)) \tag{92}$$

Let the set of all homogeneous coarsening maps be $\mathcal{B}_h$. When only homogeneous coarsenings are allowed, the planner faces the general problem that satisfies Assumption 3:

$$\mathcal{B}_h^* = \arg\max_{\Xi \in \mathcal{B}_h} Z(g_\Xi) \tag{93}$$

where g_Ξ is the allocation induced by Ξ. We first prove a proposition which shows that appropriately revised versions of Theorems 1 and 2 continue to hold in this environment:

Proposition 8. *$\mathcal{B}_h^*$ is non-empty. There exists a homogeneous coarsening with $2|\mathcal{C}|$ cutoffs such that $\Xi \in \mathcal{B}_h^*$.*

Proof. Fix an arbitrary finite homogeneous coarsening Ξ. The coarsening Ξ takes any student with score $s_c^{\tilde{\theta}}$ to a partition, hence $\Xi_c(s_c^{\tilde{\theta}}) \in \{P_1^c, P_2^c, \ldots, P_n^c\}$ with $P_1^c < P_2^c < \ldots < P_n^c$ for all $\tilde{\theta} \in \tilde{\Theta}$. By Theorem 1, for any $c \in \tilde{\mathcal{C}}$, there can be at most one lottery class.

Without loss of generality, assume the lottery regions are enumerated as $P_{x(1)} < P_{x(2)} < \ldots < P_{x(|\mathcal{C}|)}$. Let $P_n < P_{n+1} < \ldots < P_m$ denote the indifference classes that are between two consecutive lottery regions $P_{x(z)}$ and $P_{x(z+1)}$. Define Ξ' by merging $P_n < P_{n+1} < \ldots < P_m$.[52] Note that $g_\Xi(c|\theta) = g_\Xi(c|\theta)$ for all $\theta \in \Theta$.[53] This process can be repeated for all consecutive lottery regions without changing the allocation. This results in a coarsening rule Ξ' where there is only 1 partition between any two lottery regions for all $c \in \mathcal{C}$. Hence the total number of partitions are at most $2|\mathcal{C}| + 1$ and cut-offs are $2|\mathcal{C}|$ for all $c \in \mathcal{C}$.

Next, assume the homogeneous coarsening is the identity function, in other words, there is no coarsening.[54] As the economy has full support, there is a unique stable matching. Let

[52]This can be done by setting $\Xi'_c(P_n) = \ldots = \Xi'_c(P_m) = \Xi_c(P_n)$.

[53]The argument for this is exactly same as the one made in the proof of Theorem 1.

[54]Note that the case where some schools have a finite coarsening while others have no coarsening is ruled out by assuming the coarsening must be homogeneous.

μ denote this stable matching. For any school c and distribution, let t_c denote the threshold that the students must clear in order to gain admission to that school. Formally,

$$t_c = \min\{s_c^{\tilde{\theta}} : \mu(\tilde{\theta}) = c\} \tag{94}$$

Let $T = \cup_{c \in \mathcal{C}} t_c$. Next, define Ξ' in the following way:

$$\Xi'_c(s_c^{\tilde{\theta}}) = \max\{t \in T : s_c^{\tilde{\theta}} \geq t\} \tag{95}$$

Note that from the stability of μ and the definition of matching (in particular, property (iv) from Footnote 23)) there cannot be a student k such that $c' \succ_k \mu_i(k)$ and $s_{c'}^k \geq t_{c'}$. To see that μ is stable under Ξ', assume that student pair l, j (with scores $s_{c'}^l$ and $s_{c'}^j$) blocks it in school c'. Then we have $c' \succ_l \mu(l)$, $\mu(j) = c'$ and $\Xi'_{c'}(l) \geq \Xi'_{c'}(j)$. As $\mu(j) = c'$, we have $\Xi'_{c'}(j) \geq t_c^i$. As $\Xi'_{c'}(l) \geq \Xi'_{c'}(j)$ and $\Xi'_{c'}(j) \geq t_c^i$, we have that $s_{c'}^l \geq t_{c'}^i$, which contradicts the stability of μ under Ξ as $c' \succ_l \mu(l)$. By the same argument as in the first case, it follows that $g_\Xi = g_{\Xi'}$. This shows that a homogeneous coarsening Ξ' with $|\mathcal{C}|$ cut-offs attains the same allocation as no coarsening.

Then any homogeneous coarsening Ξ can be represented by a vector of cut-offs $v = P_1, P_2, \ldots, P_{|2\mathcal{C}|}$. The existence of an optimum then follows from the same argument as in Theorem 2. □

Like Theorems 1 and 2, Proposition 8 reduces a possibly infinite dimensional problem to a $2|\mathcal{C}|$-dimensional problem and allows us to compute optimal homogeneous coarsenings.

We next study the welfare implications of imposing homogeneity in coarsenings. We show that there is a loss from imposing homogeneity whenever in the optimal trinary coarsening the lottery regions of two schools intersect without being equal. To this end, we define a subclass of problems:

Definition 5. *A problem satisfies no overlap across schools if there exists an optimal coarsening such that for all lottery regions P^c and $P^{c'}$ either $[\overline{P^c}, \underline{P^c}] \cap [\overline{P^{c'}}, \underline{P^{c'}}] = \emptyset$ or $[\overline{P^c}, \underline{P^c}] = [\overline{P^{c'}}, \underline{P^{c'}}]$.*

We can now characterize exactly the case when imposing homogeneity results in a welfare loss. This is stated formally in Proposition 9:

Proposition 9. *There is no loss from imposing homogeneity if and only if the problem satisfies no overlap across schools.*

