THREE ESSAYS IN MECHANISM DESIGN

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Abstract

In the first and second chapters we study whether the current trend of using stronger solution concepts is justified for the optimal mechanism design. In the first chapter, we take a simple auction model and allow for type-dependent outside options. We argue that Bayesian foundation for dominant strategy mechanisms is valid when symmetry conditions are satisfied. This contrasts with monotonicity constraints used before in the literature. In the second chapter we develop the idea further by looking into the practical application of type-dependency of outside options in auctions - namely, a possibility of collusion between agents. We show that in this environment for a certain range of primitives no maxmin foundation for dominant strategy mechanisms will exist. Finally, in the last chapter we study a voting environment and non-transferable utility mechanism design. We argue that strategic voting as opposed to truthful voting may lead to higher total welfare through better realization of preference intensities in the risky environment. We also study optimal mechanisms rules, that are sufficiently close to the first best for the uniform distribution, and argue that strategic voting may be a proxy for information transmission if the opportunities to communicate preference intensities are scarce.

Acknowledgments

To my mother who imbued me with a strong sense of fairness and my father who taught me to always strive for the ideal.

To my adviser who always believed in this research.

To all my classmates from whom I learned a lot.

To all the living and dead poets, who seek to convey the deepest meaning by the limited amount of words and motivate us to go on.

Contents

Chapter 1

Generalized Individual Rationality Constraints and a Bayesian Foundation for Dominant Strategy Mechanisms.

1.1 Introduction

In line with the Wilson doctrine, the literature in mechanism design has been trying to come up with mechanisms robust to the relaxation of common knowledge assumptions. In particular, it has been important to relax the assumption that distribution of payoff types of different players is fixed and perceived by everybody to be the same and that to each payoff type corresponds exactly one possible belief of the types of the other players coming from the common prior (the "naive type space" assumption). Generally speaking, this relaxation from the naive type space to universal type space, a space allowing all possible beliefs and higher-order beliefs about other players types, might give rise to mechanisms that can be very complicated. However, the number (the power of the set) of constraints a mechanism designer has to satisfy also increases dramatically. Thus, a recent paper (Bergemann and Morris (2005)) shows that if efficiency of the mechanism is to be maintained one has to design an ex post incentive compatible mechanism in various environments and for different solution concepts.

Nevertheless, the gap between possible mechanisms that would be optimal rather than efficient and between ex post incentive compatible mechanisms (dominant strategy mechanisms in private value environments) was harder to bridge. Indeed, it is not clear why despite the standard logic of mechanism design literature, a mechanism designer should not ask all the relevant information including higher order beliefs if he is to maximize total revenue. One of the attempts to bridge this gap has been Chung and Ely (2007): in an auction environment, they try to establish a "maxmin" foundation for dominant strategy mechanisms - an idea that if a mechanism designer is extremely risk-averse and believes that he should only consider worst possible beliefs and higher order beliefs for the game he designed, than any Bayesian incentive compatible mechanism would perform worse than a dominant strategy mechanism which is independent of any beliefs. In fact, the authors prove even a stronger result in an environment satisfying their regularity condition. They are able to show that there exists a single belief about other players' types, such that if a mechanism designer holds this belief the best mechanism he can come up with is again a dominant strategy mechanism. Then, it can be argued that dominant strategy mechanism is optimal in a Bayesian sense for a mechanism designer holding this particular belief.

Our departing point from that paper is to allow type-dependent outside options. Type-dependent utilities have resulted naturally in many practical applications covering labor contracts with type-dependent reservation wages (Kahn (1985), Moore (1985)), regulation of electricity bypass (Laffont and Tirole (1990)), regulation of monopoly with fixed costs (Lewis and Sappington (1989)), models of international trade (Brainard and Martimort (1996)), models of corruption (Saha (2001)) and models of competition between several principals (Biglaiser and Mezzetti (1993, 2000)). Our main motivation for type-dependent outside options will be a possibility of collusion between bidders, the problem we study in the next chapter. Generally type-dependent outside options make the models harder to analyze since it will be no longer the case that only incentive compatibility constraints of the most efficient types will be binding. Indeed, if an outside option of efficient types is high enough, mechanism designer will have to increase transfers to that type, which may make a contract for the efficient type attractive to the inefficient types. This idea that incentive compatibility constraints of the less efficient types may be binding in type-dependent outside options models has received the name of "countervailing incentives" (coined by Lewis and Sappington (1989)).

Thus, the purpose of this chapter is to extend the auction environment allowing type-dependent outside options and investigate whether Bayesian and maxmin foundations continue to hold for such an environment. The regularity condition of Chung and Ely was necessary for them to show that monotonicity constraints in the dominant strategy mechanism never bind and so under some conjecture of a mechanism designer about beliefs and higher order neliefs of players and after some simplifications the best Bayesian mechanism has the same objective function as the objective function of the best dominant strategy mechanism. We argue that what matters for establishing Bayesian and maxmin foundation is not whether monotonicity constraints bind, but whether there is a symmetry in binding individual rationality and incentive compatibility constraints for a bidder across different types of the other bidders. In Chung and Ely's environment, if monotonicity constraints do not bind, it is always the case that the binding constraints of the dominant strategy mechanism are incentive compatibility constraints between a high type and a type directly lower and individual rationality constraint of the lowest type independent of the types of the other bidders. Only when monotonicity constraints do bind, there can be an asymmetry. It is this asymmetry, we argue, that prevents us from establishing maxmin foundation. To prove our point we consider only such valuations that regularity condition is always satisfied, but there can be asymmetry in binding constraints. We also construct an example for the asymmetric case that shows that there exist no Bayesian foundation for any conjecture of the auctioneer on the universal type space. Since in our environment monotonicity constraints never bind and since the break-up of monotonicity constraints in Chung and Ely (2007) environment implies asymmetry of binding constraints, we emphasize that symmetry conditions rather than regularity condition are crucial to understanding of whether there exist a Bayesian and maxmin foundation for dominant strategy mechanisms.

The structure of the paper is the following. In section 2, we introduce the model and relevant concepts, In section 3, we review dominant strategy mechanisms and find a best one for the generalized individual rationality constraints. In section 4, we solve the "minmax" problem (finding worst possible beliefs to which a mechanism designer can respond with the best possible mechanism) for payoff type spaces arguing that finite mechanism designer revenue is achieved only on a rectangle of beliefs of the low type and high type of a player and that minimal revenue is actually a minimum of those achieved at three specific belief points. We also derive conditions under which Bayesian and "maxmin" foundation can be established. In section 5, we construct an example that shows that when symmetry conditions are not satisfied dominant strategy mechanism may not have a Bayesian foundation. In section 6, we review monotonicity conditions of dominant strategy mechanisms in the case of standard individual rationality constraints and argue that when they are binding incentive compatibility and individual rationality constraints for different values of one player may bind differently across different values of the other player.

1.2 Preliminaries

1.2.1 Auction environment

A single indivisible unit is offered in an auction. There are two risk-neutral bidders and each bidder has a low and a high types. Valuations are assumed to be private with the sets of possible valuation $V_i = \left\{v_i^l, v_i^h\right\}$. The bidders valuations are distributed with some probability distribution $\nu \in \Delta\left(\times_i V_i\right)$. We assume that ν has full support. A bidder's expected utility U_i is given by $U_i = p_i v_i - t_i$, where p_i is the probability of getting the object and t_i an expected transfer to the mechanism designer.

1.2.2 Types

To finish the description of environment in which bidders compete for the object, one must specify not just valuations and distribution from which they come from, but also beliefs and higher-order beliefs of the bidders. We model beliefs and higher-order beliefs by using type spaces $\Omega = (\Omega_i, \theta_i, \pi_i)_{i=1,2}$, where Ω_i is a measurable set of types, $\theta_i : \Omega_i \to V_i$ denotes a measurable mapping from types to valuations of these types and $\pi_i : \Omega_i \to \Delta\Omega_i$ is a measurable mapping determining beliefs of every type.

For our purposes it will be sufficient to deal only with payoff and universal type spaces. Payoff type spaces, usually with an additional common prior property, were used almost exclusively in auction theory and mechanism design until recently. The payoff type space has the property that for every valuation v_i, there exist just one type ω^{v_i}, such that $\theta_i(\omega^{v_i}) = v_i$ and, thus, for every valuation there exist just one belief. As have been shown in Bergmann and Morris (2005), Neeman (2004) such an assumption is with loss of generality and Wilson Doctrine dictates us to avoid making it.

Thus, the response in recent literature was to maintain narrow type spaces, but limit dangers of such assumption by employing stronger solution concepts, such as dominant strategy mechanisms. Chung and Ely (2007) investigate whether such an approach is reasonable by stripping all implicit assumptions about type spaces and considering the most general type space, a universal type space as in Mertens and

Zamir (1985), where $\Omega^* = (\Omega_i^*, \theta_i^*, \pi_i^*)_i$ with every Ω_i being a compact topological space and with the property that for every valuation v_i and for every infinite hierarchy of beliefs $\hat{\pi}_i$, there exist a type of player i, ω_i, such that $\theta_i(\omega_i) = v_i$ and $\pi_i(\omega_i) = \hat{\pi}_i$. In their auction environment they investigate whether there may exist a conjecture of an auctioneer about types, a distribution μ over Ω^*, which rationalizes in some sense (which will be discussed later) the use of dominant strategy mechanisms. If this is the case they say that there exist a Bayesian foundation for using a dominant strategy mechanism.

In our paper we will try to investigate whether such rationalization exist by relaxing assumptions about auction environment.[1] In particular, we will consider tighter individual rationality constraints and use either payoff or universal type spaces.

1.2.3 Mechanisms

Once a type space is fixed, the mechanism designer has to come up with a best possible mechanism.

Definition 1.1. A mechanism consists of a set of messages M_i for each bidder i, an allocation rule $p: \ M \to [0;1]^N$ and a transfer function $t: \ M \to R^N$

Thus, every bidder chooses a message in M_i, and according to the total profile of messages m, receives an object with probability $p_i(m)$ and pays to an auctioneer $t_i(m)$.

[1]Although we consider only two bidders, two valuations model, it is not important to our conclusions: when symmetry conditions are satisfied Bayesian and maxmin foundations will continue to exist and if they are not satisfied it will be just as easy to construct an example when no Bayesian foundation exists.

We assume that outside options $a_1(V_1, V_2)$ and $a_2(V_1, V_2)$ of every player depend only on payoff valuations and are given by the set of $\{a_1^{11}, a_1^{12}, a_1^{21}, a_1^{22}\}$ and $\{a_2^{11}, a_2^{12}, a_2^{21}, a_2^{22}\}$, where the first lower index indicates whether the first player is low or high (1 for the low valuation) and the second index indicates whether the second player is low or high (1 for the low valuation). We also assume that every set of messages M_i must have a message of $\emptyset_i$, such that if some $m_i = \{\emptyset_i\}$, $p_i(m) = 0$ and $t_i(m) = 0$, while $U_i = a_i(V_1, V_2)$ so that player i 'quits' the mechanism and receives his outside option. We would also require that both players have to participate in the mechanism, so that we could abstract from the analysis of when it is optimal to serve just one player. Therefore, it is possible to imagine a situation when an object is sold to one player, but the other has to get a positive surplus. As we would see in the next chapter this is natural in environments where collusion threat is real since eliminating bidders in auctions may lower reservation price and, hence, a mechanism designer may want to attract more bidders. In other environments that feature type-dependent outside options and utility functions that are non-linear in the quantity of the allocated resource, it has been shown that it is possible in an optimum to reach a situation when a mechanism designer sells positive quantity to a low type of a player, this type gets a positive surplus because of binding incentive constraints and it is optimal for a mechanism designer to still serve this type (see, e.g., a bypass problem analyzed by Laffont and Tirole (1990) and a problem of providing insurance contracts by monopoly analyzed by Stiglitz (1997)). In our environment the quantity of the allocated resource is the probability of receiving the object and since utilities are linearly dependent on it, we would often get that the optimal 'quantity' is zero for one player, but that player would still get a positive surplus.

A direct revelation mechanism for a given type space Ω is such where $M_i = \Omega_i \cup \{\emptyset_i\}$. For the payoff type space, the mechanism depends only on types' valuations and we simplify notation by setting $p_i\left(v_1^l, v_2^l\right) = p_i^{11}$, $p_i\left(v_1^l, v_1^h\right) = p_i^{12}$, $p_i\left(v_1^h, v_2^l\right) = p_i^{21}$, $p_i\left(v_1^h, v_2^h\right) = p_i^{22}$ and similarly for the transfers.

For a given mechanism under a fixed type space, we have a game of incomplete information and the mechanism designer has to adopt a solution concept and, then, try to design a mechanism that will maximize total revenue under this solution concept. To minimize dangers of using simple type spaces, the common approach has been to adopt a strong solution concept that wouldn't depend on the fine details of that simple type space. In particular, for independent valuations, a dominant strategy mechanism is often used.

Definition 1.2. A direct revelation mechanism is dominant strategy incentive compatible under type space Ω, if for each bidder and every type profile $\omega \in \Omega$ and for each possible misreport of a type $\hat{\omega}_i$ individual rationality and incentive compatibility constraints are satisfied.

$$p_i\left(\omega\right)\theta_i\left(\omega_i\right) - t_i\left(\omega\right) \geq a_i\left(\theta_i\left(\omega_i\right), \theta_i\left(\omega_{-i}\right)\right)$$

$$p_i\left(\omega_i, \omega_{-i}\right)\theta_i\left(\omega_i\right) - t_i\left(\omega_i, \omega_{-i}\right) \geq p_i\left(\hat{\omega}_i, \omega_{-i}\right)\theta_i\left(\omega_i\right) - t_i\left(\hat{\omega}_i, \omega_{-i}\right)$$

When Ω is a payoff type space the above constraints depend only on valuations and, thus, we don't have to specify beliefs on this type space when we say that a mechanism is dominant strategy incentive compatible for a payoff type space.

To provide a foundation for using stronger solution concepts when trying to avoid any assumptions on type spaces, Chung and Ely eliminate any assumptions on beliefs (i.e. use a universal type space), while keeping the standard solution concept of Bayesian equilibrium.

Definition 1.3. A direct-revelation mechanism for type space $\Omega = (\Omega_i, \theta_i, \pi_i)$ is Bayesian incentive compatible (BIC) if for each bidder i and every type $\omega_i \in \Omega_i$

$$\int_{\Omega_{-i}} (p_i(\omega)\,\theta_i(\omega_i) - t_i(\omega))\,\pi_i(\omega_i)\,d\omega_{-i} \geq \int_{\Omega_{-i}} a_i(\theta_i(\omega_i), \theta_{-i}(\omega_{-i}))\,\pi_i(\omega_i)\,d\omega_{-i}$$

$$\int_{\Omega_{-i}} [(p_i(\omega_i, \omega_{-i})\,\theta_i(\omega_i) - t_i(\omega_i, \omega_{-i})) - (p_i(\hat{\omega}_i, \omega_{-i})\,\theta_i(\omega_i) - t_i(\hat{\omega}_i, \omega_{-i}))]\,\pi_i(\omega_i)\,d\omega_{-i} \geq 0$$

The class of all BIC mechanisms is Ψ.

In our paper we will investigate whether dominant strategy mechanism can be optimal in the class of all BIC nechanisms under a Bayesian or maxmin criteria.

1.2.4 Decision-making. Bayesian and maxmin foundations.

Given the valuation distribution ν (assumed to be known to the auctioneer), the mechanism designer makes a conjecture about types μ that is consistent with valuation distribution. Thus, the marginal of μ on V must be equal to ν. We call the compact subset of all such conjectures $M(\nu)$. Under every conjecture we can calculate mechanism designer's revenue under a mechanism Γ.

$$R_\mu(\Gamma) = \int_{\Omega^*} (t_1(\omega) + t_2(\omega))\,d\mu(\omega)$$

Definition 1.4. A mechanism Γ has a Bayesian foundation if there exist such a conjecture $\mu \in M(\nu)$ that

$$R_{\mu}(\Gamma) = \sup_{\Gamma' \in \Psi} R_{\mu}\left(\Gamma'\right)$$

In other words mechanism Γ is the best a mechanism designer can do if he has a conjecture μ about types in the universal type space.

Definition 1.5. A mechanism Γ has a maxmin foundation if it is the mechanism that solves the following problem

$$\sup_{\Gamma'} \inf_{\mu} R_{\mu}\left(\Gamma'\right)$$

We can interpret, then, a decision to use such a mechanism Γ as one of a cautious mechanism designer who is unsure about distribution of types in the universal type space and who considers a worst possible case. Alternatively, one can think about this in the following terms. When a mechanism designer chooses a mechanism Γ', nature chooses the worst possible distribution of types $\mu\left(\Gamma'\right)$ to minimize his revenue. Knowing this a mechanism designer chooses optimally a mechanism Γ.

It is immediate to realize that a mechanism designer can do no worse than the optimal dominant strategy mechanism. However, Chung and Ely (2007) also showed that if some regularity condition on distribution and values is satisfied, there always exists a conjecture $\mu^{*}(\nu)$ under which a mechanism designer can do no better. Then, if $\Pi^{D}(\nu)$ is the maximum revenue under optimal dominant strategy mechanism,

$$\Pi^{D}(\nu) = \sup_{\Gamma \in \Psi} R_{\mu^{*}}(\Gamma) \geq \inf_{\mu \in M(\nu)} \sup_{\Gamma \in \Psi} R_{\mu}(\Gamma) \geq \sup_{\Gamma \in \Psi} \inf_{\mu \in M(\nu)} R_{\mu}(\Gamma)$$

and, hence, if a regularity condition is satisfied, dominant strategy mechanism has both a Bayesian foundation and a maxmin foundation.

We begin our analysis by studying dominant strategy mechanisms in the enlarged environment of potentially tighter individual rationality constraints.

1.3 Dominant strategy mechanism with generalized individual rationality constraints.

For generalized IR constraints it is no longer the case that individual rationality constraints of a low type and incentive compatibility constraints of a high type are always binding . The next lemma shows that depending on individual rationality bounds, there can exist three cases: when both IR constraints bind, when an IR constraint of a low type and IC constraint of a high type bind, and finally when an IC constraint of a low type and IR constraint of a high type bind.

Lemma 1.1. *If a mechanism designer maximizes revenue obtained from a first player (similarly for a second player)* $\alpha t_1^{11}+\beta t_1^{12}+\gamma t_1^{21}+\delta t_1^{22}$ *in a dominant strategy mechanism that satisfies generalized* IR *constraints:*

$$p_1^{11}v_1^l - t_1^{11} \geq a_1^{11} \tag{1.1}$$

$$p_1^{12}v_1^l - t_1^{12} \geq a_1^{12} \tag{1.2}$$

$$p_1^{21}v_1^h - t_1^{21} \geq a_1^{21} \tag{1.3}$$

$$p_1^{22} v_1^h - t_1^{22} \geq a_1^{22} \tag{1.4}$$

and IC constraints:

$$p_1^{11} v_1^l - t_1^{11} \geq p_1^{21} v_1^l - t_1^{21} \tag{1.5}$$

$$p_1^{21} v_1^h - t_1^{21} \geq p_1^{11} v_1^h - t_1^{11} \tag{1.6}$$

$$p_1^{12} v_1^l - t_1^{12} \geq p_1^{22} v_1^l - t_1^{22} \tag{1.7}$$

$$p_1^{22} v_1^h - t_1^{22} \geq p_1^{12} v_1^h - t_1^{12} \tag{1.8}$$

then t_1^{1j}, t_1^{2j}, where j denotes the low or high type of another player, are determined by:

$$\text{If } a_1^{2j} - a_1^{1j} \leq p_1^{1j}\left(v_1^h - v_1^l\right), \text{ then } \begin{cases} t_1^{1j} = p_1^{1j} v_1^l - a_1^{1j} \\ t_1^{2j} = p_1^{2j} v_1^h - p_1^{1j}\left(v_1^h - v_1^l\right) - a_1^{1j} \end{cases}$$

$$\text{If } p_1^{1j}\left(v_1^h - v_1^l\right) \leq a_1^{2j} - a_1^{1j} \leq p_1^{2j}\left(v_1^h - v_1^l\right), \text{ then } \begin{cases} t_1^{1j} = p_1^{1j} v_1^l - a_1^{1j} \\ t_1^{2j} = p_1^{2j} v_1^h - a_1^{2j} \end{cases}$$

$$\text{If } a_1^{2j} - a_1^{1j} \geq p_1^{2j}\left(v_1^h - v_1^l\right), \text{ then } \begin{cases} t_1^{1j} = p_1^{1j} v_1^l + p_1^{2j}\left(v_1^h - v_1^l\right) - a_1^{2j} \\ t_1^{2j} = p_1^{2j} v_1^h - a_1^{2j} \end{cases}$$

Proof. First of all, notice that the problem can be treated as two separate problems, with the first one maximizing $\alpha t_1^{11} + \gamma t_1^{21}$ under constraints (1.1), (1.3), (1.5), (1.6) and the second one is maximizing $\beta t_1^{12} + \delta t_1^{22}$ under the constraints (1.2), (1.4), (1.7), (1.8). So we can concentrate on the first problem with the second one being symmetric. Second, notice that at least one constraint out of any pair of constraints (1.1) and (1.5), (1.3) and(1.6) must be binding as otherwise it would be possible to increase t_1^{11} or t_1^{21}. Also notice that when (1.5) and(1.6) are both binding (and

$t_1^{11} = t_1^{21}$, while $p_1^{11} = p_1^{21}$) either (1.1) or (1.3) must still be binding, as otherwise we could increase both t_1^{11} and t_1^{21}. Hence, without loss of generality we could conclude that there could be only three cases. When at least both IR constraints are binding, at least IR of a low type and at least IC for a high type constraints are binding, and IR of a high type and IC of a low type constraints are binding.

