14.452: Introduction to Economic Growth
Problem Set 3

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Due date: December 5th, Friday

Please only hand in Question 3, which will be graded. The rest will be reviewed in the recitation but should not be handed in.

Question 1: Consider the following continuous-time neoclassical growth model:

$$U(0) = \int_0^\infty \exp(-\rho t) \frac{(c(t))^{1-\theta} - 1}{1-\theta},$$

with production function

$$Y(t) = A\left[L(t)^{\frac{\sigma-1}{\sigma}} + K(t)^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}}.$$

1. Define a competitive equilibrium for this economy.
2. Set up the current-value Hamiltonian for an individual and characterize the necessary conditions for consumer maximization. Combine these with equilibrium factor prices and derive the equilibrium path.
3. Prove that the equilibrium is Pareto optimal in this case.
4. Show that if $\sigma \leq 1$, sustained growth is not possible.
5. Show that if A and σ are sufficiently high, this model generates asymptotically sustained growth due to capital accumulation. Interpret this result.
6. Characterize the transitional dynamics of the equilibrium path.
7. What is happening to the share of capital in national income? Is this plausible? How would you modify the model to make sure that the share of capital in national income remains constant?
8. Now assume that returns from capital are taxed at the rate τ. Determine the asymptotic growth rate of consumption and output.

Question 2: Consider the following endogenous growth model due to Uzawa and Lucas. The economy admits a representative household and preferences are given by

$$\int_0^{\infty} \exp(-\rho t) \frac{C(t)^{1-\theta} - 1}{1-\theta} dt,$$

where $C(t)$ is consumption of the final good, which is produced as

$$Y(t) = AK(t)^{\alpha} H_P^{1-\alpha}(t)$$

where $K(t)$ is capital and $H(t)$ is human capital, and $H_P(t)$ denotes human capital used in production. The accumulation equations are as follows:

$$\dot{K}(t) = I(t) - \delta K(t)$$

for capital and

$$\dot{H}(t) = BH_E(t) - \delta H(t)$$

where $H_E(t)$ is human capital devoted to education (further human capital accumulation), and the depreciation of human capital is assumed to be at the same rate, δ, as physical capital for simplicity. The resource constraints of the economy are

$$I(t) + C(t) \leq Y(t)$$

and

$$H_E(t) + H_P(t) \leq H(t).$$

1. Interpret the second resource constraint.

2. Denote the fraction of human capital allocated to production by $\phi(t)$ (so that $\phi(t) \equiv H_P(t)/H(t)$) and calculate the growth rate of final output as a function of $\phi(t)$ and the growth rates of accumulable factors.

3. Assume that $\phi(t)$ is constant, and characterize the BGP of the economy (with constant interest rate and constant rate of growth for capital and output). Show that in this BGP, $r^* \equiv B - \delta$ and the growth rate of consumption, capital, human capital and output are given by $g^* \equiv (B - \delta - \rho)/\theta$. Show also that there exists a unique value of $k^* \equiv K/H$ consistent with BGP.

4. Determine the parameter restrictions to make sure that the transversality condition is satisfied.

5. Now analyze the transitional dynamics of the economy starting with K/H different from k^* [Hint: look at dynamics in three variables, $k \equiv K/H$, $\chi \equiv C/K$ and ϕ, and consider the cases $\alpha < \theta$ and $\alpha \geq \theta$ separately].

Question 3: Consider an infinite-horizon economy that admits a representative household with preferences at time 0 given by

$$\int_0^\infty \exp(-\rho t) \frac{C(t)^{1-\theta} - 1}{1-\theta} dt.$$

Population is constant at L and labor is supplied inelastically. The unique final good is produced with the production function

$$Y(t) = \frac{1}{1-\beta} \left[\int_0^{N(t)} x(\nu,t)^{1-\beta} d\nu \right] L^\beta,$$

where $\beta \in (0,1)$, $x(\nu,t)$ denotes intermediate goods of type ν used in final good production at time t, and $N(t)$ is the number of intermediate good types available at time t. Once a particular type of intermediate good is invented, it can be produced by using ψ units of final good. The innovation possibilities frontier of the economy is

$$\dot{N}(t) = \eta Z(t),$$

where $Z(t)$ is total amount of R&D spending, and resource constraint of the economy is $C(t) + X(t) + Z(t) \leq Y(t)$, where $X(t)$ is spending on intermediate goods. There is free entry into research and a firm that invents a new intermediate good type receives a patent and becomes the monopolist producer of that good until the patent expires. Patents expire at the Poisson rate $\iota < \infty$, and once the patent on an intermediate good of a particular type is expired, it is produced competitively. The economy starts with $N(0) > 0$ intermediate goods without patents.

1. Define the equilibrium and balanced growth path (BGP) allocations.
2. Characterize the BGP. [Hint: be explicit about the distribution of intermediate goods between firms with and without patents].
3. Show that starting at time $t = 0$, the economy converges to the BGP. Does it always grow at a constant rate?
4. Show that an increase in ι reduces the BGP growth rate. Is this a realistic prediction? What types of alternatives for generalizations would you need to consider in order to reverse this prediction?
5. Can an increase in ι be welfare improving? Explain your answer intuitively.

Question 4: Consider the following endogenous growth model. Population at time t is $L(t)$ and grows at the constant rate n (i.e., $\dot{L}(t) = nL(t)$). All agents have preferences given by

$$\int_0^\infty \exp(-\rho t) \frac{C(t)^{1-\theta} - 1}{1-\theta} dt,$$

where C is consumption defined over the final good of the economy. This good is produced as

$$Y(t) = \left[\int_0^{N(t)} y(\nu, t)^{\beta} \, d\nu \right]^{1/\beta},$$

where $y(\nu, t)$ is the amount of intermediate good ν used in production at time t and $N(t)$ denotes the number of intermediate goods available at time t. The production function of each intermediate is

$$y(\nu, t) = l(\nu, t)$$

where $l(\nu, t)$ is labor allocated to this good at time t. New goods are produced by allocating workers to the R&D process, with the production function

$$\dot{N}(t) = \eta N^{\phi}(t) L_R(t)$$

where $\phi \leq 1$ and $L_R(t)$ is labor allocated to R&D at time t. So labor market clearing requires $\int_0^{N(t)} l(\nu, t) \, dv + L_R(t) = L(t)$. Risk-neutral firms hire workers for R&D. A firm who discovers a new good becomes the monopoly supplier, with a perfectly and indefinitely enforced patent.

1. Characterize the BGP in the case where $\phi = 1$ and $n = 0$, and show that there are no transitional dynamics. Why is this? Why does the long-run growth rate depend on θ? Why does the growth rate depend on L? Do you find this plausible?

2. Now suppose that $\phi = 1$ and $n > 0$. What happens? Interpret.

3. Now characterize the BGP when $\phi < 1$ and $n > 0$. Does the growth rate depend on L? Does it depend on n? Why? Do you think that the configuration $\phi < 1$ and $n > 0$ is more plausible than the one with $\phi = 1$ and $n = 0$?