Proof. In this environment we can define an object similar to the vector we defined earlier to represent the coarsenings as a vector when there are n_i cut-offs for each school i:

$$\begin{aligned} w &= [P_1^1, P_2^1, \ldots, P_{n_1}^1 | P_1^2, \ldots, P_{n_2}^2 | \ldots | P_1^{|\mathcal{C}|}, \ldots, P_{n_{|\mathcal{C}|}}^{|\mathcal{C}|}] \\ \text{s.t.} \quad & w \in \mathcal{W} = \{w \in [0,1]^{\sum_i n_i}, P_j^c > P_l^c, \forall c, j > l\} \end{aligned} \tag{96}$$

Let v be an optimal (i.e. non homogeneous) trinary coarsening that satisfies no overlap across schools. As above, v is of the form $v = \{v_1^1, v_2^1 | \ldots | v_2^{|\mathcal{C}|}, v_2^{|\mathcal{C}|}\}$ as v is a trinary coarsening. Enumerate the $2|\mathcal{C}|$ elements of v in increasing order and let z be the vector obtained by doing this, hence $z = \{z_1, \ldots, z_{2|\mathcal{C}|}\}$ and it is increasing. Therefore it is a well-defined set of coarsening cut-offs. Now, form the alternative coarsening v' by using the cut-offs in z for all schools:

$$v' = [z_1, \ldots, z_{2|\mathcal{C}|} | \ldots | z_1, \ldots, z_{2|\mathcal{C}|}] \tag{97}$$

v' is a homogeneous coarsening as it uses the same cut-offs. Furthermore, v and v' induce the same allocation.[55] Hence there is a homogeneous coarsening v' that induces the same allocation as the optimal trinary coarsening v and there is no loss from imposing homogeneity.

For the converse result, fix an arbitrary coarsening w and suppose that the problem does not satisfy the no overlap across schools condition. Consider the set of coarsenings that uses $v_c = w$ for all $c \in \mathcal{C}$. Notice that this set trivially has no overlap. Clearly, w and $\{v_c\}$ attain the same utility value for the mechanism designer. As the problem does not satisfy the no overlap across schools assumption, there must be a set of coarsenings $\{v_c'\}$ that attains a strictly higher utility than $\{v_c\}$ and w. As w is arbitrary, there is welfare loss from homogeneity regardless of its choice. □

As no-overlap is an extremely strenuous condition, Proposition 9 would seem to indicate that in practical contexts, there will very likely be a welfare loss from imposing homogeneity. As a result, rationalizing the observed homogeneity of many systems is challenging in this environment. Indeed, it may be the case that one has to appeal to concerns for simplicity in order to understand observed homogeneity.

[55] To see why this is true, notice that v can be obtained from v' by merging indifference classes that are above and below each lottery class for any school, using the construction in the proof of Theorem 1.

E. Solving the Dynamic Model of Housing Assignment

In this Appendix, we provide the prerequisites for determining the structure of the optimal priority design by first solving the dynamic matching model. Relative to the static matching models we have so far considered, this features two complications. First, we have to determine the agents' optimal stopping rule for when to accept a given house as a function of their individual characteristics and the allocation. Second, we have to find the steady-state distribution of agents who are unmatched (which is endogenous to the stopping rule) to find the steady-state allocation.

We first fix an allocation and solve for the agent's optimal stopping policy. Denoting by $P(s, v)$ the steady-state probability that an agent with income s is matched to a house in any given period given a priority design v, we have that the value function of an unmatched agent is given by:

$$\begin{aligned} V(\kappa, P(s,v)) = \kappa + \beta\delta P(s,v)\mathbb{E}_{\tilde{v}}\left[\max\left\{\frac{\tilde{v}}{1-\beta\delta}, V(\kappa, P(s,v))\right\}\right] \\ + \beta\delta(1 - P(s,v))V(\kappa, P(s,v)) \end{aligned} \tag{98}$$

where the first term is simply the agents value of going unmatched this period: their outside option. The second term is the discounted expected value of being matched to a house, weighted by the chance that that the agent is matched. The final term simply says that if the agent goes unmatched, they receive their discounted value of being unmatched.

The agent's optimal stopping rule is a threshold strategy indexed by $\bar{v}(\kappa, P(s, v))$ such that agents accept any house with $v \geq \bar{v}(\kappa, P(s, v))$ and reject otherwise. It follows that this reservation value is the unique solution to the following equation:

$$\begin{aligned} \bar{v}(\kappa, P(s,v)) &= (1-\beta\delta)\kappa \\ &+ \beta\delta P(s,v)\left[\bar{v}(\kappa, P(s,v))\Lambda(\bar{v}(\kappa, P(s,v))) + \int_{\bar{v}(\kappa,P(s,v))}^{v_{\max}} \tilde{v}\,d\Lambda(\tilde{v})\right] \\ &+ \beta\delta(1-P(s,v))\bar{v}(\kappa, P(s,v)) \end{aligned} \tag{99}$$

In general this cannot be solved in closed form.[56] However, it is simple to see that $\bar{v}(\kappa, P(s, v))$ features the following comparative statics: it is increasing in the agent's outside option κ; it is increasing in the agent's patience β; and it is increasing in the probability that an agent is matched to a house in any given period $P(s, v)$.

Having solved for the agent's optimal stopping rule as a function of the assignment

[56]One exception to this is the case where $\tilde{v} \sim U[v_{\min}, v_{\max}]$. In this case, Equation 99 becomes quadratic in $\bar{v}(\kappa, P(s, v))$ and can be solved analytically.

probability, it remains to solve for the assignment probability as a function of a given stopping rule. To do this, one first needs to find the steady-state distribution of unmatched agents and the steady state measure of occupied housing. To this end, define the steady-state measure of unmatched agents as the joint density $k(\kappa, s; v)$ with marginal cumulative measure over incomes given by $K_s(s; v)$ and let $M(v)$ be the steady-state measure of matched agents.

