

## 14.773 Problem Set 3

Due Date: March 22, 2013

March 10, 2013

### Question 1

Suppose that the economy consists of two groups, the elite and the producers. Suppose that both groups are of equal size (each consisting of a continuum of agents). Both groups have instantaneous utility  $u(c) = \log c$  and discount factor  $\beta$ . Producers have access to the production technology  $f(k) = Ak^\alpha$ , where  $k$  is capital. The elite impose a linear tax rate of  $\tau_t$  on production at time  $t$  and consume the proceeds. The capital stock for time  $t$  must be chosen at time  $t$  after the tax rate  $\tau_t$  has been announced. There is full depreciation of capital.

1. Given a tax sequence, set up the dynamic optimization problem of entrepreneurs and show that the evolution of the capital stock is given by

$$k_{t+1} = \alpha\beta(1 - \tau_t) Ak_t^\alpha. \quad (1)$$

Explain why the capital stock for time  $t+1$  does not depend on the current tax rate but only on the past tax rate. [Hint: to derive this, set up the maximization problem of the entrepreneur is a dynamic program and conjecture a decision rule of the form  $k_{i,t+1} = \kappa(1 - \tau)y_{i,t}$  for entrepreneur  $i$ , where  $y_{i,t}$  is his output at time  $t$ ].

2. To determine the Markov Perfect Equilibrium tax rates, write the value to a representative elite agent at time  $t+1$  as a function of the tax rate  $\tau = \tau_{t+1}$ , taking into account that the capital stock of entrepreneurs at date  $t+1$ ,  $k = k_{t+1}$  is given from (1). Show that this value function takes the form

$$W(k) = \max_{\tau \in [0,1]} \{ \log[\tau Ak^\alpha] + \beta W(\alpha\beta(1 - \tau) Ak^\alpha) \}. \quad (2)$$

Use standard dynamic programming arguments show that  $W$  is strictly concave and differentiable for  $k > 0$  (and denote the derivative by  $W'$ ). Show that the Euler equation for the elite is

$$\frac{1}{\tau} = \beta^2 \alpha A k^\alpha W'(k') = \beta \frac{k' W'(k')}{1 - \tau}.$$

- Now, conjecturing that  $W(k) = \eta + \gamma \log k$  and using the Envelope condition, show that  $\gamma = \alpha / (1 - \alpha\beta)$  and derive the law of motion of the capital stock of each entrepreneur (and the aggregate capital stock). Explain the role of logarithmic preferences in this result.

### Question 2

Consider an economy populated by  $\lambda$  rich agents who initially hold power, and  $1 - \lambda$  poor agents who are excluded from power, with  $\lambda < 1/2$ . All agents are infinitely lived and discount the future at the rate  $\beta \in (0, 1)$ . Each rich agent has income  $\theta/\lambda$  while each poor agent has income  $(1 - \theta)/(1 - \lambda)$  where  $\theta > \lambda$ . The political system determines a linear tax rate,  $\tau$ , the proceeds of which are redistributed lump-sum. Each agent can hide their money in an alternative non-taxable production technology, and in the process they lose a fraction  $\phi$  of their income. There are no other costs of taxation. The poor can undertake a revolution, and if they do so, in all future periods, they obtain a fraction  $\mu(t)$  of the total income of the society (i.e., an income of  $\mu(t)/(1 - \lambda)$  per poor agent). The poor cannot revolt against democracy. The rich lose everything and receive zero payoff after a revolution. At the beginning of every period, the rich can also decide to extend the franchise to the poor, and this is irreversible. If the franchise is extended, the poor decide the tax rate in all future periods.