If both IR constraints bind, we have

$$t_1^{1j} = p_1^{1j} v_1^l - a_1^{1j}$$

$$t_1^{2j} = p_1^{2j} v_1^h - a_1^{2j}$$

to have this solution the inequalities from IC must be satisfied

$$\left(p_1^{2j} - p_1^{1j}\right) v_1^l \leq t_1^{2j} - t_1^{1j} = p_1^{2j} v_1^h - a_1^{2j} - p_1^{1j} v_1^l + a_1^{1j} \leq \left(p_1^{2j} - p_1^{1j}\right) v_1^h$$

or

$$p_1^{1j}\left(v_1^h - v_1^l\right) \leq a_1^{2j} - a_1^{1j} \leq p_1^{2j}\left(v_1^h - v_1^l\right) \tag{1.9}$$

If IR constraint of a low type and IC constraint of a high type are binding, we have

$$t_1^{1j} = p_1^{1j} v_1^l - a_1^{1j}$$

$$t_1^{2j} = p_1^{2j} v_1^h - p_1^{1j} v_1^h + t_1^{1j} = p_1^{2j} v_1^h - p_1^{1j}\left(v_1^h - v_1^l\right) - a_1^{1j}$$

and the IR constraint of a high type and IC constraint of a low type must be satisfied

$$p_1^{2j} v_1^h - p_1^{1j}\left(v_1^h - v_1^l\right) - a_1^{1j} \leq p_1^{2j} v_1^h - a_1^{2j} \Leftrightarrow a_1^{2j} - a_1^{1j} \leq p_1^{1j}\left(v_1^h - v_1^l\right) \tag{1.10}$$

$$\left(p_1^{2j}-p_1^{1j}\right)v_1^h \geq \left(p_1^{2j}-p_1^{1j}\right)v_1^l$$

If IR constraint of a high type and IC constraint of a low type is binding

$$t_1^{2j}=p_1^{2j}v_1^h-a_1^{2j}$$

$$t_1^{1j}=-\left(p_1^{2j}-p_1^{1j}\right)v_1^l+t_1^{2j}=p_1^{1j}v_1^l+p_1^{2j}\left(v_1^h-v_1^l\right)-a_1^{2j}$$

and the IR constraint of a low type and IC constraint of a high type must be satisfied

$$p_1^{1j}v_1^l+p_1^{2j}\left(v_1^h-v_1^l\right)-a_1^{2j} \leq p_1^{1j}v_1^l-a_1^{1j} \Longleftrightarrow a_1^{2j}-a_1^{1j} \geq p_1^{2j}\left(v_1^h-v_1^l\right) \quad (1.11)$$

$$\left(p_1^{2j}-p_1^{1j}\right)v_1^l \leq \left(p_1^{2j}-p_1^{1j}\right)v_1^h$$

We conclude thus that a) monotonicity constraints $p_1^{2j} \geq p_1^{1j}$ must be satisfied and b) inequalities (1.9), (1.10), (1.11) determine the solution since these inequalities are mutually exclusive and completely exhaustive. □

Next lemma establishes that the optimal probabilities are such that monotonicity constraints are always satisfied.

Lemma 1.2. *If values are such that values such that $v_1^l < v_2^l < v_1^h < v_2^h$, then regardless of a_1^{ij} and a_2^{ij}, $p_1^{21}=1$ and $p_2^{12}=1$. Hence, it is always the case that optimal probabilities are such that $p_1^{21} \geq p_1^{11}$, $p_1^{22} \geq p_1^{12}$, $p_2^{12} \geq p_2^{11}$ and $p_2^{22} \geq p_2^{21}$.*

Proof. First of all, notice from lemma 1 that transfers are continuous functions of probabilities. Next, notice that the coefficient before p_1^{21} is either equal to γv_1^h if $a_1^{21}-a_1^{11} \leq p_1^{21}\left(v_1^h-v_1^l\right)$ or $\gamma v_1^h+\alpha\left(v_1^h-v_1^l\right)$ if $a_1^{21}-a_1^{11} \geq p_1^{21}\left(v_1^h-v_1^l\right)$. As for the coefficient before p_2^{21} it is either equal to γv_2^l if $b_{22}-b_{21} \geq p_2^{21}\left(v_2^h-v_2^l\right)$ or $\gamma v_2^l-\delta v_2^h$

if $b_{22}-b_{21} \leq p_2^{21}\left(v_2^h-v_2^l\right)$. Since $\gamma v_1^h > \gamma v_2^l$ according to our assumption, it's always optimal to increase p_1^{21} as much as possible.

The proof that $p_2^{12}=1$ is similar with the minimum coefficient for p_2^{12} equal to βv_2^h and a maximum coefficient before p_1^{12} equal to βv_1^l. □

These two lemmas essentially establish that the optimal revenue from the best dominant strategy mechanism can be expressed in the following way

$$\begin{aligned} R_1^{dsm}\left(p_1^{11},p_1^{12},p_1^{21},p_1^{22}\right) &= \alpha \min\left\{p_1^{11}v_1^l-a_1^{11},p_1^{11}v_1^l+p_1^{21}\left(v_1^h-v_1^l\right)-a_1^{21}\right\}+ \\ &+ \beta \min\left\{p_1^{12}v_1^l-a_1^{12},p_1^{12}v_1^l+p_1^{22}\left(v_1^h-v_1^l\right)-a_1^{22}\right\}+ \\ &+ \gamma \min\left\{p_1^{21}v_1^h-a_1^{21},p_1^{21}v_1^h-p_1^{11}\left(v_1^h-v_1^l\right)-a_1^{11}\right\}+ \\ &+ \delta \min\left\{p_1^{22}v_1^h-a_1^{22},p_1^{22}v_1^h-p_1^{12}\left(v_1^h-v_1^l\right)-a_1^{12}\right\} \end{aligned}$$

with the second term under the minimum sign never chosen at the same time for low and high type of the first player under the same type of the second player.

1.4 Maxmin foundation for dominant strategy mechanism.

We start investigating whether there exist a maxmin foundation for dominant strategy mechanisms by considering simple payoff-type spaces. As was discussed in the preliminaries, we can then represent a mechanism by a set of probabilities $\{p_i^{11},p_i^{12},p_i^{21},p_i^{22}\}$and transfers $\{t_i^{11},t_i^{12},t_i^{21},t_i^{22}\}$. Also, assume a simple structure of beliefs on the payoff type space. Suppose the belief of a low (high) player i that player $-i$ is a low type is given by μ_i (λ_i). Then, we extend naturally the higher order beliefs. For example, a low type of player 1 believes with probability μ_1 that the second player is a low type who believes with probability μ_2 that the first player is a low type and with probability $(1-\mu_2)$ that the first player is a high type. With probability $(1-\mu_1)$ the low type of the first player believes that the second player is a high type who believes with probability λ_2 that the first player is a low type and with remaining probability $(1-\lambda_2)$ that the first player is a high type. The higher order believes are determined in a similar fashion. As was shown in Chung and Ely (2007) considering such simplistic type spaces is sufficient for establishing maxmin foundation of dominant strategies. We study whether considering these type spaces is enough to establish a maxmin foundation for generalized auction environment.

It will be shown that generally revenue from an optimal Bayesian mechanism on such type spaces is non-smaller than revenue from the optimal dominant strategy mechanism. However, for some conditions on bounds, two revenues are the same. That means that for these conditions there exist very simple conjectures on types in the universal type space that rationalize dominant strategy mechanisms and, thus, establish maxmin foundation. Hence the focus of this section is to find such μ_1 and λ_1 that will minimize the revenue of the mechanism designer.

Similarly to the section on dominant strategy mechanism, we will first find optimal transfers as functions of probabilities to receive a good. As before, at this stage maximizing revenue obtained from the first player is independent of maximizing revenue from the second player. The sub-problem of maximizing transfers obtained from the first player is characterized by an objective function of $R_1^{B.m} = \alpha t_1^{11} + \beta t_1^{12} + \gamma t_1^{21} + \delta t_1^{22}$ subject to the following IC and IR constraints:

$$\mu_1 t_1^{11} + (1-\mu_1)\, t_1^{12} \leq \mu_1 \left(p_1^{11} v_1^l - a_1^{11}\right) + (1-\mu_1)\left(p_1^{12} v_1^l - a_1^{12}\right) \tag{1.12}$$

$$\lambda_1 t_1^{21} + (1-\lambda_1)\, t_1^{22} \leq \lambda_1 \left(p_1^{21} v_1^h - a_1^{21}\right) + (1-\lambda_1)\left(p_1^{22} v_1^h - a_1^{22}\right) \tag{1.13}$$

$$\mu_1 t_1^{11} + (1-\mu_1)\, t_1^{12} \leq \mu_1 t_1^{21} + (1-\mu_1)\, t_1^{22} + v_1^l \left(\mu_1 \left(p_1^{11} - p_1^{21}\right) + (1-\mu_1)\left(p_1^{12} - p_1^{22}\right)\right) \tag{1.14}$$

$$\lambda_1 t_1^{21} + (1-\lambda_1)\, t_1^{22} \leq \lambda_1 t_1^{11} + (1-\lambda_1)\, t_1^{12} + v_1^h \left(\lambda_1 \left(-p_1^{11} + p_1^{21}\right) + (1-\lambda_1)\left(-p_1^{12} + p_1^{22}\right)\right) \tag{1.15}$$

Next lemma derive a necessary condition for beliefs so that the optimal revenue is finite.

Lemma 1.3. *For maximal revenue obtained from a mechanism above to be finite, it has to be the case that*

$$\mu_1 \in \left[\min\left\{\alpha+\gamma, \frac{\alpha}{\alpha+\beta}\right\}, \max\left\{\alpha+\gamma, \frac{\alpha}{\alpha+\beta}\right\}\right]$$

$$\lambda_1 \in \left[\min\left\{\alpha+\gamma, \frac{\gamma}{\gamma+\delta}\right\}, \max\left\{\alpha+\gamma, \frac{\gamma}{\gamma+\delta}\right\}\right]$$

Proof. Suppose we have some finite transfers t_1^* that satisfy above constraints (the domain set is always non-empty). Then, we will consider four types of perturbations to those transfers so that three of the four constraints are always satisfied. Essentially those perturbations to transfers will be bets for the low or high type of the first player (or for both types) on whether the second player will be low or high.

1. Bets for the low type of the first player. Consider the following transfers to the mechanism designer $t_1 = t_1^* + a\left(\varepsilon, -\frac{\mu_1}{1-\mu_1}\varepsilon, 0, 0\right)$, where a is some positive constant. Notice that both IR and IC constraint for a low type are always satisfied. It has to be the case that when IC constraint for a high type is satisfied (so that the high type doesn't accept this bet), the change in mechanism designer's revenue is

negative. Hence the following implication must be true

$$0 \leq \lambda_1 \varepsilon - \frac{(1-\lambda_1)\,\mu_1}{1-\mu_1}\varepsilon \Rightarrow \alpha\varepsilon - \beta\tfrac{\mu_1}{1-\mu_1}\varepsilon \leq 0$$

or

$$0 \leq \frac{\lambda_1 - \mu_1}{1-\mu_1}\varepsilon \Rightarrow \left(\alpha - \beta\tfrac{\mu_1}{1-\mu_1}\right)\varepsilon \leq 0$$

Hence it must be the case that

For $\lambda_1 > \mu_1 \Rightarrow \mu_1 \geq \frac{\alpha}{\alpha+\beta}$

For $\lambda_1 < \mu_1 \Rightarrow \mu_1 \leq \frac{\alpha}{\alpha+\beta}$

For $\lambda_1 = \mu_1 \Rightarrow \mu_1 = \frac{\alpha}{\alpha+\beta}$

2. Bets for the high type of the first player. Now, let's consider such a perturbation that both IR and IC constraint of a high type is always satisfied and $t_1 = t_1^* + a\left(0, 0, \varepsilon, -\frac{\lambda_1}{1-\lambda_1}\varepsilon\right)$. Then, the following implication must be true.

$$0 \leq \mu_1 \varepsilon - \frac{(1-\mu_1)\,\lambda_1}{1-\lambda_1}\varepsilon \Rightarrow \gamma\varepsilon - \delta\tfrac{\lambda_1}{1-\lambda_1}\varepsilon \leq 0$$

or

$$0 \leq \frac{\mu_1 - \lambda_1}{1-\lambda_1}\varepsilon \Rightarrow \left(\gamma - \delta\tfrac{\lambda_1}{1-\lambda_1}\right)\varepsilon \leq 0$$

Hence it must be the case that

For $\lambda_1 > \mu_1 \Rightarrow \lambda_1 \leq \frac{\gamma}{\gamma+\delta}$

For $\lambda_1 < \mu_1 \Rightarrow \lambda_1 \geq \frac{\gamma}{\gamma+\delta}$

For $\lambda_1 = \mu_1 \Rightarrow \lambda_1 = \frac{\gamma}{\gamma+\delta}$

3. Bets for both types of the first player.

Suppose now $t_1 = t_1^* + a\left(\varepsilon, -\frac{\mu_1}{1-\mu_1}\varepsilon, \varepsilon, -\frac{\mu_1}{1-\mu_1}\varepsilon\right)$. Notice that for this perturbation, individual rationality constraint of a low type is always satisfied and IC

constraints for both types are satisfied. Hence, it must be the case that whenever individual rationality constraint of a high type is satisfied, the change in revenue is negative.

$$0 \geq \frac{\lambda_1 - \mu_1}{1 - \mu_1}\varepsilon \Rightarrow \left((\alpha + \gamma) - (\beta + \delta) \tfrac{\mu_1}{1-\mu_1} \right) \varepsilon \leq 0$$

and hence it must be the case that

For $\lambda_1 > \mu_1 \Rightarrow \mu_1 \leq \alpha + \gamma$

For $\lambda_1 < \mu_1 \Rightarrow \mu_1 \geq \alpha + \gamma$

For $\lambda_1 = \mu_1 \Rightarrow \mu_1 = \alpha + \gamma$

4. Bets for both types of the first player. Finally, suppose

$$t_1 = t_1^* + a \left(\varepsilon, -\frac{\lambda_1}{1 - \lambda_1}\varepsilon, \varepsilon, -\frac{\lambda_1}{1 - \lambda_1}\varepsilon \right)$$

and individual rationality constraint of a high type is always satisfied. Then, it must be the case that

$$0 \geq \frac{\mu_1 - \lambda_1}{1 - \lambda_1}\varepsilon \Rightarrow \left((\alpha + \gamma) - (\beta + \delta) \tfrac{\lambda_1}{1-\lambda_1} \right) \varepsilon \leq 0$$

from which it follows

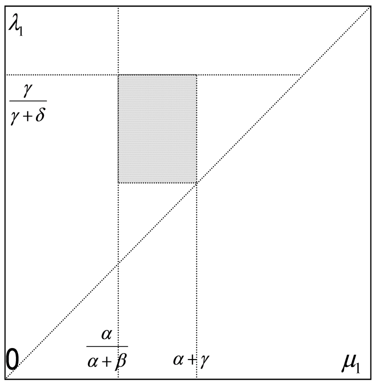
For $\lambda_1 > \mu_1 \Rightarrow \lambda_1 \geq \alpha + \gamma$

For $\lambda_1 < \mu_1 \Rightarrow \lambda_1 \leq \alpha + \gamma$

For $\lambda_1 = \mu_1 \Rightarrow \lambda_1 = \alpha + \gamma$

Taking everything together, we get exactly the conditions in the statement of the lemma. □

In theory there could be other bets that give the mechanism designer infinite profits, however for our purposes it is enough to concentrate on the beliefs specified in the graph below. We draw a case of $\frac{\gamma}{\gamma+\delta} > \frac{\alpha}{\alpha+\beta}$ with the other case being completely symmetric along the line of $\lambda_1 = \mu_1$.



Since we are interested in finding $\inf_\mu \sup_{\Gamma \text{ is b.m.}} R^\mu(\Gamma)$, we will limit attention to first-order beliefs only in the specified range. To determine the exact form of transfers as functions of optimal probabilities, we have to find out which constraints are binding when. As it will turn out, the boundary of the rectangle of the possible beliefs specified above is a set of beliefs where one or two constraints do not have to be binding. The next four lemmas establish this result.

Lemma 1.4. *IR constraint of a low type is binding unless $\lambda_1 = \alpha + \gamma$. Moreover, for this λ_1, under the condition that $p_1^{21} \geq p_1^{11}$ and $p_1^{22} \geq p_1^{12}$, mechanism designer revenue is minimal when $\mu_1 = \frac{\alpha}{\alpha+\beta}$.*

Proof. Suppose IR constraint of a low type (1.12) is not binding, then it's immediate to realize that IC constraint of a low type (1.14) must be binding as otherwise we could perturb transfers by $t_1 = t_1^* + a\,(\varepsilon, \sigma, 0, 0)$. Also, IR constraint of a high type (1.13) must be binding as otherwise we can have $t_1 = t_1^* + a\,(\varepsilon, o, \varepsilon, 0)$. As for the IC constraint of a high type (1.15), it must be binding unless no possible perturbations of transfers will increase revenue. If (1.14) and (1.13) are binding perturbations can only be of the form $\Delta = \left(-\frac{1-\mu_1}{\mu_1}\sigma + \frac{\mu_1-\lambda_1}{\mu_1(1-\lambda_1)}\varepsilon, \sigma, \varepsilon, -\frac{\lambda_1}{1-\lambda_1}\varepsilon\right)$. Then change in revenue is equal to

$$\Delta R = \sigma\left(\beta - \alpha\frac{1-\mu_1}{\mu_1}\right) + \varepsilon\left(\alpha\frac{\mu_1-\lambda_1}{\mu_1\left(1-\lambda_1\right)} + \gamma - \frac{\delta\lambda_1}{1-\lambda_1}\right)$$

and since σ and ε can be of any sign both expressions in brackets must be zero. It follows that $\mu_1 = \frac{\alpha}{\alpha+\beta}$ and $\lambda_1 = \alpha+\gamma$. Rewriting (1.14) and (1.13), we get

$$(\alpha+\gamma)\,t_1^{21} + (\beta+\delta)\,t_1^{22} = (\alpha+\gamma)\left(p_1^{21}v_1^h - a_1^{21}\right) + (\beta+\delta)\left(p_1^{22}v_1^h - a_1^{22}\right)$$

$$\alpha\left(t_1^{11} - t_1^{21}\right) + \beta\left(t_1^{12} - t_1^{22}\right) = v_1^l\left(\alpha\left(p_1^{11} - p_1^{21}\right) + \beta\left(p_1^{12} - p_1^{22}\right)\right)$$

and summing them up we get that total revenue for these beliefs and binding IR constraint of a high type and IC constraint of a low type

$$\begin{aligned} R_1^{IC,IR} &= \alpha\left(p_1^{11}v_1^l + p_1^{21}\left(v_1^h - v_1^l\right) - a_1^{21}\right) + \beta\left(p_1^{12}v_1^l + p_1^{22}\left(v_1^h - v_1^l\right) - a_1^{22}\right) + \\ &+ \gamma\left(p_1^{21}v_1^h - a_1^{21}\right) + \delta\left(p_1^{22}v_1^h - a_1^{22}\right) \end{aligned}$$

Now suppose all three constraints but IR of a low type are binding, then all the possible perturbations must be of the form Δ above for which the following condition is satisfied

$$0 = \lambda_1\left(-\frac{1-\mu_1}{\mu_1}\sigma + \frac{\mu_1-\lambda_1}{\mu_1\left(1-\lambda_1\right)}\varepsilon\right) + \left(1-\lambda_1\right)\sigma = \frac{\mu_1-\lambda_1}{\mu_1}\sigma + \frac{\lambda_1}{\mu_1}\frac{\mu_1-\lambda_1}{1-\lambda_1}\varepsilon$$

And, thus, either $\lambda_1 = \mu_1 = \alpha+\gamma$ (the only way that two beliefs may be equal to each other) or $\sigma = -\frac{\lambda_1}{1-\lambda_1}\varepsilon$ and then perturbation is equal to $\Delta = \left(\varepsilon, -\frac{\lambda_1}{1-\lambda_1}\varepsilon, \varepsilon, -\frac{\lambda_1}{1-\lambda_1}\varepsilon\right)$

and the change in revenue is equal to

$$\Delta R_1 = \left((\alpha + \gamma) - (\beta + \delta) \frac{\lambda_1}{1 - \lambda_1} \right) \varepsilon$$

and since ε can be of any sign, it must be the case that $\lambda_1 = \alpha + \gamma$.