Using our more general analysis, we can once again greatly simplify the problem: Theorem 1 directly applies to any steady state of this model. We can therefore restrict to priority designs $v = (v_1, v_2)$ that feature three tiers. Given a CDF of incomes in the population in the steady state, the allocation probability for an agent with income s is given by:

$$P(s, v) = \begin{cases} 1, s \geq v_2, \\ \frac{(Q - \delta M(v)) - (1 - K_s(v_2; v))}{K_s(v_2; v) - K_s(v_1; v)}, s \in [v_1, v_2), \\ 0, s < v_1 \end{cases} \tag{100}$$

Moreover, the stationary joint distribution at the end of a period $k^l(\kappa, s; v)$ solves the following fixed point equation which balances inflows of new agents and outflows of agents due to accepting a house and death.

$$\begin{aligned} (1 - \delta) f(\kappa, s) = &(1 - \delta) k^l(\kappa, s; v) \\ &+ (1 - \Lambda(\bar{v}(\kappa, P(s, v)))) P(s, v) \left(\delta k^l(\kappa, s; v) + (1 - \delta) f(\kappa, s) \right) \end{aligned} \tag{101}$$

Thus, the stationary joint distribution of agents at the time of matching is given by:

$$k(\kappa, s; v) = \delta k^l(\kappa, s; v) + (1 - \delta) f(\kappa, s) \tag{102}$$

Finally, the stationary measure of matched agents is given by:

$$(1 - \delta) M(v) = (1 - \Lambda(\bar{v}(\kappa, P(s, v)))) P(s, v) \left(\delta k^l(\kappa, s; v) + (1 - \delta) f(\kappa, s) \right) \tag{103}$$

Having now solved the stopping rule as a function of the allocation and the allocation as a function of the stopping rule, we have characterized the matching of agents to houses in this economy. From the steady-state policy function it is moreover simple to compute welfare from the relationship:

$$V(\kappa, P(s, v)) = \frac{\bar{v}(\kappa, P(s, v))}{1 - \beta\delta} \tag{104}$$

This discussion is formalized in Proposition 10:

Proposition 10. *For any coarsening policy v', there exists a $v = (v_1, v_2)$ that is outcome equivalent. Moreover, under this v, welfare is given by:*

$$Z(v) = \int_{\mathcal{U}} \int_0^1 \left(\frac{\frac{\bar{v}(\kappa, P(s,v))}{1-\beta\delta}}{1-\gamma} \right)^{1-\gamma} dF(\kappa, s) \tag{105}$$

where the reservation value policy solves Equation 99, the allocation is given by Equation 100, the stationary measure of unmatched agents of each type is given by Equations 101 and 102 and the stationary measure of matched agents is given by Equation 103.

Proof. In a steady state of the model, there is a time-invariant assignment probability as a function of the policy v and one's score s, $P(s, v)$. Given this, the agent has Bellman equation given by:

$$\begin{aligned} V(\kappa, P(s,v)) = \kappa + \beta\delta P(s,v)\mathbb{E}_{\tilde{v}} \left[\max \left\{ \frac{\tilde{v}}{1-\beta\delta}, V(\kappa, P(s,v)) \right\} \right] \\ + \beta\delta(1 - P(s,v))V(\kappa, P(s,v)) \end{aligned} \tag{106}$$

The first claim is that the optimal stopping solution, where 1 corresponds to accepting a house and 0 to rejecting is given by:

$$x^*(\kappa, \tilde{v}, P(s,v)) = \begin{cases} 1, \tilde{v} \geq \bar{v}(\kappa, P(s,v)), \\ 0, \tilde{v} < \bar{v}(\kappa, P(s,v)), \end{cases} \tag{107}$$

for some function $\bar{v}(\kappa, P(s,v))$. This is immediate: suppose an agent accepts $\tilde{v}$ and rejects $\tilde{v}' > \tilde{v}$. As $\tilde{v}' > \tilde{v}$, either accepting $\tilde{v}$ or rejecting $\tilde{v}'$ must be suboptimal. Moreover, we know that at this reservation value $\bar{v}(\kappa, P(s,v))$, the agent must be indifferent between accepting and rejecting. It follows that:

$$V(\kappa, P(s,v)) = \frac{\bar{v}(\kappa, P(s,v))}{1-\beta\delta} \tag{108}$$

Combining Equations 106 and 108, we obtain:

$$\begin{aligned} \frac{\bar{v}(\kappa, P(s,v))}{1-\beta\delta} = \kappa + \beta\delta P(s,v)\mathbb{E}_{\tilde{v}} \left[\max \left\{ \frac{\tilde{v}}{1-\beta\delta}, \frac{\bar{v}(\kappa, P(s,v))}{1-\beta\delta} \right\} \right] \\ + \beta\delta(1 - P(s,v))\frac{\bar{v}(\kappa, P(s,v))}{1-\beta\delta} \end{aligned} \tag{109}$$

Re-arranging this equation yields:

$$\begin{aligned}\bar{v}(\kappa, P(s,v)) = (1-\beta\delta)\kappa + \beta\delta P(s,v)\mathbb{E}_{\tilde{v}}\left[\max\left\{\tilde{v}, \bar{v}(\kappa, P(s,v))\right\}\right] \\ + \beta\delta(1-P(s,v))\bar{v}(\kappa, P(s,v))\end{aligned} \tag{110}$$

where:

$$\mathbb{E}_{\tilde{v}}\left[\max\left\{\tilde{v}, \bar{v}(\kappa, P(s,v))\right\}\right] = \bar{v}(\kappa, P(s,v))\Lambda(\bar{v}(\kappa, P(s,v))) + \int_{\bar{v}(\kappa,P(s,v))}^{v_{\max}} \tilde{v} d\Lambda(\tilde{v}) \tag{111}$$

yielding the claimed equation for the reservation value function:

$$\begin{aligned}\bar{v}(\kappa, P(s,v)) = (1-\beta\delta)\kappa + \beta\delta P(s,v)\left[\bar{v}(\kappa, P(s,v))\Lambda(\bar{v}(\kappa, P(s,v))) + \int_{\bar{v}(\kappa,P(s,v))}^{v_{\max}} \tilde{v} d\Lambda(\tilde{v})\right] \\ + \beta\delta(1-P(s,v))\bar{v}(\kappa, P(s,v))\end{aligned} \tag{112}$$

