# 14.452 Problem Set 3

Due: November 22, 2024 at 2:30PM

Please only hand in Questions 2 and 3 on Canvas, which will be graded. The rest will be reviewed in the recitation but should not be handed in.

**Question 1:** Consider the basic neoclassical growth model with CRRA preferences, but with consumer heterogeneity in initial asset holdings (you may assume no technological change if you wish). In particular, there is a set $\mathscr{H}$ of household and household $h \in \mathscr{H}$ starts with initial assets $a_h(0)$. Households are otherwise identical.

1. Characterize the competitive equilibrium of this economy and show that the behavior of per capita variables is identical to that in a representative household economy, with the representative household starting with assets $a(0) = |\mathscr{H}|^{-1} \int_{\mathscr{H}} a_h(0)\, dh$, where $|\mathscr{H}|$ is the measure (number) of households in this economy. Interpret this result and relate it to the Aggregation Theorem.

2. Show that if, instead of the no-Ponzi condition, we impose $a_h(t) \geq 0$ for all $h \in \mathscr{H}$ and for all $t$, then a different equilibrium allocation may result. In light of this finding, discuss whether (and when) it is appropriate to use a no-borrowing constraint instead of the no-Ponzi condition.

**Question 2:** Consider the standard neoclassical growth model augmented with labor supply decisions. In particular, there is a total population normalized to 1, and all individuals have utility function

$$U(0) = \int_0^{\infty} \exp(-\rho t)\, u(c(t), 1 - l(t)),$$

where $l(t) \in (0,1)$ is labor supply. In a symmetric equilibrium, employment $L(t)$ is equal to $l(t)$. Assume that the production function is given by $Y(t) = F[K(t), A(t)L(t)]$, which satisfies all the standard assumptions and $A(t) = \exp(gt)A(0)$.

1. Define a competitive equilibrium.

2. Set up the current-value Hamiltonian that each household solves taking wages and interest rates as given, and determine the necessary and sufficient conditions for the allocation of consumption over time and leisure-labor trade off.

3. Set up the current-value Hamiltonian for a planner maximizing the utility of the representative household, and derive the necessary and sufficient conditions for an optimal solution.

4. Show that the two problems are equivalent given competitive markets.

5. Show that unless the utility function is asymptotically equal to

$$u(c(t), 1-l(t)) = \begin{cases} \frac{Ac(t)^{1-\theta}}{1-\theta} h(1-l(t)) & \text{for } \theta \neq 1, \\ A \log c(t) + Bh(1-l(t)) & \text{for } \theta = 1, \end{cases}$$

for some $h(\cdot)$ with $h'(\cdot) > 0$, there will not exist a BGP with constant and equal rates of consumption and output growth, and a constant level of labor supply. Characterize such a BGP. Explain why this is the only functional form consistent with BGP.

6. Imposing the utility function in part 5 above, characterize the dynamic equilibrium path of the economy starting with an arbitrary initial condition $k(0) > 0$.

**Question 3:** Consider the two-period canonical overlapping generations model with log preferences

$$\log(c_1(t)) + \beta \log(c_2(t+1))$$

for each individual. Suppose that there is population growth at the rate $n$. Individuals work only when they are young, and supply one unit of labor inelastically. Production technology is given by

$$Y(t) = A(t) K(t)^{\alpha} L(t)^{1-\alpha},$$

where $A(t+1) = (1+g)A(t)$, with $A(0) > 0$ and $g > 0$.

1. Define a competitive equilibrium and the steady-state equilibrium.

2. Can you apply the First Welfare Theorem to this competitive equilibrium? Carefully explain your answer.

3. Characterize the steady-state equilibrium and show that it is globally asymptotically stable.

4. What is the effect of an increase in $g$ on the equilibrium path?

   In the rest of the question, assume that $g = 0$.

5. Suppose that the equilibrium involves $r^* < n$. Explain why the equilibrium is referred to as "dynamically inefficient" in this case. Show that an unfunded Social Security system can increase the welfare of *all* future generations.

6. Show that if $r^* > n$, then any unfunded Social Security system that increases the welfare of the current old generation must reduce the welfare of some future generation.

**Question 4:** Consider a variant of the neoclassical economy with preferences at time 0 given by

$$\int_0^\infty \exp(-\rho t) \frac{c(t)^{1-\theta} - 1}{1-\theta} dt.$$

Population is constant at $L$, and labor is supplied inelastically. The aggregate production function is given by

$$F(K, L) = A_K K + G(L, K),$$

where $G$ is a linearly homogeneous, smooth function that satisfies the Inada conditions, and $A_K > \rho + \delta$, and capital depreciates at rate $\delta$. Capital and labor markets are competitive.

1. Derive the differential equation system that characterizes the evolution of the capital stock and consumption in equilibrium.

2. Show that this economy generates sustained growth without technological change. What determines the asymptotic growth rate in this economy? Does this depend on the function $G$? If not, why not? [Hint: conjecture an equilibrium in which the capital stock asymptotically grows at a constant rate $g > 0$. Simplify the differential equation system obtained in part 2 under this conjecture. Solve the simplified system and verify that there is an asymptotic equilibrium with a constant growth rate.]

3. What happens to the share of labor in national income? Is this a good prediction?

4. What happens if $L$ grows at a constant rate $n \in (0, A_K - \rho - \delta]$? What happens if $n > A_K - \rho - \delta$?