Let's also find mechanism designer revenue. Rewriting IR constraint of a low type and IC constraints for $\lambda_1 = \alpha + \gamma$, we get.

$$(\alpha + \gamma)\, t_1^{21} + (\beta + \delta)\, t_1^{22} = (\alpha + \gamma) \left(p_1^{21} v_1^h - a_1^{21} \right) + (\beta + \delta) \left(p_1^{22} v_1^h - a_1^{22} \right) \quad (1.16)$$

$$\mu_1 \left(t_1^{11} - t_1^{21} \right) + (1 - \mu_1) \left(t_1^{12} - t_1^{22} \right) = v_1^l \left(\mu_1 \left(p_1^{11} - p_1^{21} \right) + (1 - \mu_1) \left(p_1^{12} - p_1^{22} \right) \right) \quad (1.17)$$

$$(\alpha + \gamma) \left(t_1^{21} - t_1^{11} \right) + (\beta + \delta) \left(t_1^{22} - t_1^{12} \right) = v_1^h \left((\alpha + \gamma) \left(-p_1^{11} + p_1^{21} \right) + (\beta + \delta) \left(-p_1^{12} + p_1^{22} \right) \right) \quad (1.18)$$

Rewriting (18), taking into account (16) we get

$$(\alpha + \gamma)\, t_1^{11} + (\beta + \delta)\, t_1^{12} = (\alpha + \gamma) \left(p_1^{11} v_1^h - a_1^{21} \right) + (\beta + \delta) \left(p_1^{12} v_1^h - a_1^{22} \right) \quad (1.19)$$

Multiplying (16), (17) and (19) by x, y, and z, where $x = \frac{\gamma - \mu_1 (\gamma + \delta)}{\alpha + \gamma - \mu_1}$, $y = \frac{\alpha - \mu_1 (\alpha + \beta)}{\alpha + \gamma - \mu_1}$, $z = \frac{\gamma - (\alpha + \gamma)(\gamma + \delta)}{\alpha + \gamma - \mu_1}$ and simplifying resulting expression, we get that mechanism designer revenue is equal to

$$\begin{aligned} R_1 &= \frac{\gamma - \mu_1 (\gamma + \delta)}{\gamma + \alpha - \mu_1} \left[(\alpha + \gamma) \left(p_1^{21} - p_1^{11} \right) + (\beta + \delta) \left(p_1^{22} - p_1^{12} \right) \right] \left(v_1^h - v_1^l \right) + \\ &+ (\alpha + \gamma) \left(p_1^{11} v_1^h - a_1^{21} \right) + (\beta + \delta) \left(p_1^{12} v_1^h - a_1^{22} \right) + \gamma \left(p_1^{21} - p_1^{11} \right) v_1^l + \delta \left(p_1^{22} - p_1^{12} \right) v_1^l \end{aligned}$$

When $\lambda_1 = \alpha + \gamma$, we know that $\mu_1 \in \left[\min \left\{ \alpha + \gamma, \frac{\gamma}{\gamma + \delta} \right\}, \max \left\{ \alpha + \gamma, \frac{\gamma}{\gamma + \delta} \right\} \right]$ and hence coefficient before the square brackets is positive. Then, if optimal probabilities are such that the expression in square brackets is positive, this revenue is minimal when $\mu_1 = \frac{\alpha}{\alpha+\beta}$, with the total revenue equal to $R^{IC,IR}$ □

Lemma 1.5. *IR constraint of a high type is binding unless $\mu_1 = \alpha + \gamma$. Moreover, for this μ_1 under the condition that $p_1^{21} \geq p_1^{11}$ and $p_1^{22} \geq p_1^{12}$, mechanism designer revenue is minimal when $\lambda_1 = \frac{\gamma}{\gamma+\delta}$.*

Proof. If IR constraint of a high type is not binding, we can immediately conclude, by an argument similar to the one in previous lemma, that IR of a low type must be binding and IC of a high type must be binding. Then, all the possible perturbations of transfers must be of the form $\Delta = \left(\varepsilon, -\frac{\mu_1}{1-\mu_1}\varepsilon, \sigma, -\frac{\lambda_1}{1-\lambda_1}\sigma + \frac{\lambda_1-\mu_1}{(1-\lambda_1)(1-\mu_1)}\varepsilon\right)$. It follows that IC of a low type does not have to be binding if $\mu_1 = \alpha + \gamma$ and $\lambda_1 = \frac{\gamma}{\gamma+\delta}$ with mechanism designer revenue (found again by adding two binding constraints) is equal to

$$\begin{aligned} R_1^{IR,IC} &= \alpha\left(p_1^{11}v_1^l - a_1^{11}\right) + \beta\left(p_1^{12}v_1^l - a_1^{12}\right) + \gamma\left(p_1^{21}v_1^h - p_1^{11}\left(v_1^h - v_1^l\right) - a_1^{11}\right) + \\ &+ \delta\left(p_1^{22}v_1^h - p_1^{12}\left(v_1^h - v_1^l\right) - a_1^{12}\right) \end{aligned}$$

If all constraints but IR constraint of a high type is binding, then we can only consider perturbations of the form $\Delta = \left(\varepsilon, -\frac{\mu_1}{1-\mu_1}\varepsilon, \varepsilon, -\frac{\mu_1}{1-\mu_1}\varepsilon\right)$ and the change in revenue can be made positive unless $\mu_1 = \alpha + \gamma$. Then, from binding constraints we can again derive

$$(\alpha+\gamma)\,t_1^{11} + (\beta+\delta)\,t_1^{12} = (\alpha+\gamma)\left(p_1^{11}v_1^l - a_1^{11}\right) + (\beta+\delta)\left(p_1^{12}v_1^l - a_1^{12}\right)$$

$$(\alpha+\gamma)\,t_1^{21} + (\beta+\delta)\,t_1^{22} = (\alpha+\gamma)\left(p_1^{21}v_1^l - a_1^{11}\right) + (\beta+\delta)\left(p_1^{22}v_1^l - a_1^{12}\right)$$

$$\lambda_1\left(t_1^{21} - t_1^{11}\right) + (1-\lambda_1)\left(t_1^{22} - t_1^{12}\right) = v_1^h\left(\lambda_1\left(p_1^{21} - p_1^{11}\right) + (1-\lambda_1)\left(p_1^{22} - p_1^{12}\right)\right)$$

And multiplying these constraints by $x = \frac{\alpha-\lambda_1(\alpha+\beta)}{\alpha+\gamma-\lambda_1}$, $y = \frac{\gamma-\lambda_1(\gamma+\delta)}{\alpha+\gamma-\lambda_1}$ and $z = \frac{\alpha-(\alpha+\gamma)(\alpha+\beta)}{\alpha+\gamma-\lambda_1}$ and, then, simplifying we obtain mechanism designer revenue as

$$\begin{aligned} R_1 &= (\alpha+\gamma)\left(p_1^{11}v_1^l - a_1^{11}\right) + (\beta+\delta)\left(p_1^{12}v_1^l - a_1^{12}\right) + \\ &+ \gamma\left(p_1^{21} - p_1^{11}\right)v_1^h + \delta\left(p_1^{22} - p_1^{12}\right)v_1^h + \\ &+ \frac{\lambda_1(\gamma+\delta)-\gamma}{\alpha+\gamma-\lambda_1}\left[(\alpha+\gamma)\left(p_1^{21}-p_1^{11}\right) + (\beta+\delta)\left(p_1^{22}-p_1^{12}\right)\right]\left(v_1^h - v_1^l\right) \end{aligned}$$

And again if the expression in square brackets is positive[2] the revenue is minimal when $\frac{\lambda_1(\gamma+\delta)-\gamma}{\alpha+\gamma-\lambda_1}$ is minimal. From lemma 3 we derive that this coefficient is positive and, hence, minimized at $\lambda_1 = \frac{\gamma}{\gamma+\delta}$ with the revenue of $R_1^{IR,IC}$ □

Lemma 1.6. *IC constraint of a low type is binding unless $\lambda_1 = \frac{\gamma}{\gamma+\delta}$. Moreover, for this λ_1, mechanism designer revenue is minimal either at $\mu_1 = \alpha+\gamma$ or at $\mu_1 = \frac{\alpha}{\alpha+\beta}$.*

Proof. If IC constraint of a low type is not binding, it is immediate to realize that IR of a low type must be binding. However, all other constraints may be not binding. We have already established in lemma 5 that IR constraint of a high type and IC constraint of a low type do not have to be binding when $\lambda_1 = \frac{\gamma}{\gamma+\delta}$ and $\mu_1 = \alpha+\gamma$. Similarly, both IC constraints do not have to be binding when a change in revenue from a perturbation $\Delta = \left(\varepsilon, -\frac{\mu_1}{1-\mu_1}\varepsilon, \sigma, -\frac{\lambda_1}{1-\lambda_1}\sigma\right)$ is always zero. Equivalently,

$$\Delta R_1 = \left(\alpha - \beta\frac{\mu_1}{1-\mu_1}\right)\varepsilon + \left(\gamma - \delta\frac{\lambda_1}{1-\lambda_1}\right)\sigma = 0$$

[2](same expression as in lemma 5)

From which it follows that $\mu_1 = \frac{\alpha}{\alpha+\beta}$, $\lambda_1 = \frac{\gamma}{\gamma+\delta}$ and revenue, derived by adding up two IR constraints, is given by

$$R_1^{IR,IR} = \alpha\left(p_1^{11}v_1^l - a_1^{11}\right) + \beta\left(p_1^{12}v_1^l - a_1^{12}\right) + \gamma\left(p_1^{21}v_1^h - a_1^{21}\right) + \delta\left(p_1^{22}v_1^h - a_1^{22}\right)$$

When all constraints but IC constraint of a low type bind, we can only consider perturbations of the form $\Delta = \left(0, 0, \sigma, -\frac{\lambda_1}{1-\lambda_1}\sigma\right)$ with the change in revenue equal to $\Delta R_1 = \left(\gamma - \delta\frac{\lambda_1}{1-\lambda_1}\right)\sigma$, from which it follows that $\lambda_1 = \frac{\gamma}{\gamma+\delta}$ so that change in revenue is zero.

For this λ_1 and binding constraints

$$\mu_1 t_1^{11} + (1-\mu_1)\, t_1^{12} = \mu_1\left(p_1^{11}v_1^l - a_1^{11}\right) + (1-\mu_1)\left(p_1^{12}v_1^l - a_1^{12}\right) \tag{1.20}$$

$$\gamma t_1^{21} + \delta t_1^{22} = \gamma\left(p_1^{21}v_1^h - a_1^{21}\right) + \delta\left(p_1^{22}v_1^h - a_1^{22}\right) \tag{1.21}$$

$$\gamma\left(t_1^{21} - t_1^{11}\right) + \delta\left(t_1^{22} - t_1^{12}\right) = \gamma\left(p_1^{21} - p_1^{11}\right)v_1^h + \delta\left(p_1^{22} - p_1^{12}\right)v_1^h \tag{1.22}$$

From (1.21) and (1.22) we derive that

$$\gamma t_1^{11} + \delta t_1^{12} = \gamma\left(p_1^{11}v_1^h - a_1^{21}\right) + \delta\left(p_1^{12}v_1^h - a_1^{22}\right) \tag{1.23}$$

and multiplying (1.20) and (1.23) by $x = \frac{\beta\gamma - \alpha\delta}{\alpha - \mu_1(\gamma+\delta)}$ and $y = \frac{a-\mu_1(\alpha+\beta)}{\gamma-\mu_1(\gamma+\delta)}$ and adding up 21, we get that the revenue is equal to

$$\begin{aligned} R_1 &= \alpha\left(p_1^{11}v_1^l - a_1^{11}\right) + \beta\left(p_1^{12}v_1^l - a_1^{12}\right) + \gamma\left(p_1^{21}v_1^h - a_1^{21}\right) + \delta\left(p_1^{22}v_1^h - a_1^{22}\right) + \\ &+ \frac{a-\mu_1(\alpha+\beta)}{\gamma-\mu_1(\gamma+\delta)}\left[\gamma p_1^{11}\left(v_1^h - v_1^l\right) + \delta p_1^{12}\left(v_1^h - v_1^l\right) - \gamma\left(a_1^{21} - a_1^{11}\right) - \delta\left(a_1^{22} - a_1^{12}\right)\right] \end{aligned}$$

And depending on whether the expression in square brackets is positive or negative the mechanism designer's revenue is minimized at either $\mu_1 = \alpha + \gamma$, or $\mu_1 = \frac{\alpha}{\alpha+\beta}$ □

Lemma 1.7. *IC constraint of a high type is binding unless* $\mu_1 = \frac{\alpha}{\alpha+\beta}$. *Moreover, for this* μ_1*, mechanism designer revenue is minimal either at* $\lambda_1 = \alpha + \gamma$ *or at* $\lambda_1 = \frac{\gamma}{\gamma+\delta}$.

Proof. Suppose IC constraint of a high type is not binding, then it's immediate to get that IR constraint of a high type must be binding. As for the other constraints we saw in lemmas 4 and 6 that for $\mu_1 = \frac{\alpha}{\alpha+\beta}$ and $\lambda_1 = \alpha + \gamma$, IR constraint of a low type doesn't have to be binding and for $\mu_1 = \frac{\alpha}{\alpha+\beta}$ and $\lambda_1 = \frac{\gamma}{\gamma+\delta}$, both IC constraints do not have to be binding.

When all other constraints are binding, we can consider perturbations in the form of $\Delta = \left(\varepsilon, -\frac{\mu_1}{1-\mu_1}\varepsilon, 0, 0\right)$, from which it will follow that $\mu_1$must be equal to $\frac{\alpha}{\alpha+\beta}$. For this μ_1 we can derive mechanism designer revenue, and it will turn out, similarly to lemma 6, that depending on the parameters revenue will be minimal at either $\lambda_1 = \alpha + \gamma$ or at $\lambda_1 = \frac{\gamma}{\gamma+\delta}$ □

Lemma 1.8. *If all constraints are binding, minimal revenue can be always found on the boundary of feasible domain, i.e. either* $\lambda_1 = \alpha + \gamma$*, or* $\lambda_1 = \frac{\gamma}{\gamma+\delta}$*, or* $\mu_1 = \alpha + \gamma$*, or* $\mu_1 = \frac{\alpha}{\alpha+\beta}$.

Proof. Multiplying IR constraints by $x_1 = \frac{\alpha+\gamma-\lambda_1}{\mu_1-\lambda_1}$, $y_1 = \frac{\alpha+\gamma-\mu_1}{\lambda_1-\mu_1}$ and IC constraints by $x_2 = \frac{\gamma-\lambda_1(\gamma+\delta)}{\lambda_1-\mu_1}$, $y_2 = \frac{\mu_1(\alpha+\beta)-\alpha}{\lambda_1-\mu_1}$, we can find total revenue. Taking a derivative with respect to λ_1, we get that it's equal to

$$\frac{dR_1}{d\lambda_1} = \frac{A}{(\lambda_1 - \mu_1)^2}$$

where A is a constant that doesn't depend on λ_1. Hence, revenue is always increasing or decreasing with λ_1 and minimal revenue can always be found on either $\lambda_1 = \alpha + \gamma$, or $\lambda_1 = \frac{\gamma}{\gamma+\delta}$. □

We conclude the set of lemmas with the following proposition.

Proposition 1.1. *Minimal revenue obtained from a first player in a Bayesian mechanism is achieved at either* $(\mu_1, \lambda_1) = \left(\alpha + \gamma, \frac{\gamma}{\gamma+\delta}\right)$ *or at* $(\mu_1, \lambda_1) = \left(\frac{\alpha}{\alpha+\beta}, \frac{\gamma}{\gamma+\delta}\right)$ *or at* $(\mu_1, \lambda_1) = \left(\frac{\alpha}{\alpha+\beta}, \alpha + \gamma\right)$ and is the minimal of the following three expressions

$$R_1 = \alpha\left(p_1^{11}v_1^l - a_1^{11}\right) + \beta\left(p_1^{12}v_1^l - a_1^{12}\right) + \gamma\left(p_1^{21}v_1^h - a_1^{21}\right) + \delta\left(p_1^{22}v_1^h - a_1^{22}\right) \quad (1.24)$$

$$\begin{aligned} R_1 &= \alpha\left(p_1^{11}v_1^l - a_1^{11}\right) + \beta\left(p_1^{12}v_1^l - a_1^{12}\right) + \gamma\left(p_1^{21}v_1^h - p_1^{11}\left(v_1^h - v_1^l\right) - a_1^{11}\right) \\ &+ \delta\left(p_1^{22}v_1^h - p_1^{12}\left(v_1^h - v_1^l\right) - a_1^{12}\right) \end{aligned} \quad (1.25)$$

$$\begin{aligned} R_1 &= \alpha\left(p_1^{11}v_1^l + p_1^{21}\left(v_1^h - v_1^l\right) - a_1^{21}\right) + \beta\left(p_1^{12}v_1^l + p_1^{22}\left(v_1^h - v_1^l\right) - a_1^{22}\right) + \\ &+ \gamma\left(p_1^{21}v_1^h - a_1^{21}\right) + \delta\left(p_1^{22}v_1^h - a_1^{22}\right) \end{aligned} \quad (1.26)$$

Proof. Lemmas 3-8 establish the result. □

We can now compare revenues from a dominant strategy mechanism to revenues from a Bayesian mechanism. From lemmas 1 and 2, we can express revenue obtained by a mechanism designer from the first player in the following way:

$$\begin{aligned} R_1^{dsm}\left(p_1^{11},p_1^{12},p_1^{21},p_1^{22}\right) &= \alpha \min\left\{p_1^{11}v_1^l - a_1^{11}, p_1^{11}v_1^l + p_1^{21}\left(v_1^h - v_1^l\right) - a_1^{21}\right\} + \\ &+ \beta \min\left\{p_1^{12}v_1^l - a_1^{12}, p_1^{12}v_1^l + p_1^{22}\left(v_1^h - v_1^l\right) - a_1^{22}\right\} + \\ &+ \gamma \min\left\{p_1^{21}v_1^h - a_1^{21}, p_1^{21}v_1^h - p_1^{11}\left(v_1^h - v_1^l\right) - a_1^{11}\right\} + \\ &+ \delta \min\left\{p_1^{22}v_1^h - a_1^{22}, p_1^{22}v_1^h - p_1^{12}\left(v_1^h - v_1^l\right) - a_1^{12}\right\} \end{aligned}$$

While proposition 1 establishes that for the worst possible belief optimal revenue from a Bayesian mechanism $R_1^{Bm}\left(p_1^{11},p_1^{12},p_1^{21},p_1^{22}\right)$ is not bigger tha

$$\begin{aligned} &\min\{\alpha\left(p_1^{11}v_1^l - a_1^{11}\right) + \beta\left(p_1^{12}v_1^l - a_1^{12}\right) + \gamma\left(p_1^{21}v_1^h - a_1^{21}\right) + \delta\left(p_1^{22}v_1^h - a_1^{22}\right), \\ &\alpha\left(p_1^{11}v_1^l + p_1^{21}\left(v_1^h - v_1^l\right) - a_1^{21}\right) + \beta\left(p_1^{12}v_1^l + p_1^{22}\left(v_1^h - v_1^l\right) - a_1^{22}\right) + \\ + \;&\gamma\left(p_1^{21}v_1^h - a_1^{21}\right) + \delta\left(p_1^{22}v_1^h - a_1^{22}\right), \\ &\alpha\left(p_1^{11}v_1^l - a_1^{11}\right) + \beta\left(p_1^{12}v_1^l - a_1^{12}\right) + \\ + \;&\gamma\left(p_1^{21}v_1^h - p_1^{11}\left(v_1^h - v_1^l\right) - a_1^{11}\right) + \delta\left(p_1^{22}v_1^h - p_1^{12}\left(v_1^h - v_1^l\right) - a_1^{12}\right)\} \end{aligned}$$

Leading to the following theorem

Theorem 1.1. *There exist a Bayesian and maxmin foundation for dominant strategy mechanisms if there is a symmetry between binding IR and IC constraints of bidder 1 across different types of bidder 2.*

Proof. Observing carefully the revenue of dominant strategy mechanism and revenues from a Bayesian mechanism, we see that

$$R_1^{dsm}\left(p_1^{11},p_1^{12},p_1^{21},p_1^{22}\right) \leq R_1^{Bm}\left(p_1^{11},p_1^{12},p_1^{21},p_1^{22}\right)$$

and the objective functions are equivalent when for any constraint that is binding for bidder 1 for a low type of bidder 2, the same constraint is binding for the high type of player 2. If there is any asymmetry the objective function in a Bayesian mechanism is strictly greater when using the same probabilities as in the optimal dominant strategy mechanism. Thus, if optimal probabilities are chosen for the Bayesian mechanism, the revenue is going to be even greater and no Bayesian foundation could be provided. □

Essentially if the conditions of theorem are satisfied this theorem helps us construct a very simple conjecture on the universal type space that will have a positive support on a subset of this type space - a payoff type space. Depending on what is the minimum of proposition 1, this conjecture will have $(\mu_1, \lambda_1) = \left(\alpha + \gamma, \frac{\gamma}{\gamma+\delta}\right)$ or $(\mu_1, \lambda_1) = \left(\frac{\alpha}{\alpha+\beta}, \frac{\gamma}{\gamma+\delta}\right)$ or $(\mu_1, \lambda_1) = \left(\frac{\alpha}{\alpha+\beta}, \alpha + \gamma\right)$.

To establish whether maxmin foundation exists when conditions of the theorem 1 is not satisfied, we have to consider more complex type spaces. We show though by the example below, that there can exist no Bayesian foundation when these conditions are broken.

1.5 Bayesian foundation for dominant strategy mechanism

Suppose that valuations of players and its distribution are represented by the following table.

	$v_2^l = 11$	$v_2^h = 15$
$v_1^l = 4$	0.1	0.2
$v_1^h = 12$	0.01	0.69

Let's, first of all, recall the standard argument for zero outside options when a maxmin foundation can be established. Let's consider a very simple type space, where for every valuation of a player there exists just one belief and these beliefs are common knowledge. Suppose for the low type of the first player we set that $\mu_1 = 0.11$ and for the high type of the first player we set $\lambda_1 = \frac{0.01}{0.7} = \frac{1}{70}$. Then, from individual rationality constraint of a low type and incentive compatibility constraint of a high type, we derive that in a Bayesian equilibrium it must be the case that

$$0.11\left(p_1^{11}v_1^l - t_1^{11}\right) + 0.89\left(p_1^{12}v_1^l - t_1^{12}\right) \geq 0$$

$$\frac{1}{70}\left(p_1^{21}v_1^h - t_1^{21}\right) + \frac{69}{70}\left(p_1^{22}v_1^h - t_1^{22}\right) \geq \frac{1}{70}\left(p_1^{11}v_1^h - t_1^{11}\right) + \frac{69}{70}\left(p_1^{12}v_1^h - t_1^{12}\right)$$

multiplying the second equation by 0.7 and summing it with the first one, we derive

$$\begin{aligned} R &= 0.1t_1^{11} + 0.2t_1^{12} + 0.01t_1^{21} + 0.69t_1^{22} \leq 0.1p_1^{11}v_1^l + 0.2p_1^{12}v_1^l + \\ &+ 0.01\left(p_1^{21}v_1^h - p_1^{11}\left(v_1^h - v_1^l\right)\right) + 0.69\left(p_1^{22}v_1^h - p_1^{12}\left(v_1^h - v_1^l\right)\right) \end{aligned}$$

which is exactly the bound one derives from an optimal dominant strategy mechanism when outside options are zero and individual rationality constraints are binding for the low type and incentive compatibility constraints are binding for the high type.