As was claimed in the text, this has a unique solution. To see this simply observe that the slope of the RHS of Equation 112 is given by (suppressing all constants):

$$RHS'(\bar{v}) = \beta\delta\left[1 - P(1-\Lambda(\bar{v}))\right] < 1 \tag{113}$$

while the slope of the LHS is 1, and $RHS(0) > LHS(0) = 0$. We additionally claimed in the text that the case with $\tilde{v} \sim U[v_{\min}, v_{\max}]$ has a closed form solution for Equation 112. To this end, see that we have:

$$\begin{aligned}\bar{v}(\kappa, P(s,v)) = (1-\beta\delta)\kappa + \beta\delta P(s,v)\left[\bar{v}(\kappa, P(s,v))\frac{\bar{v}(\kappa, P(s,v)) - v_{\min}}{v_{\max} - v_{\min}} + \frac{v_{\max}^2 - \bar{v}(\kappa, P(s,v))^2}{2(v_{\max} - v_{\min})}\right] \\ + \beta\delta(1-P(s,v))\bar{v}(\kappa, P(s,v))\end{aligned} \tag{114}$$

which can be rewritten as a quadratic in $\bar{v}(\kappa, P(s,v))$:

$$0 = a\bar{v}(\kappa, P(s,v))^2 + b\bar{v}(\kappa, P(s,v)) + c \tag{115}$$

where:

$$
\begin{aligned}
a &= \frac{\beta\delta P(s,v)}{2(v_{\max}-v_{\min})} \\
b &= -1+\beta\delta(1-P(s,v))-\beta\delta P(s,v)\frac{v_{\min}}{v_{\max}-v_{\min}} \\
c &= (1-\beta\delta)\kappa+\beta\delta P(s,v)\frac{v_{\max}^2}{2(v_{\max}-v_{\min})}
\end{aligned} \tag{116}
$$

yielding the standard solution of a quadratic for $\bar{v}(\kappa, P(s,v))$.

As a function of the steady-state assignment probability, we have solved the agents' optimal stopping problems. We now solve for the steady-state assignment probability as a function of the agents' optimal stopping problems. To this end, observe that the joint density of agents at the end of a period must match the inflow of agents to the outflow of agents:

$$
\begin{aligned}
\underbrace{(1-\delta)f(\kappa,s)}_{\text{New Agents}} = {} & \underbrace{(1-\delta)k^l(\kappa,s;v)}_{\text{Dead Agents}} \\
& + \underbrace{(1-\Lambda(\bar{v}(\kappa,P(s,v))))P(s,v)}_{\text{Agents Who Stop}}\underbrace{\left(\delta k^l(\kappa,s;v)+(1-\delta)f(\kappa,s)\right)}_{\text{Agents Available to Match This Period}}
\end{aligned} \tag{117}
$$

While the joint density of agents at the time of matching is given by:

$$
k(\kappa,s;v) = \delta k^l(\kappa,s;v)+(1-\delta)f(\kappa,s) \tag{118}
$$

Moreover, the steady state measure of occupied housing balances outflows from death and inflows from new matches:

$$
\underbrace{(1-\delta)M(v)}_{\text{Death of Old Matches}} = \underbrace{(1-\Lambda(\bar{v}(\kappa,P(s,v))))P(s,v)\left(\delta k^l(\kappa,s;v)+(1-\delta)f(\kappa,s)\right)}_{\text{New Matches}} \tag{119}
$$

Given these equations, each period there are $Q-\delta M(v)$ houses available. Given the policy v, agents are allocated houses tier by tier until the first tier such that not all agents can be allocated housing. Let the first such tier be l where agents are in l if $s \in [v_{l-1}, v_l)$. Clearly all agents with $s < v_{l-1}$ have zero assignment probability and all agents with $s \geq v_l$ have unit assignment probability. In the steady state there is a mass $1-K_s(v_l;v)$ of agents with priority above v_l. Moreover, there is a mass $K_s(v_l;v)-K_s(v_{l-1};v)$ who have score $s \in [v_{l-1}, v_l)$. Thus agents with scores $s \in [v_{l-1}, v_l)$ are allocated a house with probability given by:

$$p_L(v) = \frac{(Q - \delta M(v)) - (1 - K_s(v_l; v))}{K_s(v_l; v) - K_s(v_{l-1}; v)} \tag{120}$$

It follows that the allocation is given by:

$$P(s, v) = \begin{cases} 1, s \geq v_l, \\ p_L(v), s \in [v_{l-1}, v_l), \\ 0, s < v_{l-1} \end{cases} \tag{121}$$

Consider now a new coarsening policy $\hat{v} = (v_{l-1}, v_l)$. The allocations induced by $\hat{v}$ and v are equivalent. Thus all other equations describing the steady state are equivalent under the two policies. This proves the claim that for any v there exists a $v' = (v_1, v_2)$ that is outcome equivalent. Indeed, this is given by $v' = \hat{v}$. We have now proved all claims in the proposition. □

This highlights again the usefulness of Theorem 1 in providing a revealing structure to the optimal design: one can employ a tiered design, but it need contain at most three tiers.

E.1. An Example with Two Income Cutoffs

The main text uses the above result to explore the optimal design, giving sufficient conditions for there to be at least two income tiers in the optimum. Here, we additionally provide an explicit example that shows how three income tiers can be optimal when there is sufficient heterogeneity in outside options.

