14.452: Problem Set 1

Due date: November 7, 2013 in class.

Question 1: Consider a modified version of the continuous-time Solow growth model where the aggregate production function is

$$F(K,L,Z) = L^{\beta}K^{\alpha}Z^{1-\alpha-\beta},$$

where Z is land, available in fixed inelastic supply. Assume that $\alpha+\beta<1$, capital depreciates at the rate δ, and there is an exogenous saving rate of s.

1. First suppose that there is no population growth. Find the steady-state capital-labor ratio and the steady-state output level. Prove that the steady state is unique and globally stable.

2. Now suppose that there is population growth at the rate n, that is, $\dot{L}/L = n$. What happens to the capital-labor ratio and output level as $t \to \infty$? What happens to returns to land and the wage rate as $t \to \infty$?

3. Would you expect the population growth rate n or the saving rate s to change over time in this economy? If so, how?

Question 2: Consider the discrete-time Solow growth model with constant population growth at the rate n, no technological change and depreciation rate of capital equal to δ. Assume that the saving rate is a function of the capital-labor ratio, thus given by $s(k)$.

1. Suppose that $f(k) = Ak$ and $s(k) = s_0 k^{-1} - 1$. Show that if $A+\delta-n=2$, then for any $k(0) \in (0, As_0/(1+n))$, the economy immediately settles into an asymptotic cycle and continuously fluctuates between $k(0)$ and $As_0/(1+n) - k(0)$. [Suppose that $k(0)$ and the parameters are given such that $s(k) \in (0,1)$ for both $k = k(0)$ and $k = As_0/(1+n) - k(0)$].

2. Now consider more general continuous production function $f(k)$ and saving function $s(k)$, such that there exist $k_1, k_2 \in R_+$ with $k_1 \neq k_2$ and

$$\begin{aligned} k_2 &= \frac{s(k_1) f(k_1) + (1-\delta) k_1}{1+n} \\ k_1 &= \frac{s(k_2) f(k_2) + (1-\delta) k_2}{1+n}. \end{aligned}$$

   Show that when such (k_1, k_2) exist, there may also exist a stable steady state.

3. Prove that such cycles are not possible in the continuous-time Solow growth model for any (possibly non-neoclassical) continuous production function $f(k)$ and continuous $s(k)$.

4. What does the result in parts 1-3 imply for the approximations of discrete time by continuous time in the Solow model (suggested in Section 2.4 of the textbook)? What does this imply for the cycles in parts 1 and 2?

5. Show that if $f(k)$ is nondecreasing in k and $s(k) = k$, cycles as in parts 1 and 2 are not possible in discrete-time either.

Question 3: Consider the Solow growth model with constant saving rate s and depreciation rate of capital equal to δ. Assume that population is constant and the aggregate output is given by the CES production function

$$F(A_K(t)K(t), A_L(t)L) = \left[\gamma (A_K(t)K(t))^{\frac{\sigma-1}{\sigma}} + (1-\gamma)(A_L(t)L)^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}}$$

where $\dot{A}_L(t)/A_L(t) = g_L > 0$ and $\dot{A}_K(t)/A_K(t) = g_K > 0$. Suppose the elasticity of substitution between capital and labor is less than one, $\sigma < 1$, and capital-augmenting technological progress is faster than labor-augmenting progess, $g_K \geq g_L$. Show that as $t \to \infty$, the economy converges to a BGP where the share of labor in national income is equal to 1, and capital, output and consumption all grow at the rate g_L. In light of this result, discuss the claims in the literature that capital-augmenting technological change is inconsistent with balanced growth.

Question 4: Consider the basic Solow model in continuous time and suppose that $A(t) = A$, so that there is no technological progress of the usual kind. However, assume that the relationship between investment and capital accumulation is modified to

$$\dot{K}(t) = q(t)I(t) - \delta K(t),$$

where $[q(t)]_{t=0}^{\infty}$ is an exogenously given time-varying process. Intuitively, when $q(t)$ is high, the same investment expenditure translates into a greater increase in the capital stock. Therefore, we can think of $q(t)$ as the inverse of the relative prices of machinery to output. When $q(t)$ is high, machinery is relatively cheaper, and thus suppose that $\dot{q}(t) > 0$.

1. Suppose that $\dot{q}(t)/q(t) = \gamma_K > 0$. Show that for a general production function, $F(K, L)$, there exists no steady-state equilibrium.

2. Now suppose that the production function is Cobb-Douglas, $F(K, L) = K^{\alpha}L^{1-\alpha}$, and characterize the unique steady-state equilibrium.

3. Show that this steady-state equilibrium does not satisfy the Kaldor fact of constant K/Y. Is this a problem? [Hint: how is "K" measured in practice? How is it measured in this model?].