Suppose now that outside options are equal to zero for the second player and for the low type of the first player, but for the high type of the first player they are equal to seven regardless of the type of the second player. So, $a_2^{ij} = 0$, $a_1^{1j} = 0$, $a_1^{2j} = 7$. By lemma 2 of section 3 we know that in the optimal dominant strategy mechanism $p_1^{21} = 1$, $p_2^{12} = 1$ and since $\alpha v_1^l - \gamma\left(v_1^h - v_1^l\right) = 0.33 > 0 = \alpha v_2^l - \beta\left(v_2^h - v_2^l\right)$, it is always optimal to sell the object when both players are low to the first player and, thus, we have $p_1^{11} = 1$. Moreover, due to $\delta v_2^h > \delta v_1^h + \beta \left(v_1^h - v_1^l\right)$, it is always optimal to sell the object when both players are high to the second player. Thus, we have $p_2^{22} = 1$. Finally since outside options are such that $p_1^{22} \left(v_1^h - v_1^l\right) = 0 < a_1^{2j} - a_1^{1j} = 7 < p_1^{11} \left(v_1^h - v_1^l\right) = 8$, we know by lemma 1 that IR constraint of the first player binds when players have the same type, while IC constraints bind when they have the opposite type. It follows that $t_1^{11} = t_1^{21} = 4$, $t_1^{12} = t_1^{22} = -7$,

The optimal dominant strategy mechanism is thus represented by

	11	15
4	$p_1^{11} = 1$, $t_1^{11} = 4$, $t_2^{11} = 0$	$p_2^{12} = 1$, $t_1^{12} = -7$, $t_2^{12} = 15$
12	$p_1^{21} = 1$, $t_1^{21} = 4$, $t_2^{21} = 0$	$p_2^{22} = 1$, $t_1^{22} = -7$, $t_2^{22} = 15$

Notice that in this example, the low type of the first player gets additional surplus when the second player is high $(7 > 0)$ and the high type of the first player gets an additional surplus when the second player is low $(8 > 7)$. Notice that for a Bayesian first player we can always improve upon this mechanism unless the low type believes with probability one that the second player is low and the high type believes with probability one that the second player is high. For example, suppose the first-order belief of the low type of the first player was less than one. Then, we can increase both transfers t_1^{11} and t_1^{21} and still satisfy IR constraints on average as first player is getting surplus utility in cases when both players have different types. Thus, it is important in this construction that in the dominant strategy mechanism IR constraint of a low type when the other player is high was not binding. Moreover, since beliefs of one and zero, necessary to rationalize optimal dominant strategy mechanism, are too extreme, it will be possible for a mechanism designer to introduce Cremerer-McLean types of bets, separate high and low types and benefit from such a bet. That is our intuition why it is necessary to have symmetry in IR constraints in order to establish a Bayesian foundation.

We will show now that this dominant strategy mechanism can not be rationalizable by any conjecture η on the universal type space by contradiction. Essentially, lemma 10 shows that to rationalize optimal dominant strategy mechanism the first-order belief of a low player can not be too high as otherwise it will be possible to introduce Cremerer and McLean bets on the type of the second player, while lemma 11 shows that to rationalize it the first-order belief of the low type can not be too low as otherwise we can increase transfers t_1^{11} and t_1^{21}. Two lemmas establish the necessary contradiction. We begin with preliminary lemma 9 necessary for the proof of lemma 10.

Lemma 1.9. *If η rationalizes the optimal dominant strategy mechanism, than given some non-null subset of low types of the first player, the probability of the second type being low can not be too high. For any $x \in [0,1]$, if $\eta(\mu_1 > x) > 0$, $\eta(v_2 = 12|\mu_1 > x) \leq \frac{4}{11}$*

Proof. Suppose that for a low type of the first player we introduce a message $\mu_1 > x$ under which we sell the object to the second player regardless of his type.

	$v_2 = 11$	$v_2 = 15$
$\mu_1 \leq x$	$p_1^{11} = 1$, $t_1^{11} = 4$, $t_2^{11} = 0$	$p_2^{12} = 1$, $t_1^{12} = -7$, $t_2^{12} = 15$
$\mu_1 > x$	$p_2^{11} = 1$, $t_1^{11} = 0$, $t_2^{11} = 11$	$p_2^{12} = 1$, $t_1^{12} = -7$, $t_2^{12} = 11$
$v_1 = 12$	$p_1^{21} = 1$, $t_1^{21} = 4$, $t_2^{21} = 0$	$p_2^{22} = 1$, $t_1^{22} = -7$, $t_2^{22} = 15$

Notice that it is still a dominant strategy for all the players to report truthfully since the low type of the first player gets the same utility regardless of the message they send and since the high type finds it unprofitable to send a new message. Total difference in revenue is given by

$$\Delta R = 7\eta\left(v_2 = l|\mu_1 > x\right) - 4\left(1 - \eta\left(v_2 = l|\mu_1 > x\right)\right)$$

Hence if $\eta\left(v_2 = l|\mu_1 > x\right) > \frac{4}{11}$, there is a profitable way to increase revenue from dominant strategy mechanism. □

Lemma 1.10. *If the optimal dominant strategy is rationalizable by a conjecture η, than the belief of the low type of the first player can not be too high. Formally, $\eta\left(\mu_1 > \frac{6}{13}\right) = 0$*

Proof. We will introduce for a low type two messages "$\mu_1 \leq x$" and "$\mu_1 > x$". Compared to the optimal dominant strategy mechanism, we change only transfers for the message $\mu_1 > x$.

$x \in [0,1)$	$v_2 = 11$	$v_2 = 15$
$\mu_1 \leq x$	$p_1^{11} = 1,\ t_1^{11} = 4,\ t_2^{11} = 0$	$p_2^{12} = 1,\ t_1^{12} = -7,\ t_2^{12} = 15$
$\mu_1 > x$	$p_1^{11} = p_2^{11} = 0,\ t_1^{11} = -8,\ t_2^{11} = 0$	$p_2^{12} = 1,\ t_1^{12} = -7 + \frac{8x}{1-x},\ t_2^{12} = 15$
$v_1 = 12$	$p_1^{21} = 1,\ t_1^{21} = 4,\ t_2^{21} = 0$	$p_2^{22} = 1,\ t_1^{22} = -7,\ t_2^{22} = 15$

First of all, note that truth-telling is still a dominant strategy for bidder 2. Second, notice that the high type of the first player does not want to send a message $\mu_1 \leq x$ as before, but also does not want to send message $\mu_1 > x$ for any λ_1. Third, a low type wants to send a message $\mu_1 > x$ if and only if

$$8\mu_1 + (1 - \mu_1)\left(-\frac{8x}{1 - x}\right) > 0$$

which is equivalent to $\mu_1 > x$.

Finally, let's calculate the change in profits for the mechanism designer.

$$\Delta R = -12\eta\left(v_2^l|\mu_1 > x\right) + \left(1 - \eta\left(v_2^l|\mu_1 > x\right)\right)\left(\frac{8x}{1-x}\right)$$

simplifying we get $\Delta R > 0$ if

$$x > \frac{3\eta\left(v_2^l|\mu_1 > x\right)}{2 + \eta\left(v_2^l|\mu_1 > x\right)}$$

The right hand side is increasing in $\eta\left(v_2^l|\mu_1 > x\right)$, so we can conclude that since $\eta\left(v_2^l|\mu_1 > x\right) \leq \frac{4}{11}$ from previous lemma, for any x such that $x > \frac{6}{13}$, the change in revenue of the mechanism designer must be positive.

Thus, if a dominant strategy mechanism is rationalizable by some η, it has to be the case that beliefs of a low type of player 1 are not too high. Otherwise, the mechanism designer can profit from introducing bets for a low type of the first player on the types of the second player. □

Lemma 1.11. *If the optimal dominant strategy is rationalizable by a conjecture η, than the belief of the low type of the first player can not be too high. Formally,* $\eta\left(\mu_1 \leq \frac{7}{8}\right) = 0$

Proof. We now consider a very simple modification of the original mechanism that is clearly increasing total revenue - we increase transfers t_1^{11}, t_1^{21} from 4 to 5.

	$v_2 = 11$	$v_2 = 15$
$v_1 = 4$	$p_1^{11} = 1$, $t_1^{11} = 5$, $t_2^{11} = 0$	$p_2^{12} = 1$, $t_1^{12} = -7$, $t_2^{12} = 15$
$v_1 = 12$	$p_1^{21} = 1$, $t_1^{21} = 5$, $t_2^{21} = 0$	$p_2^{22} = 1$, $t_1^{22} = -7$, $t_2^{22} = 15$

Note that all the IC constraints are obviously satisfied. IR constraints of a second player and of a high type of a first player are also obviously satisfied since their outside options are equal to zero and seven respectively. Finally, IR constraint of the low type of the first player is satisfied if $-\mu_1 + 7(1-\mu_1) \geq 0$, which is equivalent to $\mu_1 \leq \frac{7}{8}$ and, hence it must the case that $\mu_1 > \frac{7}{8}$ for any conjecture η that rationalizes the optimal dominant strategy mechanism. □

Theorem 1.2. *The optimal dominant strategy mechanism is not rationalizable by any conjecture η of the mechanism designer.*

Proof. By lemmas 10 and 11, we get that $\frac{7}{8} < \mu_1 \leq \frac{6}{13}$, which is a contradiction. □

1.6 Standard problem and monotonicity constraints.

In this section we briefly link regularity condition of Chung and Ely (2007) to symmetry conditions of our paper.

Monotonicity constraints are binding in the sense of Chung and Ely if the following situation occurs. Suppose that in the model with zero outside options you assume that in the optimal dominant strategy mechanism individual rationality constraints are binding (satisfied with equality) only for the lowest types and incentive compatibility constraints are binding for two types that are closest to each other (the higher type does not want to pretend to be a lower type), while all other constraints are slack. With transfers derived from such an assumption derive optimal probabilities. If it is the case that $p_i\left(v_i^{j+1}, v_{-i}\right) \leq p_i\left(v_i^j, v_{-i}\right)$ then initial assumption was incorrect since some other incentive constraints are either violated or binding. Regularity conditions of Chung and Ely are necessary and sufficient to guarantee that monotonicity constraints are not binding.

Let's derive conditions when monotonicity constraints bind in the dominant strategy mechanisms for a simple 2 players, 2 values standard case. Recall that for the standard problem with zero outside opportunities, it is always the case that

IC constraint of a high type and IR constraint of a low type bind with additional monotonicity constraints. Thus, ignoring the monotonicity constraints, the optimal mechanism designer revenue is equal to the maximum of

$$\begin{aligned} R &= \alpha\left(p_1^{11}v_1^l + p_2^{11}v_2^l\right) + \beta\left(p_1^{12}v_1^l + p_2^{12}v_2^h - p_2^{11}\left(v_2^h - v_2^l\right)\right) + \\ &+ \gamma\left(p_1^{21}v_1^h - p_1^{11}\left(v_1^h - v_1^l\right) + p_2^{21}v_2^l\right) + \\ &+ \delta\left(p_1^{22}v_1^h - p_1^{12}\left(v_1^h - v_1^l\right) + p_2^{22}v_2^h - p_2^{21}\left(v_2^h - v_2^l\right)\right) \end{aligned}$$

Suppose without loss of generality that $v_1^l < v_2^l$. Then, it is immediate to obtain optimal probabilities for a low type of the first player and high type of the second player: $p_2^{12} = 1$ and $p_1^{12} = 0$. Thus, there are only two potential cases where monotonicity constraints could bind. One is when optimal probabilities from optimizing R mechanism designer's revenue ignoring monotonicity constraints are such that $p_2^{21} > p_2^{22}$. And, the other is when $p_1^{11} > p_1^{21}$.

1. $p_2^{21} > p_2^{22}$. For this it must be the case that $v_2^h \leq v_1^h$ (otherwise, $p_2^{22} = 1$) and $\gamma v_2^l - \delta\left(v_2^h - v_2^l\right) \geq \gamma v_1^h$. These two inequalities are inconsistent.

2. $p_1^{11} > p_1^{21}$. For this it must be the case $\alpha v_1^l - \gamma\left(v_1^h - v_1^l\right) \geq \alpha v_2^l - \beta\left(v_2^h - v_2^l\right)$, $\alpha v_1^l - \gamma\left(v_1^h - v_1^l\right) > 0$ (otherwise, $p_1^{11} = 0$) and $\gamma v_1^h \leq \gamma v_2^l - \delta\left(v_2^h - v_2^l\right)$ (otherwise, $p_1^{21} = 1$). In this case a monotonicity constraint binds, which implies that at the actual optimum IC constraints for both the high type and the low type of the first player bind when the second player is of the low value. Therefore, there is asymmetry in binding constraints for a low type of the first player since when the second player is low both IC and IR constraints are binding, but when the second player is high only IR constraint is binding.

1.7 Conclusion

In conclusion we would like to argue loosely that even for an environment with outside options perturbed slightly around zero with several players and valuations it is even harder to establish Bayesian and maxmin foundations. Suppose that outside options are all coming from the set of $\{0, \varepsilon\}$ with equal probability. Then, the larger is the number of players and the number of possible valuations, the more probable would be an event when for some two players and for valuations of each player that are the closest to each other we get $a_i\left(v_i^{k+1}, v_j^m, v_{-ij}\right) - a_i\left(v_i^k, v_j^m, v_{-ij}\right) > 0$, while $a_i\left(v_i^{k+1}, v_j^{m+1}, v_{-ij}\right) - a_i\left(v_i^k, v_j^{m+1}, v_{-ij}\right) < 0$. Suppose also that for all these valuations optimal probability of receiving the object for player i is zero. Then, IR constraint is binding for player i when player i has valuation v_i^{k+1} and player j has valuation v_j^m or when player i has valuation v_i^k and player j has valuation v_j^{m+1}. By the logic of our example such a situation can only be rationalized in a Bayesian mechanism when type with valuation $k+1$ of player i believes with certainty that player j has a valuation m and type with valuation k of player i believes with certainty that player j has a valuation $m+1$. However, because these believes are so extreme and opposite to each other, a mechanism designer would benefit by introducing Cremerer and McLean type of bets.

We conclude by stating that a predominance of using stronger solution concepts like a dominant strategy equilibrium as a response to the relaxation of different common knowledge assumptions may be somewhat premature. We argue that in a generalized case of tighter individual rationality constraints it is sufficiently easy to construct an example where no conjecture of a mechanism designer on the universal type space would rationalize the use of dominant strategy mechanisms. In our next chapter we take a common applied problem of a collusion threat and argue that even a stronger claim can be made - for a certain range of parameters dominant strategy mechanisms will have no maxmin foundation.

Chapter 2

Maxmin Foundations of Dominant Strategy Mechanisms under a Collusion Threat.

2.1 Introduction.

In this chapter, we will continue investigating whether there exists a "maxmin" and Bayesian foundation for dominant strategy mechanisms in an auction environment by considering a practical application of type-dependent outside options, namely a possible threat of collusion between bidders. While a collusion threat generally leads to tighter participation constraints and lower revenue for the auctioneer, a possibility of collusion also enlarges the number of possible strategies of the bidders since every type of every player has to decide whether to try to collude or not and what type to report if a collusion attempt is unsuccessful.

It turns out that this feature of the environment will be essential in proving a stronger result compared to previous chapter: for a certain range of primitives, it will be possible to show that the maximal revenue achieved in an optimal dominant strategy mechanism can be lower than the revenue achieved from some Bayesian mechanism robust to all perturbations in beliefs and higher order beliefs. Therefore, we will show that for a certain range of primitives no "maxmin" (rather than just Bayesian) foundation exists. The intuition behind this result lies at the heart of the difference between dominant strategy implementation and (Bayesian) equilibrium implementation - in a general case for a general dominant strategy mechanism one has to check that a profile of strategies where the first player uses a dominant strategy and the second player uses any strategy is preferred by the first player to the profile where the first player uses any other strategy. However, a profile where both strategies are not part of the equilibrium is never relevant to establishing equilibrium profile under Bayesian Nash equilibrium. Thus, dominant strategy equilibrium can be an extremely strict solution concept for revenue maximization, especially when the number of strategies is increased, while the number of payoff types stays the same. As it is possible to show that no "maxmin" foundation exists even for a simple case of two players with independent valuations, we will argue that a move of the recent literature to stronger solution concepts as a way of avoiding making any explicit or implicit assumptions on beliefs and higher order beliefs may have been somewhat premature.

In this chapter, we will model the ability of a mechanism designer to resist collusion formation through external rules in a reduced way. In particular, we will assume that when two bidders collude, they do so through a second price knockout auction after which only one of bidders participates in the mechanism and receives the object for the reservation price. We view, thus, the largeness of a reservation price as a proxy to whether mechanism designer can fight collusion formation through some explicit rules effectively. In this way we can abstract from all the usual modelling difficulties such as shell-bidding that arise when collusion threat is real. The use of second price knockout auctions also allows us to model the collusion game as receiving fixed collusion type-dependent payoffs, which links this chapter to the previous one.

2.2 Preliminaries

2.2.1 Timing

The timing of the game is the following.

1. An auctioneer proposes a direct mechanism $p_i(v_i, v_{-i})$ and $t_i(v_i, v_{-i})$ specifying probabilities of receiving an object and accompanying transfers as a function of reports (v_i, v_{-i}) of payoff types.

2. Every bidder decides independently whether to collude or not, voting 'yes' or 'no' to the collusion question. Votes of bidders are private information.

3. If both voters voted 'yes' they play a second-price knockout auction among themselves with the winner getting an object from the auctioneer for a reservation price r.[1]

4. If one of the voters voted 'no', bidders update their beliefs about other bidder's type and participate in the mechanism.

2.2.2 Collusion payoffs

Suppose we have the following values $v_1^l < v_2^l < v_1^h < v_2^h$ and distribution of those values

	v_2^l	v_2^h
v_1^l	α	β
v_1^h	γ	δ

If some types of both players want to collude, we model their payoffs in the following way - the two players play second price knockout auction for the right to get an object at a reservation price r.[2] One can think of this mechanism as if there is a mediator who asks players their valuations, uses second-price auction to determine payoffs and pays z_1 and z_2 in transfers for the first and the second players in such a way that it balances the budget ex ante. Under these assumptions the payoffs under collusion $C_i(v_i, v_{-i})$ can be characterized in the following way

[1]The auctioneer is not allowed to choose r as, otherwise, he would chose r to be infinity.

[2]The underlying assumption is that the mechanism that a player face when entering alone is different from the mechanism that a player face when two players participate in the mechanism

	v_2^l	v_2^h
v_1^l	$z_1,\ z_2 + v_2^l - max\{v_1^l - r, 0\} - r$	$z_1,\ z_2 + v_2^h - max\{v_1^l - r, 0\} - r$
v_1^h	$z_1 + v_1^h - max\{v_2^l - r, 0\} - r,\ z_2$	$z_1,\ z_2 + v_2^h - max\{v_1^h - r, 0\} - r$

Where z_1 and z_2 are such that budget is balanced ex-ante[3] and are determined by the following equation

$$\begin{aligned} z_1 + z_2 &= (\alpha + \beta)\, max\{v_1^l - r, 0\} + \gamma max\{v_2^l - r, 0\} + \delta max\{v_1^h - r, 0\} = \\ &= (\alpha + \beta)\, v_1^l + \gamma v_2^l + \delta v_1^h - r \end{aligned} \tag{2.1}$$

for $r \leq v_1^l$, an assumption we will maintain throughout the rest of the paper.

2.2.3 Mechanisms

In this paper we consider only mechanisms that depend on valuation reports. This class is clearly sufficient for dominant strategy implementation due to revelation principle and the fact that collusion game is essentially modeled as fixed outside options of players. However, on type spaces where any infinite hierarchy of beliefs and higher-order beliefs is possible restricting mechanisms to this class may be with loss of generality when finding maximal possible revenue. Our purpose, then, is to establish conditions under which no maxmin foundation exists. Since if

$$\Pi_{dsm} < \sup_{\Gamma \in \psi} \inf_{\eta \in M(v)} \Pi^{\eta} (\Gamma)$$

[3] If we want to balance budget ex-post, we lose the dominant strategy equilibrium

for some (narrow) class Ψ, then for a broader class of mechanisms Φ such that $\Psi \subset \Phi$, it will still be true that

$$\Pi_{dsm} < \sup_{\Gamma \in \Phi} \inf_{\eta \in M(v} \Pi^{\eta}(\Gamma)$$

Therefore, rather than reducing the type space and trying to find conditions under which maxmin foundation exists as we did in previous chapter, we reduce the class of possible mechanisms and find conditions under which maxmin foundation does not exist.[4]

In this paper we also assume that both players should get at least zero payoff, which can be modeled as an inclusion of message $\{\emptyset\}$ to the set of messages that provides zero transfer from the mechanism designer and zero probability of receiving a good.

2.2.4 Strategies

Given simple mechanisms of section 2.3. every type of every player has to decide whether to vote 'yes' or 'no' to the collusion question at the first stage and which type to report if no collusion is formed at the second stage. Throughout this paper we assume that vote of a bidder to the collusion question is private information of that bidder.[5]Therefore, a strategy for every player can be summarized by $\left(V_i^l m_{il}, V_i^h m_{ih}\right)$, where messages $m_{il}, m_{ih} \in \{l, h\}$, $V_i^l, V_i^h \in \{N, Y\}$are sent by the low and high types of player i.

[4]A standard description of (universal) type space is given in section 2.2. of Chapter 1

[5]The assumption that collusion votes are private information is not important to the analysis since it can at most change beliefs of players about each other type. However, dominant strategy mechanisms and Bayesian mechanisms that are robust to all possible beliefs and higher-order beliefs do not use this information.

We say that a voting profile $\left(V_1^l V_1^h, V_2^l V_2^h\right)$ is truthfully implemented in dominant strategies (Bayesian implemented) if strategies $\left(V_1^l l, V_1^h h\right)$ of the first player and $\left(V_2^l l, V_2^h h\right)$ of a second player are equilibrium strategies under dominant strategy mechanism (Bayesian incentive compatible mechanism), where $V_i^{l,h} \in \{N, Y\}$- a vote of 'yes' or 'no' to the collusion question.