Example 4. *Suppose that there are three levels of outside option κ_P, κ_M and κ_R where an agent with score s (or equivalently income $1 - s$) has outside option given by:*

$$\kappa = \begin{cases} \kappa_R, s < \bar{s}_1, \\ \kappa_M, s \in [\bar{s}_1, \bar{s}_2), \\ \kappa_P, s \geq \bar{s}_2. \end{cases} \tag{122}$$

where $\bar{s}_1 < \bar{s}_2$. Moreover, we suppose that outside options are such that:

$$\begin{aligned} \bar{v}(\kappa_R, 0) > \bar{v}(\kappa_M, 1) \\ \bar{v}(\kappa_M, 0) > \bar{v}(\kappa_P, 1) \end{aligned} \tag{123}$$

That is to say, an agent with a better outside option who is never assigned public housing still has higher welfare than an agent with a lower outside option when they are certain *of receiving public housing each period. Such an environment is perhaps natural when the extremely poor have much worse outside options than the moderately poor and the wealthy have very good outside options.*

For intermediate values of Q,[57] *in the $\gamma \to \infty$ limit the optimal policy is such that:*

$$v_1^* = \bar{s}_1, \quad v_2^* = \bar{s}_2 \tag{124}$$

Moreover, agents with score $e < \bar{s}_1$ have zero probability of assignment, agents with score $s \in [\bar{s}_1, \bar{s}_2)$ have intermediate probability of assignment and agents with score $s \geq \bar{s}_2$ are assigned with certainty. This is optimal because assigning agents below $\bar{s}_1$ is always welfare reducing as they are the richest agents with the best outside options. Similarly, the planner must give agents above $\bar{s}_2$ unit probability of assignment. Finally, they assign the remaining houses uniformly among those with intermediate incomes by virtue of the egalitarian motive.

[57] Formally, we require H is such that in the steady state corresponding to the described optimum, there are fewer than $Q - \delta M$ agents with score above $\bar{s}_1$ in the stationary distribution of searching agents and there are more than $Q - \delta M$ agents with score above $\bar{s}_2$ in the stationary distribution of searching agents, where M is the stationary measure of matched agents.

F. Relaxing the Full Support Assumption

Despite being a fairly weak condition, it is interesting to consider the implications for our analysis when the full support assumption (Assumption 1) fails. In particular, in this Appendix, we proceed under the assumption only that there are no mass points in the distribution of students.[58]

In this case, the interim economy after tie-breaking may have multiple stable matchings.[59] Thus, restricting the mechanism designer to use a stable matching mechanism does not pin down the allocation as there is a one-to-one correspondence between stable matchings and allocations. In general, under a stable mechanism, the selected matching may depend on the entire priority structure, i.e. a mechanism can even return two different stable matchings for two different priority structures even if the set of stable matchings is the same under those two priority structures.

To deal with these issues, we focus on two different selection rules: one is a selection rule that selects the optimal stable matching with respect to the preferences of the mechanism designer and the other is the student optimal stable matching for any given economy. We show that with mechanism-designer optimal selection, suitably modified versions of Theorems 1 and 2 continue to hold and that with student-optimal selection, Theorem 1 continues to hold but Theorem 2 fails.

F.1. The Mechanism Designer Optimal Selection

In this section, we assume whenever multiple stable matchings exist, the mechanism will implement the best matching with respect to the preferences of the mechanism designer. We show that suitably modified versions of Theorems 1 and 2 continue to hold without a full support assumption. First, we prove a modified version of Theorem 1:[60]

Theorem 3. *For any coarsening Ξ with allocation g_Ξ corresponding to the mechanism designer optimal stable matching, there is an alternative trinary coarsening Ξ' with allocation $g_{\Xi'}$ corresponding to the mechanism designer optimal stable matching such that $Z(g_{\Xi'}) \geq Z(g_\Xi)$.*

Proof. For any Ξ and matching μ that is selected by the mechanism designer optimal selection, construct a trinary coarsening $\Xi'(\mu)$ as in the proof of Theorem 1. Notice that the matching μ is still stable under $\Xi'(\mu)$. Hence the mechanism designer optimal selection picks a matching that is weakly better than μ under the coarsening $\Xi'(\mu)$, thus $Z(g_{\Xi'}) \geq Z(g_\Xi)$. □

[58] We give an example that shows the necessity of the no mass points assumption in this section.
[59] See Azevedo and Leshno (2016) for examples.
[60] Recall that Z is the objective function of the mechanism designer.

Hence, even if it is not possible to replicate each allocation via a trinary coarsening, there always exists an alternative trinary coarsening that is preferred by the mechanism designer under the mechanism designer optimal selection. Thus, restricting attention to trinary coarsenings is without loss of optimality.

The following result states that under mechanism designer optimal selection, an optimal coarsening exists, so Theorem 2 holds under this selection.

Theorem 4. *Under a stable matching mechanism with mechanism designer optimal selection, there exists an optimal trinary coarsening* Ξ*.*

Proof. From Theorem 3, we can restrict attention to trinary coarsenings, which we continue to represent by the coarsenings $v \in \mathcal{V}$, which is a compact set. Let $\psi : \mathcal{V} \rightrightarrows \mathbb{R}$ be the correspondence mapping each coarsening to the set of utility values of the mechanism designer under stable selections. We prove first that ψ is upper hemicontinuous (UHC) and then that it is closed-valued; establishing existence of an upper semi-continuous maximum selection $\mathcal{H}(v) = \max \psi(v)$. It then follows by compactness of $\mathcal{V}$ that an optimum exists.

