14.452 Problem Set 4

Due: December 10, 2024 at 1PM

Please only hand in Question 4 on Canvas, which will be graded. The rest will be reviewed in the recitation but should not be handed in.

Question 1: Consider the following continuous-time neoclassical growth model:

$$U(0) = \int_0^\infty \exp(-\rho t) \frac{(c(t))^{1-\theta} - 1}{1-\theta},$$

with production function

$$Y(t) = A\left[L(t)^{\frac{\sigma-1}{\sigma}} + K(t)^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}}.$$

1. Define a competitive equilibrium for this economy.
2. Set up the current-value Hamiltonian for an individual and characterize the necessary conditions for consumer maximization. Combine these with equilibrium factor prices and derive the equilibrium path.
3. Prove that the equilibrium is Pareto optimal in this case.
4. Show that if $\sigma \leq 1$, sustained growth is not possible.
5. Show that if A and σ are sufficiently high, this model generates asymptotically sustained growth due to capital accumulation. Interpret this result.
6. Characterize the transitional dynamics of the equilibrium path.
7. What is happening to the share of capital in national income? Is this plausible? How would you modify the model to make sure that the share of capital in national income remains constant?
8. Now assume that returns from capital are taxed at the rate τ. Determine the asymptotic growth rate of consumption and output.

Question 2: Consider the following endogenous growth model due to Uzawa and Lucas. The economy admits a representative household and preferences are given by

$$\int_0^{\infty} \exp(-\rho t) \frac{C(t)^{1-\theta} - 1}{1-\theta} dt,$$

where $C(t)$ is consumption of the final good, which is produced as

$$Y(t) = AK(t)^{\alpha} H_P^{1-\alpha}(t)$$

where $K(t)$ is capital and $H(t)$ is human capital, and $H_P(t)$ denotes human capital used in production. The accumulation equations are as follows:

$$\dot{K}(t) = I(t) - \delta K(t)$$

for capital and

$$\dot{H}(t) = BH_E(t) - \delta H(t)$$

where $H_E(t)$ is human capital devoted to education (further human capital accumulation), and the depreciation of human capital is assumed to be at the same rate, δ, as physical capital for simplicity. The resource constraints of the economy are

$$I(t) + C(t) \leq Y(t)$$

and

$$H_E(t) + H_P(t) \leq H(t).$$

1. Interpret the second resource constraint.

2. Denote the fraction of human capital allocated to production by $\phi(t)$ (so that $\phi(t) \equiv H_P(t)/H(t)$) and calculate the growth rate of final output as a function of $\phi(t)$ and the growth rates of accumulable factors.

3. Assume that $\phi(t)$ is constant, and characterize the BGP of the economy (with constant interest rate and constant rate of growth for capital and output). Show that in this BGP, $r^* \equiv B - \delta$ and the growth rate of consumption, capital, human capital and output are given by $g^* \equiv (B - \delta - \rho)/\theta$. Show also that there exists a unique value of $k^* \equiv K/H$ consistent with BGP.

4. Determine the parameter restrictions to make sure that the transversality condition is satisfied.

5. Now analyze the transitional dynamics of the economy starting with K/H different from k^* [Hint: look at dynamics in three variables, $k \equiv K/H$, $\chi \equiv C/K$ and ϕ, and consider the cases $\alpha < \theta$ and $\alpha \geq \theta$ separately].

Question 3: Consider an infinite-horizon economy that admits a representative household with preferences at time 0 given by

$$\int_0^\infty \exp(-\rho t) \frac{C(t)^{1-\theta} - 1}{1-\theta} dt.$$

Population is constant at L and labor is supplied inelastically. The unique final good is produced with the production function

$$Y(t) = \frac{1}{1-\beta} \left[\int_0^{N(t)} x(\nu, t)^{1-\beta} d\nu \right] L^\beta,$$

where $\beta \in (0,1)$, $x(\nu, t)$ denotes intermediate goods of type ν used in final good production at time t, and $N(t)$ is the number of intermediate good types available at time t. Once a particular type of intermediate good is invented, it can be produced by using ψ units of final good. The innovation possibilities frontier of the economy is

$$\dot{N}(t) = \eta Z(t),$$

where $Z(t)$ is total amount of R&D spending, and resource constraint of the economy is $C(t) + X(t) + Z(t) \leq Y(t)$, where $X(t)$ is spending on intermediate goods. There is free entry into research and a firm that invents a new intermediate good type receives a patent and becomes the monopolist producer of that good until the patent expires. Patents expire at the Poisson rate $\iota < \infty$, and once the patent on an intermediate good of a particular type is expired, it is produced competitively. The economy starts with $N(0) > 0$ intermediate goods without patents.

1. Define the equilibrium and balanced growth path (BGP) allocations.

2. Characterize the BGP. [Hint: be explicit about the distribution of intermediate goods between firms with and without patents].

3. Show that starting at time $t = 0$, the economy converges to the BGP. Does it always grow at a constant rate?

4. Show that an increase in ι reduces the BGP growth rate. Is this a realistic prediction? What types of alternatives for generalizations would you need to consider in order to reverse this prediction?