We say that a voting profile $\left(V_1^l V_1^h, V_2^l V_2^h\right)$ is optimal in dominant strategies mechanism (Bayesian incentive compatible mechanism) when revenue is maximized implementing such a profile under a dominant strategy (Bayesian equilibrium) solution concept.

2.2.5 Equilibrium concepts

Dominant strategy equilibrium

A profile of $\left(V_1^l l, V_1^h h\right)$ and $\left(V_2^l l, V_2^h h\right)$ is a truthful **equilibrium in dominant strategies** if strategy of every player is a best response to any possible alternative strategy of the other player. Hence equilibrium in dominant strategies is equivalent to satisfying the following collusion constraints, individual rationality constraints and incentive compatibility constraints.

1. **Collusion constraints**. For every type of the first or second player who voted 'no' to the collusion question, it must be the case that the mechanism payoff is at least as high as the one this player would have got in collusion. This must be the case regardless of whether the other player actually voted 'yes' since for a dominant strategy equilibrium voting 'no' must also be a best response to a strategy of voting 'yes' of the other player.[6] Since these votes are

[6] One could argue that under a less strict condition of dominance after iterated deletion of dominant strategies not all strategies can be relevant. We turn to this question in the end of the paper.

private information the mechanism can not depend directly on them. Hence, we derive the following condition.

$$p_i\left(v_i^m, v_{-i}^k\right) v_i^m - t_i\left(v_i^m, v_{-i}^k\right) \geq C_i\left(v_i^m, v_{-i}^k\right) \tag{2.2}$$

for every i, k and m such that $V_i^m = \{N\}$.

2. **Individual Rationality constraints.**

$$p_i\left(v_i^m, v_{-i}^k\right) v_i^m - t_i\left(v_i^m, v_{-i}^k\right) \geq 0 \tag{2.3}$$

for every m, k, and i (again since a player $-i$ can always vote 'no', it must be the case that under truth telling players' payoff is larger than zero for any reports of both players).

3. **Incentive compatibility constraints.** When a players votes 'no' to the collusion question he must reveal his type correctly

$$p_i\left(v_i^m, v_{-i}^k\right) v_i^m - t_i\left(v_i^m, v_{-i}^k\right) \geq p_i\left(v_i^n, v_{-i}^k\right) v_i^m - t_i\left(v_i^n, v_{-i}^k\right) \tag{2.4}$$

for all m, k and n such that $V_i^m \in \{N\}$. And when he votes 'yes' to the collusion question he must receive in collusion more than he could have got by voting 'no' and reporting a different type

$$C_i\left(v_i^m, v_{-i}^k\right) \geq p_i\left(v_i^n, v_{-i}^k\right) v_i^m - t_i\left(v_i^n, v_{-i}^k\right)$$

for all m, k and n such that $V_i^m \in \{Y\}$.[7]

[7] This condition will almost always be satisfied as it only limits the upper bound for some transfers that a mechanism designer receives.

Bayesian equilibrium

We will say that a profile of $\left(V_1^l l, V_1^h h\right)$ and $\left(V_2^l l, V_2^h h\right)$ is a truthful **Bayesian incentive compatible equilibrium** of a mechanism that depends only on reports of valuations of players with first-oder beliefs (μ_i, λ_i), where μ_i, (λ_i) is a belief of a low (high) type of a player i that the player $-i$ is of a low type, if it satisfies the following collusion constraints, individual rationality constraints and incentive compatibility constraints.[8]

1. **Collusion constraints.** For every type of the first or second player who votes 'no' to the collusion question, it must be the case that the payoff in the mechanism is at least as high as the one this player would get in collusion conditional on other player voting 'yes'. So, taking a low type of player i for example

$$\mu_i \left(p_i \left(v_i^l, v_{-i}^l\right) v_i^l - t_i \left(v_i^l, v_{-i}^l\right)\right) + (1-\mu_i)\left(p_i \left(v_i^l, v_{-i}^h\right) v_i^l - t_i \left(v_i^l, v_{-i}^h\right)\right) \geq \mu_i C_i \left(v_i^l, v_{-i}^l\right) + (1-\mu_i) C_i \left(v_i^l, v_{-i}^h\right) \quad (2.5)$$

for every i, if $V_i^l = \{N\}$, $V_{-i}^l = \{Y\}$ and $V_{-i}^h = \{Y\}$. Or

$$p_i \left(v_i^l, v_{-i}^k\right) v_i^l - t_i \left(v_i^l, v_{-i}^k\right) \geq C_i \left(v_i^l, v_{-i}^k\right) \quad (2.6)$$

for every i, if $V_{-i}^k = \{Y\}$ and $V_{-i}^{-k} = \{N\}$.

[8]Rigorously speaking, there is no reason why we should be interested only in truthful equilibria under Bayesian mechanisms as no revelation argument can be applied for using payoff-dependent mechanisms when type spaces can be arbitrarily complicated. For our purposes though, it's sufficient to find an equilibrium that will be robust to perturbations in beliefs and higher order beliefs and will be better than dominant strategy equilibria. As it is convenient to consider truthful equilibria we will settle on doing so.

2. **Individual Rationality constraints.** These constraints are only relevant for the types participating in the mechanism. Bayesian updating happens only when a player votes 'yes' and still participates in the mechanism. This means that the other player voted 'no' and, hence, it must be that (I will describe only the low type with exactly same conditions for a high type)

$$\mu_i \left(p_i \left(v_i^l, v_{-i}^l\right) v_i^l - t_i \left(v_i^l, v_{-i}^l\right)\right) + (1-\mu_i) \left(p_i \left(v_i^l, v_{-i}^h\right) v_i^l - t_i \left(v_i^l, v_{-i}^h\right)\right) \geq 0 \tag{2.7}$$

for every i, if $V_i^l = \{N\}$ or if $V_i^l = \{Y\}$ and $V_{-i}^l, V_{-i}^h = \{N\}$. And

$$p_i \left(v_i^l, v_{-i}^k\right) v_i^l - t_i \left(v_i^l, v_{-i}^k\right) \geq 0 \tag{2.8}$$

for every i, if $V_i^l = \{Y\}$ and $V_{-i}^k = \{N\}$, $V_{-i}^{-k} = \{Y\}$.

3. **Incentive compatibility constraints.** Again a player facing the mechanism must find it optimal to reveal his type correctly. For the low type of player i, we get

$$\begin{aligned} \mu_i \left(p_i \left(v_i^l, v_{-i}^l\right) v_i^l - t_i \left(v_i^l, v_{-i}^l\right)\right) + (1-\mu_i) \left(p_i \left(v_i^l, v_{-i}^h\right) v_i^l - t_i \left(v_i^l, v_{-i}^h\right)\right) &\geq \\ \mu_i \left(p_i \left(v_i^h, v_{-i}^l\right) v_i^l - t_i \left(v_i^h, v_{-i}^l\right)\right) + (1-\mu_i) \left(p_i \left(v_i^h, v_{-i}^h\right) v_i^l - t_i \left(v_i^h, v_{-i}^h\right)\right) & \end{aligned} \tag{2.9}$$

for every i, if $V_i^l = \{N\}$ or if $V_i^l = \{Y\}$ and $V_{-i}^l, V_{-i}^h = \{N\}$. And

$$p_i \left(v_i^l, v_{-i}^k\right) v_i^l - t_i \left(v_i^l, v_{-i}^k\right) \geq p_i \left(v_i^h, v_{-i}^k\right) v_i^l - t_i \left(v_i^h, v_{-i}^k\right) \tag{2.10}$$

for every i, if $V_i^l = \{Y\}$ and $V_{-i}^k = \{N\}$, $V_{-i}^{-k} = \{Y\}$. While a player reaching a collusion outcome must find it optimal not to vote 'no' and misreport one's type. For example, for a low type

$$\mu_i C_i\left(v_i^l, v_{-i}^l\right) + (1-\mu_i)\, C_i\left(v_i^l, v_{-i}^h\right) \geq$$
$$\mu_i\left(p_i\left(v_i^h, v_{-i}^l\right) v_i^l - t_i\left(v_i^h, v_{-i}^l\right)\right) + (1-\mu_i)\left(p_i\left(v_i^h, v_{-i}^h\right) v_i^l - t_i\left(v_i^h, v_{-i}^h\right)\right) \quad (2.11)$$

if $V_i^l, V_{-i}^l, V_{-i}^h = \{Y\}$. And if some type of the other player voter 'no' the condition becomes[9]

$$\mu_i C_i\left(v_i^l, v_{-i}^l\right) + (1-\mu_i)\left(p_i\left(v_i^l, v_{-i}^h\right) v_i^l - t_i\left(v_i^l, v_{-i}^h\right)\right) \geq$$
$$\mu_i\left(p_i\left(v_i^h, v_{-i}^l\right) v_i^l - t_i\left(v_i^h, v_{-i}^l\right)\right) + (1-\mu_i)\left(p_i\left(v_i^h, v_{-i}^h\right) v_i^l - t_i\left(v_i^h, v_{-i}^h\right)\right) \quad (2.12)$$

2.3 Maxmin foundation of a dominant strategy mechanism

In this section, we will consider different dominant strategy mechanisms that a mechanism designer could offer by changing an implemented voting profile. We will also compare some of them to their respective counterpart mechanisms implemented in a Bayesian equilibrium. Rather than establishing the optimal dominant strategy mechanism (or even best maxmin mechanism) our purpose will be to investigate whether there exists a maxmin foundation for dominant strategy mechanism. Thus, to prove that under some conditions no maxmin foundation exists it will be sufficient to show that all possibly optimal dominant strategy mechanisms are either not optimal in dominant strategy mechanisms class or achieve less revenue than some Bayesian mechanisms for all beliefs and higher order beliefs. The condition that we

[9]Again, constraints (2.11) and (2.12) will almost always be slack

get under which no maxmin foundation exists is fairly intuitive and requires that collusion transfers z_1 and z_2 should not be too far apart from each other, and that the probability of an event where collusion is formed in a Bayesian mechanism is not too large.

Our analysis will consist of the following five steps.

First, we will consider $NY-NN$, $YN-NN$, $NN-NY$, $NN-YN$, $NN-NN$ voting profiles implemented in dominant strategies and argue that they are always inferior to $NN-YY$ and $YY-NN$ voting profiles implemented in dominant strategies. Intuitively, it is so since $YY-NN$ and $NN-YY$ voting profiles eliminate collusion constraints on the type who voted N for the player with NY voting profile without changing instances of participation in the mechanism.

Second, we will argue that $NY-NY$, $NY-YN$, $YN-NY$, $YN-YN$ voting profiles, when every player votes "yes" to the collusion question exactly once, are inferior in the maxmin sense to their corresponding voting profiles implemented through Bayesian mechanisms.

Third, we will find optimal $NN-YY$ and $YY-NN$ implemented in dominant strategies.

Fourth, we will look into $NY-YY$, $YN-YY$, $YY-NY$, $YY-YN$, $YY-YY$ voting profiles implemented in dominant strategies and find when they are suboptimal.

Finally, we show that for some range of parameters voting profiles $NY-NY$, $NY-YN$, $YN-NY$, $YN-YN$ implemented through a Bayesian mechanism will be better than $NN-YY$ and $YY-NN$ voting profiles implemented in dominant strategies for any beliefs and higher-order beliefs.

Lemma 2.1. *$NN-NY$, $NN-YN$, $NY-NN$, $YN-NN$, $NN-NN$ voting profiles implemented in dominant strategies are never strictly optimal.*

Proof. To prove this fact it's sufficient to point out that all the constraints that have to be satisfied to implement $NN-YY$ voting profile in dominant strategies also have to be satisfied for implementing $NN-NY$, $NN-YN$ and $NN-NN$ voting profiles in dominant strategies. This includes collusion constraints for the low and high type of the first player

$$p_1^{11}v_1^l - t_1^{11} \geq z_1 \tag{2.13}$$

$$p_1^{12}v_1^l - t_1^{12} \geq z_1 \tag{2.14}$$

$$p_1^{21}v_1^h - t_1^{21} \geq z_1 + v_1^h - v_2^l \tag{2.15}$$

$$p_1^{22}v_1^h - t_1^{22} \geq z_1 \tag{2.16}$$

incentive compatibility constraints for the first and second player

$$p_1^{1j}v_1^l - t_1^{1j} \geq p_1^{2j}v_1^l - t_1^{2j} \tag{2.17}$$

$$p_1^{2j}v_1^h - t_1^{2j} \geq p_1^{1j}v_1^h - t_1^{1j} \tag{2.18}$$

$$p_2^{i1}v_2^l - t_2^{i1} \geq p_2^{i2}v_2^l - t_2^{i2} \tag{2.19}$$

$$p_2^{i2}v_2^h - t_2^{i2} \geq p_2^{i1}v_2^h - t_2^{i1} \tag{2.20}$$

for $j, i = 1, 2$. Moreover, all individual rationality constraints for the second player have to be satisfied as well with even tighter bounds on those types of the second player that vote N to the collusion question.

Similarly, all the constraints that have to be satisfied in implementing $YY - NN$ voting structure in dominant strategies have to be satisfied in implementing $YN - NN$ and $NY - NN$ voting profiles in dominant strategies. Hence, none of the voting profiles in the title of the lemma can be strictly optimal. □

While we have shown that the above dominant strategy mechanisms are never optimal, we would also avoid using Bayesian mechanisms that implement such voting profiles in establishing whether there is a maxmin foundation for other dominant strategy mechanisms making it harder to show that under certain conditions no maxmin foundation exists. The reason why we avoid using such Bayesian mechanisms is because they do not satisfy trembling hand refinement. Take for example a voting profile $NN-NN$. In a Bayesian mechanism that implements such a profile we could ignore all collusion constraints since nobody is voting yes to the collusion question, but if some type of some player starts voting with very small probability Y instead of N, it is, first, not decreasing the payoff of that player, and, second, the other player immediately wants to switch from voting N to voting Y.

We next consider voting profiles that result in exactly one Y vote.

Lemma 2.2. *Dominant strategy mechanisms that implement $NY-NY$, $NY-YN$, $YN - NY$, $YN - YN$ voting profiles never have a maxmin foundation.*

Proof. In this proof we will consider $NY - NY$ voting profile dealing with all other profiles in the appendix 1. Let's write down all the constraints of the first player for the dominant strategy implementation. According to this voting profile collusion is formed when both players are high types and all other pairs participate in the mechanism. The relevant constraints for this voting profile for the first player are the following. The low type of a player 1 must get a payoff at least as high as in collusion.

$$p_1^{11}v_1^l - t_1^{11} \geq z_1 \tag{2.21}$$

$$p_1^{12}v_1^l - t_1^{12} \geq z_1 \tag{2.22}$$

Individual rationality constraint must be satisfied for a high type

$$p_1^{21}v_1^h - t_1^{21} \geq 0 \tag{2.23}$$

and incentive compatibility constraints for a low type and a high type must satisfy the following conditions

$$p_1^{21}v_1^h - t_1^{21} \geq p_1^{11}v_1^h - t_1^{11} \tag{2.24}$$

$$p_1^{11}v_1^l - t_1^{11} \geq p_1^{21}v_1^l - t_1^{11} \tag{2.25}$$

Also, somewhat less obvious, we also have to ensure that payoff in collusion outcomes for a high type of player 1 must be bigger than payoff he would have got by participating in a mechanism as a low type, since we are seeking implementation in dominant strategies and player 2 may always vote Y to the collusion question. Thus, we have to ensure that

$$z_1 + v_1^h - v_2^l \geq p_1^{11}v_1^h - t_1^{11} \tag{2.26}$$

$$z_1 \geq p_1^{12}v_1^h - t_1^{12} \tag{2.27}$$

Observing the above constraints, we could notice that (2.23) follows from (2.24) and (2.21). Also, from (2.27) and (2.22), it follows that $p_1^{12} = 0$ and $t_1^{12} = -z_1$. As is in the standard problem with zero individual rationality constraints, we guess that (2.21) and (2.24) are binding and we will guess that (2.25) and (2.26) are satisfied in the optimum. Then, we can find optimal transfers $t_1^{11} = p_1^{11}v_1^l - z_1$, $t_1^{12} = p_1^{12}v_1^l - z_1 = -z_1$, $t_1^{21} = p_1^{21}v_1^h - p_1^{11}\left(v_1^h - v_1^l\right) + z_1$.

For the second player we get similar collusion constraints

$$p_2^{11}v_2^l - t_2^{11} \geq z_2 \tag{2.28}$$

$$p_2^{21}v_2^l - t_2^{21} \geq z_2 \tag{2.29}$$

individual rationality constraints

$$p_2^{12}v_2^h - t_2^{12} \geq 0 \tag{2.30}$$

and IC constraints[10]

$$p_2^{12}v_2^h - t_2^{12} \geq p_2^{11}v_2^h - t_2^{11} \tag{2.31}$$

$$p_2^{11}v_2^l - t_2^{11} \geq p_2^{12}v_2^l - t_2^{12} \tag{2.32}$$

$$z_2 + v_2^h - v_1^l \geq p_2^{11}v_2^h - t_2^{11} \tag{2.33}$$

$$z_2 + v_2^h - v_1^h \geq p_2^{21}v_2^h - t_2^{21} \tag{2.34}$$

with similar optimal transfers $t_2^{11} = p_2^{11}v_2^l - z_2$, $t_2^{21} = -z_2$, $t_2^{12} = p_2^{12}v_2^h - p_2^{11}\left(v_2^h - v_2^l\right) - z_2$ with (2.33) and (2.34) being slack at the optimum. Then, taking together the optimal transfers for the first and the second player, we conclude that at the optimum it must be the case that $p_2^{12} = p_1^{21} = 1$. Also, $p_2^{11} = 1$ if $\alpha v_2^l - \beta\left(v_2^h - v_2^l\right) > \alpha v_1^l - \gamma\left(v_1^h - v_1^l\right)$ and $\alpha v_2^l - \beta\left(v_2^h - v_2^l\right) > 0$, $p_1^{11} = 1$ if $\alpha v_2^l - \beta\left(v_2^h - v_2^l\right) < \alpha v_1^l - \gamma\left(v_1^h - v_1^l\right)$ and $\alpha v_1^l - \gamma\left(v_1^h - v_1^l\right) > 0$ and otherwise $p_2^{11} = p_1^{11} = 0$. We

[10] We omit the IC constraints that high type find it optimal to vote Y rather than voting N and participating as a low type as these constraints will be satisfied.

conclude that the maximal revenue in implementing $NY-NY$ in dominant strategies is equal to

$$\begin{aligned} \Pi_{d.s.m}^{NY-NY} &= \max\left\{\alpha v_2^l - \beta\left(v_2^h - v_2^l\right), \alpha v_1^l - \gamma\left(v_1^h - v_1^l\right), 0\right\} + \\ &+ \beta v_2^h + \gamma v_1^h + \delta r - (\alpha+\beta+\gamma)(z_1+z_2) \end{aligned} \quad (2.35)$$

Now let's consider the same voting profile implemented through a Bayesian mechanism. One of the important differences between a dominant strategy implementation and Bayesian implementation is that under Bayesian implementation a player thinking whether to collude or not only considers events when the other player would collude. This implies that collusion constraints are only relevant when one player votes "yes" and the other player votes "no" to the collusion question.

In the following analysis we will find in a class of valuations-dependent mechanisms that implement $NY-NY$ voting profile, one that solves the $\sup_\Gamma \inf_\mu \Pi$ problem on the universal type spaces. We could think about this problem in the following way. Suppose for any mechanism that a mechanism designer chooses, nature picks the worst possible beliefs and higher-order beliefs. What mechanism will then be chosen by a mechanism designer. Generally solving such kind of problems may be extremely hard, but since we will reduce the class of possible mechanisms to simple valuations-based mechanisms and since only two first-order beliefs will be relevant for the following analysis, it will be easy to find sufficient conditions when no maxmin foundation exists.

Following definitions in section $2-4-2$, implementing $NY-NY$ in a Bayesian equilibrium require that for low and high types of the first player with any first-order beliefs μ_1 and λ_1 the following constraints are satisfied.

$$p_1^{12} v_1^l - t_1^{12} \geq z_1 \tag{2.36}$$

$$\mu_1 \left(p_1^{11} v_1^l - t_1^{11}\right) + (1-\mu_1)\left(p_1^{12} v_1^l - t_1^{12}\right) \geq 0 \tag{2.37}$$

$$p_1^{21} v_1^h - t_1^{21} \geq 0 \tag{2.38}$$

$$p_1^{21} v_1^h - t_1^{21} \geq p_1^{11} v_1^h - t_1^{11} \tag{2.39}$$

However, in case when $\mu_1 = 1$, i.e. when the first player beliefs the second player to be the low type with certainty, there is an additional IC constraint[11]

$$p_1^{11} v_1^l - t_1^{11} \geq p_1^{21} v_1^l - t_1^{21} \tag{2.40}$$

There is also an additional constraint for the high type of a first player. Since this type can vote N rather than Y and misreport his type, it must be the case that the following constraint is satisfied[12]

$$\lambda_1 \left(p_1^{21} v_1^h - t_1^{21}\right) + (1-\lambda_1) z_1 \geq \lambda_1 \left(p_1^{11} v_1^h - t_1^{11}\right) + (1-\lambda_1)\left(p_1^{12} v_1^h - t_1^{12}\right) \tag{2.41}$$

It is obvious from (2.36)-(2.41) that the constraints are tightest for $\mu_1 = 1$ and $\lambda_1 = 0$ with the same type of constraints binding as in the dominant strategy implementation and the optimal transfers: $t_1^{11} = p_1^{11} v_1^l$, $t_1^{12} = -z_1$, $t_1^{21} = p_1^{21} v_1^h - p_1^{11}\left(v_1^h - v_1^l\right)$. Notice, first of all, that these transfers satisfy the above set of constraints for any

[11] If $\mu_1 \neq 1$, then setting $t_1^{22} = -\infty$ always satisfy IC constraint of a low type for non-degenerate distribution of types.