Denoting the set of stable matchings (hence the possible allocations) under coarsening v by $\mathcal{S}(v)$, we can write $\psi = Z \circ \mathcal{S}(v)$. Azevedo and Leshno (2016) prove that the set of stable matchings has a cut-off structure and is an upper hemicontinuous function of the economy (Proposition B1 in the appendix of Azevedo and Leshno (2016)). Moreover, by Assumption 3, Z is a continuous function of the allocation (which is the stable matching selected by the mechanism), hence ψ is a UHC correspondence of a coarsening as it is a continuous function of a UHC correspondence.[61]

We now establish that ψ is closed valued. To this end, fix a coarsening v. Now consider the set of market clearing cutoffs $\mathcal{P}$ that correspond to the set of stable matchings under v. Consider a sequence $\{z_n\}$ such that $z_n \in \psi(v)$ for all n that converges to some z^*. We wish to prove that $z^* \in \psi(v)$. For any z_n, the utility value from an admissible stable matching under v, there must exist a corresponding vector of market clearing cutoffs $P_n \in \mathbb{R}^{|\mathcal{C}|}$. Then we can find a subsequence P_{n_k} that is element-wise monotone in $\mathbb{R}^{|\mathcal{C}|}$. As the set of market clearing cut-offs forms a complete lattice (see Azevedo and Leshno (2016) Theorem A1), $\lim_{n_k} P_{n_k} = P^*$ and $P^* \in \mathcal{P}$. Define ν_n as the measure of students whose assignment is different under P_n to P^*. In the coarsened economy, there are no mass points in the distribution of students. Thus, for any $\varepsilon > 0$ there exists an N such that $\nu_{n_k} < \varepsilon$ for all $n_k > N$. Hence, as $P_{n_k} \to P^*$, we have $Z(P_{n_k}) \to Z(P^*) = z^*$. Thus, as $P^* \in \mathcal{P}$, we have shown that $z^* \in \psi(v)$. Hence, ψ is closed-valued.

Finally, as ψ is closed-valued, $\mathcal{H}(\psi(v)) = \max \psi(v)$ is well-defined. Moreover, $\mathcal{H}(v)$

[61] See Border (1989) for a proof.

corresponds to the utility value of the mechanism designer under the mechanism designer optimal selection. By the UHC property of ψ, $\mathcal{H}$ is upper-semicontinuous. Thus, as $\mathcal{V}$ is a compact set, by the extreme value theorem it follows that $\max_{v\in\mathcal{V}} \mathcal{H}(\psi(v))$ exists. □

F.2. The Student Optimal Selection

In this section we assume whenever multiple stable matchings exist, the mechanism will implement the student optimal selection. In this case, while it is without loss of optimality to restrict attention to trinary coarsenings (Theorem 1 continues to hold), we show via an example that an optimum can fail to exist in the absence of the full support assumption (Theorem 2 does not hold). First, we prove that Theorem 1 holds under the student optimal stable selection whether or not the full support assumption holds:

Theorem 5. *Under a stable matching mechanism with student optimal selection, for any coarsening Ξ that induces the allocation g_Ξ, there is an alternative trinary coarsening Ξ' such that $g_{\Xi'} = g_\Xi$.*

Proof. Fix an arbitrary coarsening Ξ. Divide the schools to two subsets: $c \in \tilde{\mathcal{C}}$ if Ξ_c is finite and $c \in \hat{\mathcal{C}}$ if Ξ_c is the identity function. For any school $c \in \tilde{\mathcal{C}}$, the coarsening Ξ takes any student with score $s_c^{\tilde{\theta}}$ to an equivalence class, hence $\Xi(s_c^{\tilde{\theta}}) \in \{P_1^c, P_2^c, \ldots, P_N^c\}$ with $P_1^c < P_2^c < \ldots < P_N^c$ for some N.

Let $\tilde{\mu}$ denote the student optimal stable matching and P_x^c be the lowest indifference class that has a student placed in school c, i.e. $\tilde{\mu}(\tilde{\theta}_\Xi, \tau) = c$ for some $(\tilde{\theta}_\Xi, \tau)$ with $s_c^{\tilde{\theta}_\Xi} = P_x^c$ and $\tilde{\mu}(\tilde{\theta}_\Xi, \tau) \neq c$ for all $(\tilde{\theta}_\Xi, \tau)$ with $s_c^{\tilde{\theta}_\Xi} = P_y^c$ where $y < x$. For all $c \in \tilde{\mathcal{C}}$, define Ξ'_c as in the proof of Theorem 1,

$$\Xi'_c(s_c^{\tilde{\theta}}) = \begin{cases} P_1^c & , \text{ if } \quad \Xi_c(s_c^{\tilde{\theta}}) = P_z^c, z < x, \\ P_x^c & , \text{ if } \quad \Xi_c(s_c^{\tilde{\theta}}) = P_x^c, \\ P_N^c & , \text{ if } \quad \Xi_c(s_c^{\tilde{\theta}}) = P_z^c, z > x. \end{cases} \tag{125}$$

Next, for all $c \in \hat{\mathcal{C}}$, as we did in the proof of Theorem 1, let t_c denote the threshold that the students must clear in order to gain admission to that school. Formally,

$$t_c = \inf\{s_c^\theta : \tilde{\mu}(\theta) = c\} \tag{126}$$

Then, for all $c \in \hat{\mathcal{C}}$, define Ξ'_c in the following way,

$$\Xi'_c(s_c^\theta) = \begin{cases} 0 & , \text{ if } \quad s_c^\theta < t_c, \\ 1 & , \text{ if } \quad s_c^\theta \geq t_c. \end{cases} \tag{127}$$

Note that Ξ' is a trinary coarsening. The stability of $\tilde{\mu}$ under Ξ' follows exactly the same arguments in Theorem 1 and is therefore omitted.