- Define MPE in this game.
- First suppose that  $\mu(t) = \mu^l$  at all times. Also assume that  $0 < \mu^l < 1 - \theta$ . Show that in the MPE, there will be no taxation when the rich are in power, and the tax rate will be  $\tau = \phi$  when the poor are in power. Show that in the MPE, there is no extension of the franchise and no taxation.
- Suppose that  $\mu^l \in (1 - \theta, (1 - \phi)(1 - \theta) + \phi(1 - \lambda))$ . Characterize the MPE in this case. Why is the restriction  $\mu^l < (1 - \phi)(1 - \theta) + \phi(1 - \lambda)$  necessary?
- Now consider the SPE of this game when  $\mu^l > 1 - \theta$ . Construct an equilibrium where there is extension of the franchise along the equilibrium path. [Hint:

first, to simplify, take  $\beta \rightarrow 1$ , and then consider a strategy profile where the rich are always expected to set  $\tau = 0$  in the future; show that in this case the poor would undertake a revolution; also explain why the continuation strategy of  $\tau = 0$  by the rich in all future periods could be part of a SPE]. Why is there extension of the franchise now? Can you construct a similar non-Markovian equilibrium when  $\mu^l < 1 - \theta$ ?

5. Explain why the MPE led to different predictions than the non-Markovian equilibria. Which one is more satisfactory?
6. Now suppose that  $\mu(t) = \mu^l$  with probability  $1 - q$ , and  $\mu(t) = \mu^h$  with probability  $q$ , where  $\mu^h > 1 - \theta > \mu^l$ . Construct a MPE where the rich extend the franchise, and from there on, a poor agent sets that tax rate. Determine the parameter values that are necessary for such an equilibrium to exist. Explain why extension of the franchise is useful for rich agents?
7. Now consider non-Markovian equilibria again. Suppose that the unique MPE has franchise extension. Can you construct a SPE equilibrium, as  $\beta \rightarrow 1$ , where there is no franchise extension?
8. Contrast the role of restricting strategies to be Markovian in the two cases above [Hint: why is this restriction ruling out franchise extension in the first case, while ensuring that franchise extension is the unique equilibrium in the second?].

### Question 3

Consider a country consisting of two ethnic groups, A and B. All agents are infinitely lived (in discrete time) and maximize the net present discounted value of their income with discount factor  $\beta \in (0, 1)$ . Suppose that both groups are of equal size, and have exogenous income levels  $y_j$ ,  $j \in \{A, B\}$  in each period. At the beginning of the period  $t$ , there are two possible political regimes  $S_{t-1}$  inherited from yesterday. A-dominance ( $S_{t-1} = A$ ) and B-dominance ( $S_{t-1} = B$ ). A secession shock  $x_t = \{0, A, B\}$  takes place with probabilities  $\{1 - q_A - q_B, q_A, q_B\}$  where  $q_A, q_B < 1/2$ , where  $x_t$  denotes the identity of the group which will have the opportunity to secede at the end of the period. Whoever has political power as determined by  $S_{t-1}$  chooses  $S_t$  which determines the group which can set policies today (and institutions  $S_{t+1}$  tomorrow), where policies consist of a policy vector  $(\tau_t^A, \tau_t^B)$ , where  $\tau_t^j$  is a lump-sum tax imposed on group  $j$ , satisfying  $\tau_t^j \leq y_j$ . Negative values of  $\tau$ 's are allowed (as transfers). Since both groups are of the same size, the government budget constraint

is

$$\tau_t^A + \tau_t^B \leq 0.$$

Following the setting of policies, if  $x_t = A$ , group A has an opportunity to secede from this country, in which case each of its members receive an income of  $\delta y_A$  from  $t$  onward, where  $\delta \in (0, 1)$  and the members of group  $B$  receive zero from  $t$  onward. If  $x_t = B$ , an analogous situation occurs where if group  $B$  secedes, all of its members receive an income of  $\delta y_B$  from  $t$  onward and the members of group  $A$  receive zero from  $t$  onward. Let  $s_t^j = \{0, 1\}$  represent the secession decision which can be taken by group  $j$  if has the opportunity where 1 denotes secession. Note that only one group can receive a secession shock at a time.