[12] The additional IC constraints are not necessarily satisfied if we ignore them and solve for optimal transfers as functions of probabilities, but after we find optimal probabilities we make sure that these IC constraints are indeed satisfied.

λ_1, μ_1 and, second, that t_1^{11} and t_1^{21} are bigger than their corresponding optimal transfers in dominant strategy implementation by z_1. This is so since the low type of the second player votes N to the collusion question, hence, we don't have to provide additional payoff for the low type of the first player to make him vote N if the second player is also low. That implies that t_1^{11} is bigger than its optimal counterpart in the dominant strategy implementation by z_1. Moreover, because of IC constraint (2.39), t_1^{21} is also bigger by z_1.

For the second player we have similar constraints:

$$p_2^{21}v_2^l - t_2^{21} \geq z_2 \tag{2.42}$$

$$\mu_2\left(p_2^{11}v_2^l - t_2^{11}\right) + (1-\mu_2)\left(p_2^{21}v_2^l - t_2^{21}\right) \geq 0 \tag{2.43}$$

$$p_2^{12}v_2^h - t_2^{12} \geq 0 \tag{2.44}$$

$$p_2^{12}v_2^h - t_2^{12} \geq p_2^{11}v_2^h - t_2^{11} \tag{2.45}$$

For $\mu_2 = 1$, i.e. for the case when the low type of the second player believes the first player to be the low type with certainty, there is an additional IC constraint

$$p_2^{11}v_2^l - t_2^{11} \geq p_2^{12}v_2^l - t_2^{12} \tag{2.46}$$

For the high type there is also an additional IC constraint

$$\lambda_2\left(p_2^{12}v_2^h - t_2^{12}\right) + (1-\lambda_2)\left(z_2 + v_2^h - v_1^h\right) \geq \lambda_2\left(p_2^{11}v_2^h - t_2^{11}\right) + (1-\lambda_2)\left(p_2^{21}v_2^h - t_2^{21}\right)$$

All the same considerations as for the first player can be applied again and for worst possible beliefs of $\mu_2 = 1$ and $\lambda_2 = 0$, we get the tightest set of constraints that results in $t_2^{11} = p_2^{11} v_2^l$, $t_2^{21} = p_2^{21} v_2^l - z_2$, $t_2^{12} = p_2^{12} v_2^h - p_2^{11} \left(v_2^h - v_2^l \right)$. Hence total revenue is equal to

$$\begin{aligned} \Pi &= p_1^{11} \left(\alpha v_1^l - \gamma \left(v_1^h - v_1^l \right) \right) + p_2^{11} \left(\alpha v_2^l - \beta \left(v_2^h - v_2^l \right) \right) + \\ &+ \beta p_1^{12} v_1^l + \beta p_2^{12} v_2^h + \gamma p_1^{21} v_1^h + \gamma p_2^{21} v_2^h - \beta z_1 - \gamma z_2 + \delta r \end{aligned}$$

and it must be the case that $p_1^{21} = 1$, $p_2^{12} = 1$ so additional IC constraints are satisfied. The total revenue is equal to

$$\Pi_{B.m.}^{NY-NY} = \max \left\{ \alpha v_1^l - \gamma \left(v_1^h - v_1^l \right), \alpha v_2^l - \beta \left(v_2^h - v_2^l \right), 0 \right\} + \beta \left(v_2^h - z_1 \right) + \gamma \left(v_1^h - z_2 \right) + \delta r$$

Thus, since $\Pi_{B.m.}^{NY-NY}$ is bigger than $\Pi_{d.s.m.}^{NY-NY}$ by $(\alpha + \gamma) z_1 + (\alpha + \beta) z_2$, which is strictly greater than zero for non-degenerate distribution of types, it becomes clear that a dominant strategy mechanism that implements $NY - NY$ voting profile never has a maxmin foundation.

For the remaining voting profiles ($NY - YN$, $YN - NY$, $YN - YN$), the proof is completely analogous. We find optimal Bayesian mechanisms in the Appendix 1. □

This lemma deserves some discussion as it will be one of the driving forces of the main result. When comparing IC, IR and collusion constraints of the above dominant strategy mechanism to those of a Bayesian mechanism for worst possible beliefs ($\mu_1 = 1$, $\lambda_1 = 0$), one could notice that one major difference is that for implementation in dominant strategies we require

$$p_1^{11} v_1^l - t_1^{11} \geq z_1$$

while for the worst case beliefs of $\mu_1 = 1$, we only require

$$p_1^{11} v_1^l - t_1^{11} \geq 0$$

This wedge is coming from the idea that in Bayesian implementation votes Y or N to the collusion question are only important if the other player also voted Y. Therefore, in Bayesian implementation in order to make a player vote N we need to provide him collusion payoffs only in situations when the other player voted Y. In the dominant strategy implementation, however, we can never exclude the strategy of other player voting Y from the set of allowed strategies. Moreover, as I show in conclusion, even when we adopt a stricter notion of iterated deletion of strictly dominated strategies, the strategy of the other player voting Y can not be excluded.

In the next lemma we introduce possible candidates to optimal dominant strategy mechanisms and, thus, to those dominant strategy mechanisms that might have a maxmin foundation.

Lemma 2.3. *The highest revenue from implementing an $NN - YY$ voting profile in a dominant strategy mechanism is equal to*

$$\begin{aligned}\Pi_{dsm}^{NN-YY} &= (\beta + \delta) v_2^h + \gamma v_2^l + \left(\frac{v_1^h - v_2^l}{v_1^h - v_1^l}\right) \max\left\{\alpha v_1^l, \alpha v_2^l - \beta\left(v_2^h - v_2^l\right)\right\} + \\ &+ \left(\frac{v_2^l - v_1^l}{v_1^h - v_1^l}\right) \max\left\{\alpha v_1^l - \gamma\left(v_1^h - v_1^l\right), \alpha v_2^l - \beta\left(v_2^h - v_2^l\right), 0\right\} - z_1\end{aligned}$$

while the highest revenue from implementing a $YY - NN$ in a dominant strategy mechanism is equal to

$$\Pi_{dsm}^{YY-NN} = (\alpha + \beta) v_1^l + (\gamma + \delta) v_1^h - z_2$$

Proof. Recall the matrix of collusion payoffs for $r \leq v_1^l$

	v_2^l	v_2^h
v_1^l	$z_1,\ z_2+v_2^l-v_1^l$	$z_1,\ z_2+v_2^h-v_1^l$
v_1^h	$z_1+v_1^h-v_2^l,\ z_2$	$z_1,\ z_2+v_2^h-v_1^h$

Hence, using notation of Chapter 1, we have $a_1^{21}-a_1^{11}=v_1^h-v_2^l$, $a_1^{22}-a_1^{12}=0$, $a_2^{ij}=0$ for implementing $NN-YY$ profile and $a_2^{12}-a_2^{11}=v_2^h-v_2^l$, $a_2^{22}-a_2^{21}=v_2^h-v_1^h$, $a_1^{ij}=0$ for $YY-NN$ voting profile. In what follows we also incorporate the results of lemma 2 of Chapter 1 that optimal probabilities p_1^{21}, p_2^{12} are such that $p_1^{21}=p_2^{12}=1$.

NN-YY voting profile

When a mechanism designer picks the first player to give him collusion payoffs, we know that since $p_1^{12}=0$ and since $0=a_1^{22}-a_1^{12}\leq p_1^{12}\left(v_1^h-v_1^l\right)$ by lemma 1 of Chapter 1, we can say that transfers of the first player are equal to

$$t_1^{12}=p_1^{12}v_1^l-a_1^{12}=-z_1$$

$$t_1^{22}=p_1^{22}v_1^h-p_1^{12}\left(v_1^h-v_1^l\right)-a_1^{22}=p_1^{22}v_1^h-z_1$$

Moreover, since $a_1^{21}-a_1^{11}=v_1^h-v_2^l<p_1^{21}\left(v_1^h-v_1^l\right)=v_1^h-v_1^l$, we can pin down transfers to the low type of the first player when the second player is low

$$t_1^{11}=p_1^{11}v_1^l-a_1^{11}=p_1^{11}v_1^l-z_1$$

Also, since $a_2^{12}=a_2^{11}=a_2^{22}=a_2^{21}=0$, we can immediately pin down all transfers from the second player:

$$t_2^{11}=p_2^{11}v_2^l$$

$$t_2^{21}=p_2^{21}v_2^l=0$$

$$t_2^{12} = p_2^{12} v_2^h - p_2^{11}\left(v_2^h - v_2^l\right) = v_2^h - p_2^{11}\left(v_2^h - v_2^l\right)$$

$$t_2^{22} = p_2^{22} v_2^h$$

It follows immediately that optimal probabilities when both players are high types is determined by $p_2^{22} = 1$, since $\delta v_2^h > \delta v_1^h$. As for t_1^{21}, the exact form of transfers will depend on whether $a_1^{21} - a_1^{11} = v_1^h - v_2^l$ is bigger than $p_1^{11}\left(v_1^h - v_1^l\right)$. If this is the case, then by lemma 1 of Chapter 1

$$t_1^{21} = v_1^h - a_1^{21} = v_2^l - z_1$$

otherwise,

$$t_1^{21} = v_1^h - p_1^{11}\left(v_1^h - v_1^l\right) - a_1^{11} = v_1^h - p_1^{11}\left(v_1^h - v_1^l\right) - z_1$$

Thus, for $p_1^{11} < \frac{v_1^h - v_2^l}{v_1^h - v_1^l}$, which is always smaller than 1, the coefficient before p_1^{11} is equal to αv_1^l, while and for $p_1^{11} > \frac{v_1^h - v_2^l}{v_1^h - v_1^l}$, the coefficient before p_1^{11} is equal to $\alpha v_1^l - \gamma\left(v_1^h - v_1^l\right)$. The coefficient for p_2^{11} is always equal to $\alpha v_2^l - \beta\left(v_2^h - v_2^l\right)$.

Summarizing the above, we have several cases:

a) If $\alpha v_1^l - \gamma\left(v_1^h - v_1^l\right) > \alpha v_2^l - \beta\left(v_2^h - v_2^l\right)$ and $\alpha v_1^l - \gamma\left(v_1^h - v_1^l\right) > 0$, then $p_1^{11} = 1$. And total revenue is equal to

$$\begin{aligned}\Pi_{dsm}^{NN-YY} &= \alpha\left(v_1^l - z_1\right) + \beta\left(-z_1 + v_2^h\right) + \gamma\left(v_1^l - z_1\right) + \delta\left(v_2^h - z_1\right) = \\ &= (\beta + \delta)\, v_2^h + (\gamma + \alpha)\, v_1^l - z_1\end{aligned}$$

b) If $0 \geq \alpha v_1^l - \gamma\left(v_1^h - v_1^l\right)$ and $0 \geq \alpha v_2^l - \beta\left(v_2^h - v_2^l\right)$, then $p_1^{11} = \frac{v_1^h - v_2^l}{v_1^h - v_1^l}$, $p_2^{11} = 0$

$$\begin{aligned}
\Pi_{dsm}^{NN-YY} &= \alpha\left(\frac{v_1^h - v_2^l}{v_1^h - v_1^l} v_1^l - z_1\right) + \beta\left(-z_1 + v_2^h\right) + \gamma\left(v_2^l - z_1\right) + \delta\left(v_2^h - z_1\right) = \\
&= (\beta + \delta) v_2^h + \gamma v_2^l + \alpha \frac{v_1^h - v_2^l}{v_1^h - v_1^l} v_1^l - z_1
\end{aligned}$$

c) If $\alpha v_1^l - \gamma\left(v_1^h - v_1^l\right) < \alpha v_2^l - \beta\left(v_2^h - v_2^l\right) < \alpha v_1^l$ and $\alpha v_2^l - \beta\left(v_2^h - v_2^l\right) > 0$, then $p_1^{11} = \frac{v_1^h - v_2^l}{v_1^h - v_1^l}$, $p_2^{11} = \frac{v_2^l - v_1^l}{v_1^h - v_1^l}$

$$\begin{aligned}
\Pi_{dsm}^{NN-YY} &= \alpha\left(\frac{v_1^h - v_2^l}{v_1^h - v_1^l} v_1^l + \frac{v_2^l - v_1^l}{v_1^h - v_1^l} v_2^l - z_1\right) + \beta\left(-z_1 + v_2^h - \frac{v_2^l - v_1^l}{v_1^h - v_1^l}\left(v_2^h - v_2^l\right)\right) + \\
&+ \gamma\left(v_2^l - z_1\right) + \delta\left(v_2^h - z_1\right) = \alpha\left(\frac{v_1^h - v_2^l}{v_1^h - v_1^l} v_1^l + \frac{v_2^l - v_1^l}{v_1^h - v_1^l} v_2^l\right) + \\
&+ (\beta + \delta) v_2^h + \beta\left(-\frac{v_2^l - v_1^l}{v_1^h - v_1^l}\left(v_2^h - v_2^l\right)\right) + \gamma v_2^l - z_1
\end{aligned}$$

d) If $\alpha v_1^l < \alpha v_2^l - \beta\left(v_2^h - v_2^l\right)$ and $\alpha v_2^l - \beta\left(v_2^h - v_2^l\right) > 0$, then $p_1^{11} = 0$, $p_2^{11} = 1$

$$\Pi_{dsm}^{NN-YY} = \alpha\left(v_2^l - z_1\right) + \beta\left(v_2^l - z_1\right) + \gamma\left(v_2^l - z_1\right) + \delta\left(v_2^h - z_1\right) = (\alpha + \beta + \gamma) v_2^l + \delta v_2^h - z_1$$

which together gives us the formula in the statement of the lemma

YY-NN voting profile

Suppose now the mechanism designer "buys out" the second player. Then since $a_1^{ij} = 0$ we have the standard case and using lemma 1 and lemma 2 of Chapter 1, we immediately pin down transfers for the first player:

$$t_1^{11} = p_1^{11} v_1^l$$

$$t_1^{21} = p_1^{21} v_1^h - p_1^{11}\left(v_1^h - v_1^l\right) = v_1^h - p_1^{11}\left(v_1^h - v_1^l\right)$$

$$t_1^{12} = p_1^{12} v_1^l = 0$$

$$t_1^{22} = p_1^{22} v_1^h$$

Since $a_2^{12} - a_2^{11} = v_2^h - v_2^l \geq p_2^{12}\left(v_2^h - v_2^l\right)$ and $a_2^{22} - a_2^{21} = v_2^h - v_1^h \geq p_2^{21}\left(v_2^h - v_2^l\right) = 0$ by using lemma 1 and 2 of Chapter 1, the transfers of the second player are expressed by

$$t_2^{11} = p_2^{11} v_2^l + p_2^{12}\left(v_2^h - v_2^l\right) - a_2^{12} = p_2^{11} v_2^l + v_1^l - v_2^l - z_2$$

$$t_2^{12} = p_2^{12} v_2^h - a_2^{12} = v_1^l - z_2$$

$$t_2^{22} = p_2^{22} v_2^h - a_2^{22} = p_2^{22} v_2^h - v_2^h + v_1^h - z_2$$

As for t_2^{21}, it formula depends on whether $a_2^{22} - a_2^{21} = v_2^h - v_1^h$ is bigger than $p_2^{22}\left(v_2^h - v_2^l\right)$. If this is the case, then

$$t_2^{21} = p_2^{22}\left(v_2^h - v_2^l\right) - a_2^{22} = p_2^{22}\left(v_2^h - v_2^l\right) - v_2^h + v_1^h - z_2$$

and otherwise,

$$t_2^{21} = p_2^{21} v_2^l - a_2^{21} = -z_2$$

Solving for optimal probabilities, we note, first of all, that $p_2^{11} = 1$ since $\alpha v_2^l > \alpha v_1^l$. Secondly, the coefficient before p_1^{22}, which is equal to δv_1^h, is always smaller than the minimal coefficient before p_2^{22}, which is equal to δv_2^h. Hence, $p_2^{22} = 1$. Thus, total revenue in this case is equal to

$$\begin{aligned}\Pi_{dsm}^{YY-NN} &= \alpha\left(v_1^l - z_2\right) + \beta\left(v_1^l - z_2\right) + \gamma\left(v_1^h - z_2\right) + \delta\left(v_1^h - z_2\right) = \\ &= (\alpha + \beta)\, v_1^l + (\gamma + \delta)\, v_1^h - z_2\end{aligned}$$

□

Implementing these voting profiles in dominant strategies may be a best thing for a mechanism designer who is anxious about possible beliefs and higher-order beliefs of the players. Moreover, it can be shown by Theorem 1 of Chapter 1 that there exist a maxmin foundation if the set of voting profiles possible for implementation is limited by $\{NN - YY,\ YY - NN\}$ (in other words there exists such a conjecture of a mechanism designer on the universal type space that under Bayesian implementation provides revenue at most equal to that of dominant strategy implementation). However, since our purpose is to find conditions under which no maxmin foundation exists, we omit this result.

We now turn to the remaining voting profiles of dominant strategy implementation.

Lemma 2.4. *$NY - YY$, $YY - NY$, $YY - YY$ voting profiles are either not optimal dominant strategy mechanisms or do not have a maxmin foundation.* Voting profiles YN-YY, YY-YN *are either not optimal dominant strategy mechanisms or do not have a maxmin foundation for a range of parameters where* $(\alpha + \beta)\, v_1^l + \gamma v_2^l + \delta v_1^h < v_2^l$.

Proof. First of all, notice that the revenue from implementing $NN - YY$ voting structure in dominant strategies has a lower bound that is bigger than r

$$\begin{aligned}\Pi_{d.s.m}^{NN-YY} &= (\beta+\delta)\, v_2^h + \gamma v_2^l + \left(\frac{v_1^h - v_2^l}{v_1^h - v_1^l}\right) \max\left\{\alpha v_1^l, \alpha v_2^l - \beta\left(v_2^h - v_2^l\right)\right\} + \\ &+ \left(\frac{v_2^l - v_1^l}{v_1^h - v_1^l}\right) \max\left\{\alpha v_1^l - \gamma\left(v_1^h - v_1^l\right), \alpha v_2^l - \beta\left(v_2^h - v_2^l\right), 0\right\} - z_1 > \\ &> \delta v_2^h + (\gamma+\alpha+\beta)\, v_2^l - z_1 > r = \Pi_{d.s.m}^{YY-YY}\end{aligned}$$

as

$$r + z_1 < r + z_1 + z_2 = (\alpha + \beta)\, v_1^l + \gamma v_2^l + \delta v_1^h < \delta v_2^h + (\gamma + \alpha + \beta)\, v_2^l$$

Thus, implementing voting profile $YY - YY$ is never optimal among dominant strategy mechanisms.

Second, consider $NY - YY$ voting structure. Because of the following collusion and incentive compatibility constraints

$$p_1^{11} v_1^l - t_1^{11} \geq z_1 \tag{2.47}$$

$$p_1^{12} v_1^l - t_1^{12} \geq z_1 \tag{2.48}$$

$$z_1 + v_1^h - v_2^l \geq p_1^{11} v_1^h - t_1^{11} \tag{2.49}$$

$$p_2^{12} v_2^h - t_2^{12} \geq p_2^{11} v_2^h - t_2^{11} \tag{2.50}$$

the upper bound of the mechanism designer is given by

$$\Pi_{d.s.m}^{NY-YY} \leq (\gamma + \delta)\, r + \beta \left(v_2^h - z_1\right) - \alpha z_1 + \max \left\{ \alpha v_1^l \frac{v_1^h - v_2^l}{v_1^h - v_1^l}, \alpha v_2^l - \beta \left(v_2^h - v_2^l\right) \right\}$$

with the maximum coming from the incentive compatibility constraints (2.50) and (2.49). Comparing this upper bound to the revenue achieved from implementing optimal $NY - YN$ in a Bayesian equilibrium, we derive

$$\begin{aligned}
\Pi_{B.m.}^{NY-YN} &= \frac{v_1^h - v_2^l}{v_1^h - v_1^l} \max\left\{\alpha v_1^l, a v_2^l - \beta\left(v_2^h - v_2^l\right)\right\} + \\
&+ \frac{v_2^l - v_1^l}{v_1^h - v_1^l} \max\left\{a v_2^l - \beta\left(v_2^h - v_2^l\right), 0\right\} + \beta v_2^h + \delta\left(v_1^h - z_2\right) - \alpha z_1 + \gamma r > \\
&> \max\left\{\alpha v_1^l \frac{v_1^h - v_2^l}{v_1^h - v_1^l}, a v_2^l - \beta\left(v_2^h - v_2^l\right)\right\} + \beta v_2^h + \delta\left(v_1^h - z_2\right) - \alpha z_1 + \gamma r > \\
&> \max\left\{\alpha v_1^l \frac{v_1^h - v_2^l}{v_1^h - v_1^l}, \alpha v_2^l - \beta\left(v_2^h - v_2^l\right)\right\} + (\gamma + \delta) r + \beta\left(v_2^h - z_1\right) - \alpha z_1 \geq \\
&\geq \Pi_{d.s.m}^{NY-YY}
\end{aligned}$$

since $r + z_2 < v_1^h$ by equation (2.1). Thus, implementing $NY - YN$ in dominant strategies can't have a maxmin foundation.