5. Can an increase in ι be welfare improving? Explain your answer intuitively.

Question 4: Consider a model of direct technological change with two sectors, and the final good produced from these two sectors according to the technology:

$$Y(t) = F[Y_L(t), Y_H(t)]$$

where F is linear homogeneous, differentiable and increasing in both of its arguments. Sector $j = L, H$ has the production function

$$Y_j(t) = \frac{1}{1-\beta}\left(\int_0^{N_j(t)} x_j(\nu, t)^{1-\beta}\, d\nu\right) Q_j^{\beta},$$

with $Q_L = L$ and $Q_H = H$, and x's corresponding quantities of machines that depreciate fully after use. The innovation possibilities frontier of the economy takes the form

$$\dot{N}_L(t) = \eta_L Z_L(t) \text{ and } \dot{N}_H(t) = \eta_H Z_H(t),$$

and the resource constraint is $C(t) + X(t) + Z_L(t) + Z_H(t) \leq Y(t)$. The household side is represented by a representative household with CRRA preferences defined over the final good, $Y(t)$. Throughout, the total supplies of the two factors, L and H, are taken as constants.

1. Solve the maximization problem of sectors L and H and show that in equilibrium

$$Y_j(t) = \frac{1}{1-\beta} p_j(t)^{\frac{1-\beta}{\beta}} N_j(t) Q_j,$$

   where $p_j(t)$ is the price of the output of sector j at time t.

2. Define a Balanced Growth Path (BGP) equilibrium as an allocation in which the final good grows at a constant rate and $p_L(t)$ and $p_H(t)$ are constant. Derive a condition for the BGP as a function of L, H, p_L and p_H. Derive relative wage ratio w_H/w_L as a function of L, H, p_L and p_H. Then solve out for p_H/p_L as a function of H/L (Hint: first derive the relationship $p_H/p_L = (Y_H/Y_L)^{-1/\varepsilon}$ where ε is the local elasticity of substitution (of the function F) between Y_H and Y_L, defined as $\varepsilon = \partial \ln(Y_H/Y_L)/\partial \ln(p_H/p_L)$).

3. What is the effect of an increase in H/L on w_H/w_L? Can an increase in H/L raise w_H/w_L? (If not, why not? If yes, under what conditions?). Can you define the notion of induced bias of technology and show that an increase in H/L always makes technology further bias towards factor H? (Hint: first establish this result assuming that the local elasticity of substitution, ε, is constant, and then argue that even if it is not constant, the same result applies).

4. Suppose now that final good producers face a tax rate of τ when purchasing Y_H (i.e., instead of p_H, they pay $(1+\tau)p_H$), the proceeds of which are rebated lump-sum to the representative household. Taking N_L and N_H as given (and constant), find the relation between τ and p_H/p_L. Now, starting from a BGP with $\tau = 0$, consider a permanent increase to $\tau > 0$. Show that this will at first change p_H/p_L but then in finite time p_H/p_L will return to its initial level (to its BGP level with $\tau = 0$). Is taxing sector H inducing further biased towards factor H? Explain and provide the intuition.

Question 5: Consider a world economy that consists of J countries, indexed by $j = 1,..,J$. Each country admits a representative household with preferences at time 0 given by

$$\int_0^\infty \exp(-\rho t) \frac{C_j(t)^{1-\theta} - 1}{1-\theta} dt.$$

Population in each country is constant and given by L_j. Labor is supplied inelastically. The unique final good in each country is produced with the production function

$$Y_j(t) = \frac{1}{1-\beta} \left[\int_0^{N_j(t)} x_j(\nu,t)^{1-\beta} d\nu \right] L_j^\beta,$$

where $\beta \in (0,1)$, $x_j(\nu,t)$ denotes intermediate goods of type ν used in final good production at time t, and $N_j(t)$ is the number of intermediate good types available in country j at time t. Once a particular type of intermediate good is invented, it can be produced by using $\psi \equiv 1-\beta$ units of final good. The innovation possibilities frontier of country j is

$$\dot{N}_j(t) = \eta_j \left(\frac{N(t)}{N_j(t)} \right)^\phi Z_j(t), \tag{1}$$

where $\eta_j > 0$, $\phi > 0$, $Z_j(t)$ is total amount of R&D spending, and

$$N(t) = \Phi(N_1(t), ..., N_J(t)) \tag{2}$$

represents the average technology in the world, where Φ is a linearly homogeneous function (in its J arguments). The resource constraint in each country is $C_j(t)+X_j(t)+Z_j(t) \leq Y_j(t)$, where $X_j(t)$ is spending on intermediate goods. There is free entry into research and a firm that invents a new intermediate good type receives a perpetual patent and becomes the monopolist producer of that good. Each country starts with $N_j(0) > 0$ intermediate goods at time $t = 0$.

1. Define a world equilibrium.

2. Consider a BGP equilibrium in which the growth rate in each country is the same and constant at g. Characterize the BGP equilibrium output and monopolist profits in each country, and determine the free entry condition. Use the free-entry condition to solve for the relative level of output $v_j \equiv \frac{N_j(t)}{N(t)}$ in each country j, as a function of the world growth rate g. Show that countries with a greater η_j and L_j have greater relative levels of output on the BGP.

3. Is the allocation characterize in 2 the world equilibrium? Is it asymptotically stable? [Hint: you can answer these questions without doing any algebra, just giving the high-level idea.] Why is it that countries with different levels of η_j and L_j differ in terms of their relative incomes but not their growth rates?

4. Suppose that $\Phi(N_1(t), ..., N_J(t)) = \max\{N_1(t), ..., N_J(t)\}$. Derive the world growth rate. Now suppose that $J = 2$ and both countries are identical. How does this change the equilibrium? Is the world equilibrium "symmetric" (meaning the two countries investing same amount in R&D and achieving the same level of income)? Now suppose that the level of R&D in each country is determined by a country-welfare maximizing social planner (so that we have a game between two country social planners). Argue (without providing algebraic details) whether you expect the world equilibrium in this modified to still be symmetric.