The timing for the game is as follows:

- The group that was in power in the previous period starts out in power, and the secession shock  $x_t$  is realized.
- The group in power determines taxes and whether to transfer power to the other group.
- A secession decision is made if a group has the ability to secede.

1. Define the payoff-relevant state vector, the strategies and a Markov Perfect Equilibrium (MPE) in this game.
2. Show that if  $\delta = 0$ , there exists a unique MPE such that if  $S_t = A$  then  $(\tau_t^A = -y_B, \tau_t^B = y_B)$  and if  $S_t = B$  then  $(\tau_t^A = y_A, \tau_t^B = -y_A)$ .
3. Explain why the above strategy profile may not be an equilibrium when  $\delta > 0$ .
4. Construct an equilibrium in which for  $\delta \in (\underline{\delta}, \bar{\delta})$ , there exists an MPE in which whenever group  $j$  has the opportunity to secede, the political regime switches to group  $j$ -dominance (unless it is already under group  $j$ -dominance). Explain why this equilibrium needs both parameter conditions  $\delta > \underline{\delta}$  and  $\delta < \bar{\delta}$ .
5. How does this theory relate to and differs from other models of equilibrium institutional change?
6. How would to enrich this model to make it applicable more broadly to situations of within-country ethnic conflict?

**Question 4**

Consider the following jury problem. Each of  $n$  individuals have a prior  $\pi$  that a defendant is guilty, denoted by  $\theta = G$ . The alternative is  $\theta = I$  (for innocent). Each individual receives a signal  $s = \{g, i\}$  (for example, from their reading of the evidence presented at the trial). Suppose that the signals are conditionally independent and identically distributed and satisfy

$$\Pr(s = g|\theta = G) = p \text{ and } \Pr(s = i|\theta = I) = q; \quad q, p > 0.5$$

Suppose that the group requires unanimity to take a decision  $x = G$ . Let the vote of juror  $j$  be denoted by  $v_j \in \{g, i\}$ . Suppose also that each member  $j$  of the group has the following payoff:

$$u_j(x, \theta) = \begin{cases} 0 & \text{if } x = \theta \\ -z & \text{if } x = G \text{ and } \theta = I \\ -(1 - z) & \text{if } x = I \text{ and } \theta = G \end{cases}$$

where  $z \in [0; 1]$ .

1. Show that the "optimal" decision is

$$x = I \text{ if } \Pr(\theta = G|\text{information set}) \leq z$$

Interpret this condition.

2. Let us now focus on the Bayesian Nash Equilibrium. Suppose first that all jurors vote "sincerely" and consider the problem of juror 1 who has received signal  $s_1 = i$ . Explain why the only relevant probability to consider is  $\Pr(\theta = G|s_j = g \text{ for all } j \neq 1 \text{ and } s_1 = i)$ .
3. Show that under sincere voting, this probability is

$$\Pr(\theta = G|s_j = g \text{ for all } j \neq 1 \text{ and } s_1 = i) = \frac{1}{1 + \frac{q}{1-p} \left(\frac{1-q}{p}\right)^{n-1} \frac{1-\pi}{\pi}}$$

Show that for  $n$  large, when all other jurors are voting sincerely, it is a best response for juror 1 to vote  $v_j = g$ . Explain why in this case there does not exist an equilibrium with sincere voting.

4. Now for the case in which there does not exist an equilibrium with sincere voting, characterize the symmetric mixed strategy equilibrium in which conditional on  $s_j = i$ , juror  $j$  votes  $v_j = g$  with probability  $\alpha$  (and votes  $v_j = g$  with probability 1 when  $s_j = g$ ). [Hint: derive the analog of the above expression under this mixed strategy profile and then use the fact that each juror has to be indifferent conditional on  $s_j = i$ , which implies  $\Pr(\theta = G | v_j = g \text{ for all } j \neq 1 \text{ and } s_1 = i) = z$ ].