Third, we turn to $YN - YY$ voting profile. Incorporating collusion constraints, we can notice that the upper bound to revenue is given by

$$\Pi_{d.s.m}^{YN-YY} \leq (\alpha + \beta) r + \gamma\left(v_2^l - z_1\right) + \delta\left(v_2^h - z_1\right)$$

Comparing this bound to the revenue derived from the optimal $NN - YY$ voting profile implemented in dominant strategies we notice that

$$\Pi_{d.s.m}^{NN-YY} = (\alpha + \beta + \gamma) v_2^l + \delta v_2^h - z_1 > (\alpha + \beta) r + \gamma\left(v_2^l - z_1\right) + \delta\left(v_2^h - z_1\right) \geq \Pi_{d.s.m}^{YN-YY}$$

since by our assumption

$$r + z_1 < r + z_1 + z_2 = (\alpha + \beta) v_1^l + \gamma v_2^l + \delta v_1^h < v_2^l$$

Fourth, let's consider $YY - NY$ case. Incorporating collusion constraints on the low type of a second player, we derive upper bound on the revenue

$$\Pi_{d.s.m}^{YY-NY} \leq (\beta + \delta) r + \alpha \left(v_1^l - z_2\right) + \gamma \left(v_1^h - z_2\right)$$

and comparing this bound to the revenue from an optimal Bayesian mechanism, we derive

$$\begin{aligned} \Pi_{B.m.}^{YN-NY} &= \alpha \left(v_1^l - z_2\right) + \beta r + \gamma v_1^h + \delta \left(v_2^h - z_1\right) > \\ &> (\beta + \delta) r + \alpha \left(v_1^l - z_2\right) + \gamma \left(v_1^h - z_2\right) \geq \Pi_{d.s.m}^{YY-NY} \end{aligned}$$

as

$$\delta \left(v_2^h - r - z_1\right) + \gamma z_2 > 0$$

Thus, implementing $YY - NY$ voting profile in dominant strategies can never have a maxmin foundation.

Finally, let's consider $YY - YN$ case. Incorporating collusion constraints on the high type of the second player, we derive upper bound on the revenue

$$\Pi_{d.s.m}^{YY-YN} \leq (\alpha + \gamma) r + \beta \left(v_1^l - z_2\right) + \delta \left(v_1^h - z_2\right)$$

and comparing it to an optimal Bayesian mechanism that implements $YN - YN$, we notice that

$$\Pi_{B.m.}^{YN-YN} = \alpha r + \beta \left(v_1^l - z_2\right) + \gamma \left(v_2^l - z_1\right) + \delta v_2^h > (\alpha + \gamma) r + \beta \left(v_1^l - z_2\right) + \delta \left(v_1^h - z_2\right)$$

since $v_2^l - r - z_1 > 0$. □

Having covered all possible voting profiles that could be implemented in dominant strategies, we can finally establish conditions under which no maxmin foundation could exist. Suppose as in the statement of lemma we consider the range of parameters to satisfy condition $(\alpha+\beta)v_1^l+\gamma v_2^l+\delta v_1^h<v_2^l$. We can interpret it as saying that the probability of both types being high should not be too large. Summarizing previous four lemmas (and results of Appendix 1) we get the following two tables for revenues from robust Bayesian mechanisms

Profiles	Revenues from Bayesian mechanisms that are robust to maxmin criterion
$NY-NY$	$\max\left\{\alpha v_1^l-\gamma\left(v_1^h-v_1^l\right),\alpha v_2^l-\beta\left(v_2^h-v_2^l\right),0\right\}+$ $+\beta\left(v_2^h-z_1\right)+\gamma\left(v_1^h-z_2\right)+\delta r$
$NY-YN$	$\frac{v_1^h-v_2^l}{v_1^h-v_1^l}\max\left\{\alpha v_1^l,av_2^l-\beta\left(v_2^h-v_2^l\right)\right\}+\frac{v_2^l-v_1^l}{v_1^h-v_1^l}\max\left\{av_2^l-\beta\left(v_2^h-v_2^l\right),0\right\}+$ $+\beta v_2^h+\delta\left(v_1^h-z_2\right)-\alpha z_1+\gamma r$
$YN-NY$	$\alpha\left(v_1^l-z_2\right)+\beta r+\gamma v_1^h+\delta\left(v_2^h-z_1\right)$
$YN-YN$	$\alpha r+\beta\left(v_1^l-z_2\right)+\gamma\left(v_2^l-z_1\right)+\delta v_2^h$

and revenues from dominant strategy mechanisms that are candidates to having maxmin foundation

Voting profiles	Revenues from dominant strategy mechanisms
$NN-YY$	$(\beta+\delta)v_2^h+\gamma v_2^l+\left(\frac{v_1^h-v_2^l}{v_1^h-v_1^l}\right)\max\left\{\alpha v_1^l,\alpha v_2^l-\beta\left(v_2^h-v_2^l\right)\right\}+$ $+\left(\frac{v_2^l-v_1^l}{v_1^h-v_1^l}\right)\max\left\{\alpha v_1^l-\gamma\left(v_1^h-v_1^l\right),\alpha v_2^l-\beta\left(v_2^h-v_2^l\right),0\right\}-z_1$
$YY-NN$	$(\alpha+\beta)v_1^l+(\gamma+\delta)v_1^h-z_2$

Which leads us to the following theorem.

Theorem 2.1. *When $(\alpha+\beta)v_1^l+\gamma v_2^l+\delta v_1^h<v_2^l$ and the maximum of the four revenues from robust Bayesian mechanisms is bigger than the maximum of the two revenues from dominant strategy mechanisms, dominant strategy mechanisms have no maxmin foundation.*

Proof. Lemmas 1-4 establish the result. □

We would like now to give a simple sufficient condition when no maxmin foundation exists. We would pick a voting profile $NY-NY$ implemented through Bayesian implementation for all possible beliefs and higher-order beliefs and find conditions when revenue from implementing this mechanism is bigger than revenue from the above dominant strategy mechanisms. Note that this doesn't imply that such a Bayesian mechanism will be optimal, but it does imply non-existence of the maxmin foundation. Intuitively, the sufficient condition that we get says that probability of both players being high should not be too large (as otherwise the relative loss of getting just a reservation price rather than first or second player high valuation would not be compensated by the relaxed collusion constraints when one of the players is not a high type) and that collusion transfers that players get should not be too far apart from each other (as otherwise mechanism designer will find it optimal to "buy out" a player who doesn't get too much in collusion)

Corollary 2.1. *If collusion transfers z_1, z_2 are not too different from each other $-d < z_2 - z_1 < c$ and if the probability of both players being high is not too large, $\delta < \bar{\delta}$, there exists no maxmin foundation for dominant strategy mechanisms.*

Proof. Comparing $\Pi_{B.m.}^{NY-NY}$ to $\Pi_{d.s.m.}^{NN-YY}$, we derive that $\Delta\Pi_1 = \Pi_{B.m.}^{NY-NY} - \Pi_{d.s.m.}^{NN-YY}$ is equivalent to

$$\begin{aligned}
\Delta\Pi_1 &= \left(\frac{v_1^h - v_2^l}{v_1^h - v_1^l}\right)\left(\max\left\{\alpha v_1^l - \gamma\left(v_1^h - v_1^l\right), \alpha v_2^l - \beta\left(v_2^h - v_2^l\right), 0\right\}\right) + \\
&- \left(\frac{v_1^h - v_2^l}{v_1^h - v_1^l}\right)\max\left\{\alpha v_1^l, \alpha v_2^l - \beta\left(v_2^h - v_2^l\right)\right\} + \alpha z_1 + \gamma\left(v_1^h - v_2^l + z_1 - z_2\right) + \\
&+ \delta\left(r + z_1 - v_2^h\right) \geq \alpha z_1 + \gamma\left(z_1 - z_2\right) + \delta\left(r + z_1 - v_2^h\right)
\end{aligned}$$

Therefore, if $z_1 > z_2 - c$, where

$$c = \frac{0.5\alpha\left(\alpha+\beta\right)v_1^l + \gamma v_2^l + \delta v_1^h - r}{0.5\alpha + \gamma} > 0$$

we get from equation (1) that

$$\left(\alpha+\beta\right)v_1^l + \gamma v_2^l + \delta v_1^h = r + z_1 + z_2 < r + c + 2z_1$$

which implies

$$z_1 > 0.5\left(\left(\alpha+\beta\right)v_1^l + \gamma v_2^l + \delta v_1^h - r - c\right)$$

Hence, there exists δ such that for $\delta < \bar{\delta}_1$, $\Delta\Pi_1 > 0$. To see this set

$$\bar{\delta}_1 = \frac{\alpha z_1 + \gamma\left(z_1 - z_2\right)}{v_2^h - r - z_1}$$

and notice that this threshold for δ is bigger than zero since

$$\alpha z_1 + \gamma\left(z_1 - z_2\right) > 0.5\alpha\left(\left(\alpha+\beta\right)v_1^l + \gamma v_2^l + \delta v_1^h - r\right) - \left(0.5\alpha + \gamma\right)c = 0$$

Now let's compare $\Pi_{B.m.}^{NY-NY}$ to $\Pi_{d.s.m.}^{YY-NN}$. For $\Delta\Pi_2 = \Pi_{B.m.}^{NY-NY} - \Pi_{d.s.m.}^{NN-YY}$ we derive

$$\begin{aligned}\Delta\Pi_2 &= \max\left\{\alpha v_1^l - \gamma\left(v_1^h - v_1^l\right), \alpha v_2^l - \beta\left(v_2^h - v_2^l\right), 0\right\} + \beta\left(v_2^h - v_1^l + z_2 - z_1\right) + \\ &+ \alpha\left(z_2 - v_1^l\right) + \delta\left(r + z_2 - v_1^h\right) \geq \beta\left(v_2^l - v_1^l + z_2 - z_1\right) + \\ &+ \alpha\left(z_2 + v_2^l - v_1^l\right) + \delta\left(r + z_2 - v_1^h\right)\end{aligned}$$

Suppose $z_2 > z_1 - d$, where

$$d = \frac{(\alpha+\beta)\left(v_2^l - v_1^l\right) + 0.5\alpha\left((\alpha+\beta)v_1^l + \gamma v_2^l + \delta v_1^h - r\right)}{\beta + 0.5\alpha} > 0$$

then using again equation (1), we observe that

$$z_2 > 0.5\left((\alpha+\beta)v_1^l + \gamma v_2^l + \delta v_1^h - r - d\right)$$

Hence, for $\delta < \bar{\delta}_2$, $\Delta\Pi_2 > 0$ where

$$\bar{\delta}_2 = \frac{\beta\left(v_2^l - v_1^l + z_2 - z_1\right) + \alpha\left(z_2 + v_2^l - v_1^l\right)}{v_1^h - r - z_2} > 0$$

since

$$\begin{aligned}\beta\left(v_2^l - v_1^l + z_2 - z_1\right) + \alpha\left(z_2 + v_2^l - v_1^l\right) &> (\alpha+\beta)\left(v_2^l - v_1^l\right) + \\ +0.5\alpha\left((\alpha+\beta)v_1^l + \gamma v_2^l + \delta v_1^h - r\right) &- (0.5\alpha+\beta)d = 0\end{aligned}$$

Not taking $\bar{\delta} = \min\left\{\bar{\delta}_1, \bar{\delta}_2, \frac{(\alpha+\beta)\left(v_2^l - v_1^l\right)}{v_1^h}\right\}$, and collusion transfers z_1, z_2 such that $-d < z_2 - z_1 < c$, we obtain the result. □

A possible critique of this result may be that assumption of common knowledge of collusion transfers z_1 and z_2 may be too strong and contrary to the spirit of our exercise. Although we don't have a formal result for the environment when these transfers are private information, we make the following observation. If a mechanism designer does not know collusion transfers, then dominant strategy implementation would require even stricter conditions to be satisfied, essentially lowering revenue from a dominant strategy mechanism to a much smaller revenue since the correspondence between collusion transfers and mechanism designer payoffs is one to one and since a mechanism designer can not predict whether it is beneficial to "buy out" player one or player two. However, Bayesian mechanisms that we consider require that collusion payoffs are guaranteed for both players only at a particular instances (if, for example, $NY - NY$ voting profile is implemented we need to provide collusion payoffs for player 1 when the first player is low and the second player is high and for player 2 when the first player is high and the second player is low). Since probabilities of facing such uncertainties will be smaller for Bayesian mechanisms that we consider, it will be even less appealing to use dominant strategy mechanisms.

2.4 Conclusion

We conclude by stating that in practical environments where strategy spaced can be sufficiently rich like in auction environments where collusion threat is real, dominant strategy implementation may require too many conditions to be satisfied even compared to Bayesian equilibrium implementation robust to perturbations in beliefs and higher order beliefs. The difference between two solution concepts is too strong even if we somewhat relax dominant strategy implementation to implementation in dominant strategies after iterated deletion of weakly dominated strategies. Take for example a voting profile of $NY - NY$. Under such a voting profile in dominant strategy mechanism that we used collusion payoff has to be guaranteed to the low type of the first player even when the second player is also low, since it is possible that the low type of the second player may vote Y to the collusion question. But in dominant strategy implementation after iterated deletion of strictly dominated strategies, the strategy of voting Y for the low type of the second player can only be excluded if the strategy of voting Y for the low type of the first player has been excluded before. But this is not possible when a second player can still vote Y. We believe, thus, that this difference between dominant strategy implementation and equilibrium implementation is too large in practical applications to be bridged by demanding robustness of Bayesian equilibria to all possible beliefs and higher order beliefs.

2.5 Appendix 1

We derive below optimal $NY-YN$, $YN-NY$, $YN-YN$ implemented in a Bayesian equilibrium.

Implementing $NY-YN$ voting structure in a Bayesian mechanism In this case high type of player 1 and low type of player 2 vote "yes" to the collusion question, so the constraints will be the following

$$p_1^{11}v_1^l - t_1^{11} \geq z_1 \tag{2.51}$$

$$\mu_1\left(p_1^{11}v_1^l - t_1^{11}\right) + (1-\mu_1)\left(p_1^{12}v_1^l - t_1^{12}\right) \geq 0 \tag{2.52}$$

$$p_1^{22}v_1^h - t_1^{22} \geq 0 \tag{2.53}$$

$$p_1^{22}v_1^h - t_1^{22} \geq p_1^{12}v_1^h - t_1^{12} \tag{2.54}$$

However, in case when $\mu_1 = 0$, i.e. the first player beliefs the second player to be the high type with certainty, there is an additional IC constraint

$$p_1^{12}v_1^l - t_1^{12} \geq p_1^{22}v_1^l - t_1^{22} \tag{2.55}$$

Also, high type of a player 1, must not vote 'no' and misreport himself to be of a low type

$$\lambda_1\left(z_1 + v_1^h - v_2^l\right) + (1-\lambda_1)\left(p_1^{22}v_1^h - t_1^{22}\right) \geq \lambda_1\left(p_1^{11}v_1^h - t_1^{11}\right) + (1-\lambda_1)\left(p_1^{12}v_1^h - t_1^{12}\right) \tag{2.56}$$

It can be noticed that for $\mu_1 > 0$, the revenue must be increasing with respect to μ_1, as for higher μ_1 it is possible to increase t_1^{12} and relax (2.54). Then, similarly to the case of $NY - NY$ voting profile, we set $\mu_1 = 0$. It follows then that (2.53) will be always satisfied due to (2.52) and (2.54). Also, (2.55) can be assumed to be satisfied at the optimum. Also (2.56) is the tightest when $\lambda_1 = 1$ because of (2.54). Thus, simplifying (2.56) we get

$$t_1^{11} \geq p_1^{11} v_1^h - z_1 - v_1^h + v_2^l \tag{2.57}$$

So, we can solve for optimal transfers at $\mu_1 = 0$: $t_1^{11} = p_1^{11} v_1^l - z_1$, $t_1^{12} = p_1^{12} v_1^l$, $t_1^{22} = p_1^{22} v_1^h - p_1^{12}\left(v_1^h - v_1^l\right)$. With (2.57) satisfied when $p_1^{11} \leq \frac{v_1^h - v_2^l}{v_1^h - v_1^l}$.

For the second player the analysis is almost the same except for binding IC constraints. The constraints for the second player are:

$$p_2^{22} v_2^h - t_2^{22} \geq z_2 + v_2^h - v_1^h \tag{2.58}$$

$$\lambda_2\left(p_2^{12} v_2^h - t_2^{12}\right) + (1 - \lambda_2)\left(p_2^{22} v_2^h - t_2^{22}\right) \geq 0 \tag{2.59}$$

$$p_2^{11} v_2^l - t_2^{11} \geq 0 \tag{2.60}$$

$$p_2^{11} v_2^l - t_2^{11} \geq p_2^{12} v_2^l - t_2^{12} \tag{2.61}$$

When $\lambda_2 = 1$, i.e. the high type of the second player believes the first player to be the low type with certainty, there is an additional IC constraint[13]

$$p_2^{12} v_2^h - t_2^{12} \geq p_2^{11} v_2^h - t_2^{11} \tag{2.63}$$

[13]There is also an additional constraint for the low type of a second player not to vote 'no' and misreport to be a high type

$$\mu_2\left(p_2^{11} v_2^l - t_2^{11}\right) + (1 - \mu_2) z_2 \geq \mu_2\left(p_2^{12} v_2^l - t_2^{12}\right) + (1 - \mu_2)\left(p_2^{22} v_2^l - t_2^{22}\right) \tag{2.62}$$

which is the tightest for $\mu_2 = 0$ and which will be satisfied if $p_2^{22} = 1$ (this will be the case).

Again $\lambda_2 = 1$ gives the tightest set of constraints. However, now it will be an additional incentive constraint $IC_{H\to L}$ that will be binding and $IC_{L\to H}$ can be assumed to be satisfied. Also, it is clear that (2.59) will be satisfied from (2.63) and (2.60) and so, we can solve for transfers: $t_2^{11} = p_2^{11}v_2^l$, $t_2^{22} = p_2^{22}v_2^h - z_2 - v_2^h + v_1^h$, $t_2^{12} = p_2^{12}v_2^h - p_2^{11}\left(v_2^h - v_2^l\right)$. Note that $IC_{L\to H}$ is satisfied as long as $p_2^{12} \geq p_2^{11}$.

The total profit is, thus, equal to

$$\begin{aligned}
\Pi_{B.m.}^{NY-YN} &= p_1^{11}\left(\alpha v_1^l\right) + p_2^{11}\left(av_2^l - \beta\left(v_2^h - v_2^l\right)\right) + p_1^{12}\left(\beta v_1^l - \delta\left(v_1^h - v_1^l\right)\right) + \\
&+ p_2^{12}\left(\beta v_2^h\right) + p_1^{22}\left(\delta v_1^h\right) + p_2^{22}\left(\delta v_2^h\right) - \alpha z_1 - \delta\left(z_2 + v_2^h - v_1^h\right) + \gamma r
\end{aligned}$$

Hence, $p_2^{12} = p_2^{22} = 1$ and total revenue is equal to

$$\begin{aligned}
\Pi_{B.m.}^{NY-YN} &= \frac{v_1^h - v_2^l}{v_1^h - v_1^l}\max\left\{\alpha v_1^l, av_2^l - \beta\left(v_2^h - v_2^l\right)\right\} + \\
&+ \frac{v_2^l - v_1^l}{v_1^h - v_1^l}\max\left\{av_2^l - \beta\left(v_2^h - v_2^l\right), 0\right\} + \beta v_2^h + \delta\left(v_1^h - z_2\right) - \alpha z_1 + \gamma r
\end{aligned}$$

Implementing $YN - NY$ voting structure in a Bayesian mechanism The constraints for the first and the second player are the following:

$$p_1^{22}v_1^h - t_1^{22} \geq z_1 \tag{2.64}$$

$$\lambda_1\left(p_1^{21}v_1^h - t_1^{21}\right) + (1-\lambda_1)\left(p_1^{22}v_1^h - t_1^{22}\right) \geq 0$$

$$p_1^{11}v_1^l - t_1^{11} \geq 0$$

$$p_1^{11}v_1^l - t_1^{11} \geq p_1^{21}v_1^l - t_1^{21}$$

with an additional constraint for $\lambda_1 = 1$[14]

$$p_1^{21} v_1^h - t_1^{21} \geq p_1^{11} v_1^h - t_1^{11}$$

And for the second player

$$p_2^{11} v_2^l - t_2^{11} \geq z_2 + v_2^l - v_1^l \tag{2.65}$$

$$\mu_2 \left(p_2^{11} v_2^l - t_2^{11}\right) + (1 - \mu_2) \left(p_2^{21} v_2^l - t_2^{21}\right) \geq 0$$

$$p_2^{22} v_2^h - t_2^{22} \geq 0$$

$$p_2^{22} v_2^h - t_2^{22} \geq p_2^{21} v_2^h - t_2^{21}$$

with an additional constraint for $\mu_2 = 0$[15]

$$p_2^{21} v_2^l - t_2^{21} \geq p_2^{22} v_2^l - t_2^{22}$$

Again, the revenue is minimized at $\lambda_1 = 1$, $\mu_2 = 0$, i.e. when high type of a first player believe the second player to be the low type and low type of second

[14]There is an extra IC constraint for a low type not to vote 'no' and misreport himself to be a high type

$$\mu_1 \left(p_1^{11} v_1^l - t_1^{11}\right) + (1 - \mu_1) z_1 \geq \mu_1 \left(p_1^{21} v_1^l - t_1^{21}\right) + (1 - \mu_1) \left(p_1^{22} v_1^l - t_1^{22}\right)$$

which is the tightest at $\mu_1 = 0$ and which will be always satisfied if (2.64) is binding (which will be the case)

[15]There is also an additional IC constraint for a high type of a second player not to vote "no" and misreport his type to be that of a low one.