Let μ' denote the student optimal stable matching under Ξ'. To see why μ' is stable under Ξ, assume student pair i, j (with scores s_c^i and s_c^j and tie-breakers τ_i and τ_j) blocks it in school c', i.e. $c' \succ_i \mu'(i)$, $\mu'(j) = c'$ and $\Xi_{c'}(s_{c'}^i) > \Xi_{c'}(s_{c'}^j)$ or $\Xi_{c'}(s_{c'}^i) = \Xi_{c'}(s_{c'}^j)$ and $\tau_i > \tau_j$.

First, assume $c' \in \tilde{\mathcal{C}}$ and $\Xi_{c'}(s_{c'}^i) > \Xi_{c'}(s_{c'}^j)$. Then $\mu'(j) = c'$ and $\Xi_{c'}(s_{c'}^i) > \Xi_{c'}(s_{c'}^j)$ imply that $\Xi'_{c'}(s_{c'}^i) = P_N^{c'}$, which means that i belongs to a class strictly higher than $P_x^{c'}$. From the definition of $P_x^{c'}$, we know that there exists k such that $\Xi'_{c'}(k) = P_x^{c'}$ and $\mu'(k) = c'$, which is a contradiction as i and c' will block μ' under Ξ'.

Next, assume $c' \in \tilde{\mathcal{C}}$, $\Xi_{c'}(s_{c'}^i) = \Xi_{c'}(s_{c'}^j)$ and $\tau_i > \tau_j$. There are two cases, either $\Xi'_{c'}(s_{c'}^i) > P_x^{c'}$ or $\Xi'_{c'}(s_{c'}^i) = P_x^{c'}$. In the first case, from the definition of $P_x^{c'}$, there exists k such that $\tilde{\mu}(k) = c'$ and $\Xi'_{c'}(s_{c'}^k) = P_x^{c'}$, which is a contradiction as i and c' will block μ' under Ξ'. In the second case, as the lottery classes are same under Ξ and Ξ', $\Xi'_{c'}(s_{c'}^i) = P_x^{c'}$ and $\Xi_{c'}(s_{c'}^i) = \Xi_{c'}(s_{c'}^j)$ imply that $\Xi'_{c'}(s_{c'}^j) = P_x^{c'}$. However, as $\tau_i > \tau_j$, this is a contradiction: i and c' will block μ' under Ξ'.

Finally, if $c' \in \hat{\mathcal{C}}$, then Ξ is the identity function. There are two cases, $s_{c'}^i = s_{c'}^j$ and $\tau_i > \tau_j$ or $s_{c'}^i > s_{c'}^j$. In the first, case, $\mu'(j) = c'$ implies that $s_{c'}^i = s_{c'}^j \geq t_{c'}$, which implies $\Xi'_{c'}(s_{c'}^i) = \Xi'_{c'}(s_{c'}^j) = 1$. As $\tau_i > \tau_j$ and $c' \succ_i \mu'(i)$, this contradicts the stability of μ' under Ξ'. In the second case, as $s_{c'}^i > t_{c'}$, it must be that $\tilde{\mu}(i) \succeq_i c'$ (where $c \succeq_i c'$ if $c \succ_i c'$ or $c = c'$), as otherwise $\tilde{\mu}$ will not be stable under Ξ. Moreover, we know that $\tilde{\mu}$ is stable under Ξ'. As μ' is the student optimal stable matching under Ξ', all students must weakly prefer their matching under μ' to $\tilde{\mu}$ which is a contradiction as $\tilde{\mu}(i) \succeq_i c' \succ_i \mu'(i)$. Thus, μ' is stable under Ξ.

As μ' is stable under Ξ, $\tilde{\mu}$ is stable under Ξ' and both are student optimal stable matchings, they must be equivalent. Thus the same matching is used in the construction of g_Ξ and $g_{\Xi'}$ under the student optimal selection and $g_\Xi = g_{\Xi'}$. This proves the result as Ξ' is a trinary coarsening. $\square$

This result implies that even without the full support assumption, if the student optimal stable mechanism is used to compute the stable matching in the interim economy, then it is still without loss of optimality to restrict attention to trinary coarsenings.

However, in the absence of the full support assumption, an optimal coarsening can fail to exist. The intuition for why this is true is that in the limit as the mechanism designer changes the coarsening thresholds, there can exist multiple stable matchings and the student optimal selection from the limit set of stable matchings may be worse from the mechanism designer's perspective than the matchings attained with arbitrarily close coarsening thresholds. Example 5 provides a concrete example of this phenomenon.

Example 5. *There are four schools $\mathcal{C} = \{c_0, c_1, c_2, c_3\}$, capacities $Q = (4, 1, 1, 1)$, and four ordinal types of students, each of measure one, with preferences:*

$$\begin{aligned} \succ_{t_1}&: c_1, c_2, c_0 \\ \succ_{t_2}&: c_2, c_1, c_0 \\ \succ_{t_3}&: c_3, c_1, c_0 \\ \succ_{t_4}&: c_3, c_0 \end{aligned} \tag{128}$$

The marginal score distributions in the acceptable schools are given by:

$$\begin{aligned} s^{c_1}(t_2) \sim U[2,3],\ s^{c_1}(t_3) \sim U[1,2],\ &s^{c_1}(t_1) \sim U[0,1] \\ s^{c_2}(t_1) \sim U[1,2],\ &s^{c_2}(t_2) \sim U[0,1] \\ s^{c_3}(t_3) \sim U[1,2],\ &s^{c_3}(t_4) \sim U[0,1] \end{aligned} \tag{129}$$

Assume the mechanism designer maximizes the following utility function:

$$Z(g) = \sum_t \sum_{c \in \{c_1, c_2, c_3\}} u_t(c) g(c|t) \tag{130}$$

with:

$$u_{t_1}(c_i) = \begin{cases} 1 & \text{if} \quad i = 2 \\ 0 & \text{if} \quad i = 1 \end{cases} \tag{131}$$

$$u_{t_2}(c_i) = \begin{cases} 1 & \text{if} \quad i = 1 \\ 0 & \text{if} \quad i = 2 \end{cases} \tag{132}$$

$$u_{t_3}(c_i) = 0 \quad \text{for} \quad i = 1, 3 \tag{133}$$

$$u_{t_4}(c_3) = -1 \tag{134}$$

This utility function assumes that the designer prefers assigning type t_1 students to school c_2 and type t_2 students to the school c_1, while assigning the minimal amount of type t_4 students to school c_3. The first best outcome for the designer is to match all t_1 students to c_2, all t_2 students to c_1 and all t_3 students to c_3. This yields a payoff of $Z^{FB} = 2$. Note that without priority design, the student optimal stable matching assigns all students of types t_1 to c_1, t_2 to c_2, t_3 to c_3, and t_4 to c_0. The yields a payoff of $Z^{SOSM} = 0$.

We note that the first-best matching is not attainable as the student-optimal stable match-

ing under any coarsening. To match all t_1 students to c_2 and all t_2 students to c_1 in a stable matching, a necessary and sufficient condition is to reject a positive measure of t_3 students from school c_3 (and assign a positive measure of type t_4 students to c_3). But this necessarily leads to a payoff strictly smaller than 2 as some positive measure t_4 students are assigned to c_3.

However, the payoff from the first-best matching can be arbitrarily well approximated through coarsening. Define for any $\epsilon \geq 0$, the coarsening:

$$\Xi_{c_3}(s_{c_3};\epsilon) = \begin{cases} 0, s_{c_3} < 1-\epsilon \\ 1, s_{c_3} \in [1-\epsilon, 1+\epsilon), \\ 2, s_{c_3} \geq 1+\epsilon \end{cases} \tag{135}$$

and $\Xi_c(s_c;\epsilon) = s_c$ for all $c \neq c_3$.

Under this coarsening for $\epsilon > 0$, the unique stable matching assigns $\frac{\epsilon}{2}$ measure type t_4 students are assigned to c_3. Thus, $\frac{\epsilon}{2}$ measure t_3 students are assigned to either c_1 or c_0. Given that t_2 types have the highest priority at c_1 and t_1 types have the highest priority at c_2, it is not possible under any stable matching that any type t_1 or t_2 student is assigned to c_0. Therefore, $\frac{\epsilon}{2}$ type t_3 students are assigned to c_0. Hence, it is not possible that any t_1 student is assigned to c_1 or c_1 and any t_3 type would form a blocking pair, as t_3 students have higher priority at c_1. Thus, in the unique stable matching, all t_1 types are assigned to c_2, all t_2 types are assigned to c_1, and measure $\frac{\epsilon}{2}$ of t_4 types are assigned to c_3. For $\epsilon = 0$, the outcome is the outcome without coarsening. Thus, the payoff of the mechanism designer is given by:

$$Z(\epsilon) = \begin{cases} 2 - \frac{\epsilon}{2}, \epsilon > 0, \\ 0, \epsilon = 0. \end{cases} \tag{136}$$

To conclude, the first-best is not attainable but can be arbitrarily well approximated by coarsening. Thus, the problem of the designer is not upper semi-continuous and an optimum does not exist under the student-optimal selection.

As a result, an optimum can cease to exist in the absence of full support under the student optimal stable matching. This notwithstanding, given the earlier results, it remains true that if the mechanism-designer optimal and student optimal selections coincide, then Theorem 2 holds. This is, for instance, true in the natural case with a utilitarian mechanism designer.

F.3. The Necessity of the No Mass Points Assumption for Existence of an Optimum

While we relaxed the full support assumption in this appendix, we retained the no mass points assumption. To see the necessity of the no mass points assumption for the existence of an optimum, we introduce the following example. Intuitively, with mass points, changing the coarsening cutoffs by an arbitrarily small amount can alter the allocation of students to schools in a way that adversely impacts the objective function of the mechanism designer.

Example 6. *Suppose that there is one school c with capacity of unit measure and an outside option. Suppose all students prefer c to the outside option. There are three types of students, each of unit measure. Students with score $s \in [0.5, 1]$ who give the mechanism designer utility of 1 when they are assigned to the school (Type 1), those with score $s \in [0, 0.5)$, who give the mechanism designer utility of 2 (Type 2), and those with score $s = 0$ who give utility -M to the mechanism designer (Type 3). The first two types are uniformly distributed over their respective domains. See that there is a mass point of students with score zero.*

It is clear here that under any coarsening, if there exists another coarsening such that more type 2 students attend the school while no type 3 students are admitted, then that initial coarsening must not be optimal. In particular, suppose that those with scores $s \geq \underline{v} > 0$ are coarsened into the same indifference class. The mechanism designer's utility is:

$$U(\underline{v}) = \frac{0.5}{1-\underline{v}} \times 1 + \frac{0.5-\underline{v}}{1-\underline{v}} \times 2 = \frac{1.5-2\underline{v}}{1-\underline{v}},\ \underline{v} \in (0,1] \tag{137}$$

which is decreasing in $\underline{v}$ for all $\underline{v} > 0$ so no $\underline{v} > 0$ can be optimal. Now suppose that the mechanism designer fully coarsens the students so that $\underline{v} = 0$. Their utility is:

$$U(0) = \frac{1}{3} \times 1 + \frac{1}{3} \times 2 - \frac{1}{3}M \tag{138}$$

Hence, a sufficient condition that there will not exist an optimal coarsening is that setting $\underline{v} = 0.5$ (i.e. not coarsening) is better than full coarsening. That is:

$$U(0) = 1 - \frac{1}{3}M < 1 = U(0.5) \tag{139}$$

which is satisfied for any $M > 0$. Hence, whenever $M > 0$, there does not exist an optimal coarsening.