$$\lambda_2 \left(z_2 + v_2^h - v_1^l\right) + (1 - \lambda_2) \left(p_2^{22} v_2^h - t_2^{22}\right) \geq \lambda_2 \left(p_2^{11} v_2^h - t_2^{11}\right) + (1 - \lambda_2) \left(p_2^{21} v_2^h - t_2^{21}\right)$$

which is the tightest for $\lambda_2 = 1$ and which is always satisfied when (2.65) is binding (which will be the case) player believe the second player to be the high type. In this case we can easily solve for transfers as before: $t_1^{22} = p_1^{22} v_1^h - z_1$, $t_1^{21} = p_1^{21} v_1^h - p_1^{11}\left(v_1^h - v_1^l\right)$, $t_1^{11} = p_1^{11} v_1^l$, $t_2^{22} = p_2^{22} v_2^h - p_2^{21}\left(v_2^h - v_2^l\right)$, $t_2^{21} = p_2^{21} v_2^l$, $t_2^{11} = p_2^{11} v_2^l - z_2 - v_2^l + v_1^l$. And the total revenue is equal to

$$\begin{aligned}\Pi_{B.m.}^{YN-NY} &= p_1^{11}\left(\alpha v_1^l - \gamma\left(v_1^h - v_1^l\right)\right) + p_2^{11}\left(\alpha v_2^l\right) + p_1^{21}\left(\gamma v_1^h\right) + p_2^{21}\left(\gamma v_2^l - \delta\left(v_2^h - v_2^l\right)\right) + \\ &+ p_1^{22}\left(\delta v_1^h\right) + p_2^{22}\left(\delta v_2^h\right) - \delta z_1 - \alpha\left(z_2 + v_2^l - v_1^l\right) + \beta r\end{aligned}$$

and hence $p_2^{11} = p_1^{21} = p_2^{22} = 1$ and maximal revenue is given by

$$\Pi_{B.m.}^{YN-NY} = \alpha\left(v_1^l - z_2\right) + \beta r + \gamma v_1^h + \delta\left(v_2^h - z_1\right)$$

Implementing $YN-YN$ voting structure in a Bayesian mechanism Constraints in this case are the following:

$$p_1^{21} v_1^h - t_1^{21} \geq z_1 + v_1^h - v_2^l \tag{2.66}$$

$$\lambda_1\left(p_1^{21} v_1^h - t_1^{21}\right) + (1-\lambda_1)\left(p_1^{22} v_1^h - t_1^{22}\right) \geq 0$$

$$p_1^{12} v_1^l - t_1^{12} \geq 0$$

$$p_1^{12} v_1^l - t_1^{12} \geq p_1^{22} v_1^l - t_1^{22}$$

With an additional IC constraint for $\lambda_1 = 0$[16]

$$p_1^{22}v_1^h - t_1^{22} \geq p_1^{12}v_1^h - t_1^{12}$$

$$p_2^{12}v_2^h - t_2^{12} \geq z_2 + v_2^h - v_1^l \tag{2.67}$$

$$\lambda_2\left(p_2^{12}v_2^h - t_2^{12}\right) + (1-\lambda_2)\left(p_2^{22}v_2^h - t_2^{22}\right) \geq 0$$

$$p_2^{21}v_2^l - t_2^{21} \geq 0$$

$$p_2^{21}v_2^l - t_2^{21} \geq p_2^{22}v_2^l - t_2^{22}$$

with an additional IC constraint for $\lambda_2 = 0$[17]

$$p_2^{22}v_2^h - t_2^{22} \geq p_2^{21}v_2^h - t_2^{21}$$

Again, taking $\lambda_1 = \lambda_2 = 0$, we solve for transfers: $t_1^{12} = p_1^{12}v_1^l$, $t_1^{21} = p_1^{21}v_1^h - z_1 - v_1^h + v_2^l$, $t_1^{22} = p_1^{22}v_1^h - p_1^{12}\left(v_1^h - v_1^l\right)$, $t_2^{12} = p_2^{12}v_2^h - z_2 - v_2^h + v_1^l$, $t_2^{21} = p_2^{21}v_2^l$, $t_2^{22} = p_2^{22}v_2^h - p_2^{21}\left(v_2^h - v_2^l\right)$. Thus, we can immediately see that it will be optimal to have $p_2^{12} = p_1^{21} = p_2^{22} = 1$ and the optimal revenue is equal to

[16]There is also an additional IC constraint for a low type of a player 1 not to vote "no" and misreport itself to be of a high type

$$\mu_1 z_1 + (1-\mu_1)\left(p_1^{12}v_1^l - t_1^{12}\right) \geq \mu_1\left(p_1^{21}v_1^l - t_1^{21}\right) + (1-\mu_1)\left(p_1^{22}v_1^l - t_1^{22}\right)$$

which is the tightest for $\mu_1 = 1$ and which will be satisfied for binding (2.66) and $p_1^{21} = 1$

[17]There is also an additional IC constraint for a low type of a player 2 not to vote "no" and misreport itself to be a low type

$$\mu_2\left(z_2 + v_2^l - v_1^l\right) + (1-\mu_2)\left(p_2^{21}v_2^l - t_2^{21}\right) \geq \mu_2\left(p_2^{12}v_2^l - t_2^{12}\right) + (1-\mu_2)\left(p_2^{22}v_2^l - t_2^{22}\right)$$

which is the tightest for $\mu_2 = 1$ and which will be satisfied for binding (2.67) and $p_2^{12} = 1$

$$\Pi_{B.m.}^{YN-YN} = \alpha r + \beta \left(v_1^l - z_2\right) + \gamma \left(v_2^l - z_1\right) + \delta v_2^h$$

Chapter 3

Strategic Voting, Welfare and Non-transferable Utility Mechanism Design.

3.1 Introduction

The Gibbard-Satterthwaite theorem showed that there exists no resolute voting rule except for dictatorial ones for three or more alternatives that is strategy proof, is defined for universal domain of preferences and satisfies the Pareto condition. Several directions of research followed this result. First, the domain was restricted to find non-manipulable voting rules (Kalai and Muller (1977), Barbie, Puppe and Tasnadi (2003)). Second, the attention was focused on degrees of manipulability of different voting rules resulting in both analytical (Nitzan (1985), Leppeley and Mbih (1987), Favardin, Leppeley, Serais (2002) and Leppeley and Valgones (2003)) and simulational (Nitzan (1985), Kelly (1993) and Smith (1999)) results. These papers concentrated on calculating the percentage of preference profiles that leads to untruthful voting. While the question of manipulability may be of its own interest, the more relevant question seems to be a normative one: how strategic voting affects total welfare and which rules affect it the most when strategic voting is taken into consideration? Moreover, a recent paper (Maskin (2008)) shows that if one assumes non-manipulability and some other basic assumptions, no optimal rule can be cardinal. This result provides an important starting point for this chapter, the main idea of which is to show that strategic voting may lead to higher total welfare through better realization of preference intensities. We prove in Theorem 1 that for any distribution function that generates values of different alternatives, strategic voting, when it exists, leads to ex ante Pareto improvement compared to sincere voting. The reason why this is true is the risky nature of strategic voting - when somebody casts a strategic ballot, she tries to increase the probability of a favorable alternative, but it also increases the probability of unfavorable alternatives, which were regarded by this individual as not likely to win. Thus, only those with high enough stakes would want to engage in strategic voting. This introduces a cardinal component into the voting rules that may lead to greater total ex ante welfare.

The above idea also links strategic voting to the literature on costly voting (Ledyard (1984), Myerson (2000)), which establishes that despite positive costs it is always possible to achieve a Pareto efficient outcome in a large election with two alternatives and majority voting. Although only those with costs going to zero are participating in an election in that model, since costs and utilities are uncorrelated, those for whom stakes are higher abstain less, which enables us to achieve Pareto efficiency.

The importance of realization of cardinal intensities in a voting model, also renders interest to a mechanism design problem with non-transferable utility, where the strength of the preferences has to be communicated to the mechanism designer, but where there are no actual payments in money between the parties. It turns out that just as in the voting setting, exposing agents to uncertainty can help to elicit their preferences, and in the end achieve a second-best outcome that looks very close to the first best. We derive in Theorem 2 the second best for the uniform distribution case and characterize optimal rules that are very close to the first best. Hence, we conclude that in the settings where the uncertainty is high and when information transmission is high, optimal mechanisms will almost achieve a Pareto outcome. If, however, opportunities to communicate preference intensities are scarce, strategic voting becomes a proxy for such information transmission.

Our model on mechanism design is close to the one studied by Borgers and Postl (2009). However, they study a situation where agents' preference rankings are the opposite of each other and so two players have to figure out whether they should compromise or whether somebody's first choice will be picked. In our model we allow for all the possible rankings. It also turns out that differently from the optimal rules in their model, optimal rules in our model look similar to the standard optimal rules in the public good provision literature, i.e. in a setting with transferable utility. We describe the potential similarities when we derive those rules.

In section 2 we analyze a simple symmetric model of voting, emphasizing that strategic voting leads to welfare improvement, whenever equilibria with strategic voting exist. We also characterize the nature of equilibria that involve strategic voting. In section 3, we talk about similarities to the two alternatives, costly voting literature. In section 4, we consider a mechanism design problem in a similar environment to the one in the first part. Section 5 concludes.

3.2 A Voting Model

3.2.1 Assumptions and Structure of Equilibria

Suppose there are 2 voters, $i = 1, 2$ and 3 alternatives, $j = x, y, z$. Utility of a given alternative for a given voter is represented by v^i_j. We assume that v^i_j are iid according to some distribution function $G()$.

The outcome is determined by a scoring rule of the type $(1, A, 0)$ and, in particular, $0.5 < A < 1$. As it will be shown below this rule is convenient for consideration since for a every preference profile, there can be at most two voting ballots of preference rankings that could be cast optimally. To see this, let's consider the following table of outcomes, where rows are ballots cast by a first player and columns are ballots cast by a second player.

	α	β	γ	δ	ε	$1-\alpha-\beta-\gamma-\delta-\varepsilon$
	xyz	xzy	yxz	yzx	zxy	zyx
xyz	x	x	$\widetilde{xy}$	y	x	y
xzy	x	x	x	z	$\widetilde{xz}$	z
yxz	$\widetilde{xy}$	x	y	y	x	y
yzx	y	z	y	y	z	$\widetilde{yz}$
zxy	x	$\widetilde{xz}$	x	z	z	z
zyx	y	z	y	$\widetilde{yz}$	z	z

Here $\widetilde{xy}$ denotes a tie between the first and the second alternative. Observing the table of outcomes, it becomes clear that it's never optimal for a type xyz to cast a vote different from xyz or xzy. For example, it is never optimal to cast a vote of yxz. Voting yxz can be optimal when a voter expects a "close race" between second and third alternatives and puts his second best alternative on top of the ballot to prevent his least liked alternative from winning. However, with this scoring rule any type can guarantee that at least his second-placed alternative wins by voting sincerely, so there is no need to misreport one's type as yxz.

We now turn to analyzing all equilibria of the game. As it will turn out there will always be a sincere voting equilibrium (which is not surprising since this environment is characterized by high uncertainty), but depending on the distribution function $G()$, there could be also a couple of equilibria that involve strategic voting.

Suppose that probabilities of voting a particular profile ballot for the second player is given by $(\alpha, \beta, \gamma, \delta, \epsilon, 1-\alpha-\beta-\gamma-\delta-\varepsilon)$. Finding the differences in utilities from voting in two possible ways (sincerely and strategically) and suppressing the index of the voter, we derive the following expressions

$$U_{xyz} - U_{xzy} = v_x^1 \left(\frac{\varepsilon}{2} - \frac{\gamma}{2}\right) + v_y^1 \left(1 - \alpha - \beta - \varepsilon - \frac{\gamma}{2}\right) - v_z^1 \left(1 - \alpha - \beta - \gamma - \frac{\varepsilon}{2}\right)$$

$$U_{yxz} - U_{yzx} = v_x^1 \left(\beta + \varepsilon + \frac{\alpha}{2}\right) + v_y^1 \left(\frac{1 - 2\alpha - \beta - \gamma - \delta - \varepsilon}{2}\right) - v_z^1 \left(\frac{1 - \alpha + \beta - \gamma - \delta + \varepsilon}{2}\right)$$

$$U_{zxy} - U_{zyx} = v_z^1 \left(\frac{\delta}{2} - \frac{\beta}{2}\right) + v_x^1 \left(\alpha + \gamma + \frac{\beta}{2}\right) - v_y^1 \left(\alpha + \gamma + \frac{\delta}{2}\right)$$

Thus, the misreporting for types $xyz,\ yxz,\ zxy$ from xyz to xzy, from yxz to yzx, from zxy to zyx occur whenever in a corresponding order

$$\frac{1}{2} v_x^1 (\gamma - \varepsilon) > v_y^1 \left(1 - \alpha - \beta - \varepsilon - \frac{\gamma}{2}\right) - v_z^1 \left(1 - \alpha - \beta - \gamma - \frac{\varepsilon}{2}\right) \tag{3.1}$$

$$v_y^1 \left(\frac{-1 + 2\alpha + \beta + \gamma + \delta + \varepsilon}{2}\right) > v_x^1 \left(\beta + \epsilon + \frac{\alpha}{2}\right) - v_z^1 \left(\frac{1 - \alpha + \beta - \gamma - \delta + \varepsilon}{2}\right) \tag{3.2}$$

$$v_z^1 \left(\frac{\beta}{2} - \frac{\delta}{2}\right) > v_x^1 \left(\alpha + \gamma + \frac{\beta}{2}\right) - v_y^1 \left(\alpha + \gamma + \frac{\delta}{2}\right) \tag{3.3}$$

Note, first of all, that these bounds are always tighter than corresponding conditional ones (i.e the ones that are determined by conditional inequalities, e.g. $v_x^1 > v_y^1 > v_z^1$ for a type xyz). For a type xyz

$$\frac{1}{2} v_y^1 (\gamma - \varepsilon) < v_y^1 \left(1 - \alpha - \beta - \varepsilon - \frac{\gamma}{2}\right) - v_z^1 \left(1 - \alpha - \beta - \gamma - \frac{\varepsilon}{2}\right)$$

is equivalent to

$$(v_y^1 - v_z^1) \left(\alpha + \beta + \gamma + \frac{\varepsilon}{2} - 1\right) < 0$$

and this always holds. For a type yxz

$$v_x^1\left(\frac{-1+2\alpha+\beta+\gamma+\delta+\varepsilon}{2}\right) < v_x^1\left(\beta+\varepsilon+\frac{\alpha}{2}\right) - v_z^1\left(\frac{1-\alpha+\beta-\gamma-\delta+\varepsilon}{2}\right)$$

is equivalent to

$$(v_x^1 - v_z^1)\left(\frac{\alpha-\beta+\gamma+\delta-\varepsilon-1}{2}\right) < 0$$

and finally for a type zxy

$$v_x^1\left(\frac{\beta}{2}-\frac{\delta}{2}\right) < v_x^1\left(\alpha+\gamma+\frac{\beta}{2}\right) - v_y^1\left(\alpha+\gamma+\frac{\delta}{2}\right) \iff (v_1 - v_2)\left(-\alpha-\gamma-\frac{\delta}{2}\right) < 0$$

Note also that misreporting occurs if and only if the highest utility is multiplied by a positive number, e.g. when $\gamma - \epsilon$ is positive in the first inequality. To see this we can express these inequalities, e.g. (3.9) in the following way:

$$\frac{1}{2}(v_x^1 - v_y^1)(\gamma - \varepsilon) > (v_y^1 - v_z^1)\left(1-\alpha-\beta-\gamma-\frac{\epsilon}{2}\right)$$

and as the right hand side is always positive $\gamma - \epsilon$ has to be positive if a type xyz wants to cast a ballot xzy.

Let's define η, κ and λ to be the probabilities with which (3.1), (3.2), (3.3) occur, then summarizing the above, we derive the following structure of best responses for all types. For η, κ, λ depending on α, β, γ, δ, ϵ

xyz : votes xzy with probability η and xyz with probability $1-\eta$ if $\gamma > \varepsilon$ and o/w always votes xyz

xzy : votes xyz with probability η and xzy with probability $1-\eta$ if $\gamma < \varepsilon$ and o/w always votes xzy

yxz : votes yzx with probability κ and yxz with probability $1-\kappa$ if $\alpha > 1-\alpha-\beta-\gamma-\delta-\varepsilon$ and o/w always votes yxz

yzx : votes yxz with probability κ and yzx with probability $1-\kappa$ if $\alpha < 1-\alpha-\beta-\gamma-\delta-\varepsilon$ and o/w always votes yzx

zxy : votes zyx with probability λ and zxy with probability $1-\lambda$ if $\beta > \delta$ and o/w always votes zxy

zyx : votes zxy with probability λ and zyx with probability $1-\lambda$ if $\beta < \delta$ and o/w always votes zyx

Let's now impose equilibrium symmetry conditions on these probabilities. Suppose $\gamma > \epsilon$, then, since the same distribution function generates all utilities and since xyz-type votes xzy with probability η we get that $\alpha = \frac{1}{6}(1-\eta)$, $\beta = \frac{1}{6}(1+\eta)$, $\eta > 0$. Assume that $\alpha \geq 1-\alpha-\beta-\gamma-\delta-\epsilon$, then, we have $\gamma = \frac{1}{6}(1-\kappa)$, $\delta = \frac{1}{6}(1+\kappa)$, $k \geq 0$ and, thus

$$\begin{aligned} \frac{1}{6}(1-\kappa) &> \epsilon \\ \frac{1}{6}(1-\eta) &\geq \frac{1}{3}-\epsilon \end{aligned}$$

Summing these inequalities up we get $-\frac{\kappa}{6}-\frac{\eta}{6} > 0$, $\eta > 0$, $k \geq 0$ which is a contradiction. So, we must have $\alpha < 1-\alpha-\beta-\gamma-\delta-\epsilon$ and $\gamma = \frac{1}{6}(1+\kappa)$, $\delta = \frac{1}{6}(1-\kappa)$, $k > 0$. Moreover, as $\beta > \delta$, we get $\varepsilon = \frac{1}{6}(1-\lambda)$, $1-\alpha-\beta-\gamma-\delta-\varepsilon = \frac{1}{6}(1+\lambda)$, $\lambda > 0$. Therefore, in equilibrium we get a cyclical permutation of types (xzy, yxz, zyx) that are never engaged in strategic voting and a cycle of types (xyz, zxy, yzx) that do vote strategically.

Using inequalities above, we can express κ, η, λ in the following way.

$$\eta = \Pr\left[\frac{1}{2}v_x^1\frac{\kappa+\lambda}{6} > v_y^1\left(\frac{5}{12}+\frac{\lambda}{6}-\frac{\kappa}{12}\right)+v_z^1\left(-\frac{5}{12}+\frac{\kappa}{6}-\frac{\lambda}{12}\right)|v_x^1 > v_y^1 > v_z^1\right] \tag{3.4}$$

$$\kappa = \Pr\left[\frac{1}{2}v_y^1\frac{\eta+\lambda}{6} > v_z^1\left(\frac{5}{12}+\frac{\eta}{6}-\frac{\lambda}{12}\right)+v_x^1\left(-\frac{5}{12}+\frac{\lambda}{6}-\frac{\eta}{12}\right)|v_y^1 > v_z^1 > v_x^1\right] \tag{3.5}$$

$$\lambda = \Pr\left[\frac{1}{2}v_z^1\frac{\kappa+\eta}{6} > v_x^1\left(\frac{5}{12}+\frac{\kappa}{6}-\frac{\eta}{12}\right)+v_y^1\left(-\frac{5}{12}+\frac{\eta}{6}-\frac{\kappa}{12}\right)|v_z^1 > v_x^1 > v_y^1\right] \tag{3.6}$$

Lemma 3.1. *This symmetrical system of equations (3.4)-(3.6) might have at most one solution, in which $\kappa = \lambda = \eta$.*

Proof. Let's rewrite inequalities in the following way

$$\eta = \Pr\left[v_x^1 - v_y^1 > \left(\frac{5+\lambda-2\kappa}{\kappa+\lambda}\right)\left(v_y^1 - v_z^1\right)|v_x^1 > v_y^1 > v_z^1\right]$$

$$\kappa = \Pr\left[v_y^1 - v_z^1 > \left(\frac{5+\eta-2\lambda}{\eta+\lambda}\right)\left(v_z^1 - v_x^1\right)|v_y^1 > v_z^1 > v_x^1\right]$$

$$\lambda = \Pr\left[v_z^1 - v_x^1 > \left(\frac{5+\kappa-2\eta}{\eta+\kappa}\right)\left(v_x^1 - v_y^1\right)|v_z^1 > v_x^1 > v_y^1\right]$$

Suppose, for example, that $\eta > \kappa$ (any other strict inequalities are covered in the same way since the system is symmetric), then since all utilities are distributed according to the same distribution function, it has to be the case that

$$\frac{5+\lambda-2\kappa}{\kappa+\lambda} < \frac{5+\eta-2\lambda}{\eta+\lambda}$$

From which it follows that $5(\eta-\kappa)+3(\lambda^2-\eta\kappa)<0$, which implies $\lambda<\eta$. But then it follows that

$$\frac{5+\lambda-2\kappa}{\kappa+\lambda}<\frac{5+\kappa-2\eta}{\eta+\kappa}$$

and so $5(\eta-\lambda)+3(-\kappa^2+\eta\lambda)$¡0, which in tern implies that it must be the case that $\kappa>\lambda$. This implies

$$\frac{5+\eta-2\lambda}{\eta+\lambda}<\frac{5+\kappa-2\eta}{\eta+\kappa}$$

or equivalently $5(\kappa-\lambda)+3(\eta^2-\lambda\kappa)<0$, which implies $\eta<\lambda$ and we get a contradiction. □

If we assume that $\gamma=\varepsilon$ by same analysis we would get a sincere equilibrium of $\alpha=\beta=\gamma=\delta=\varepsilon=\frac{1}{6}$. So sincere voting is always an equilibrium of this incomplete information game. If we assume $\gamma<\varepsilon$, we get a similar equilibrium, where two cyclical sets of preference ranks with probabilities higher than $\frac{1}{6}$ and lower than $\frac{1}{6}$ are switching.

If there is strategic voting the probability of it is equal to η and is determined by the following equation.

$$\eta=\Pr\left[v_1>\frac{5+\eta}{2\eta}v_2+\frac{\eta-5}{2\eta}v_3|v_1>v_2>v_3\right] \tag{3.7}$$

where v_1, v_2 and v_3 are maximal, middle and minimum elements drawn from a distribution $G()$, or

$$\eta=6\int\limits_{-\infty}^{+\infty}\int\limits_{v_3}^{+\infty}\int\limits_{\frac{5+\eta}{2\eta}v_2+\frac{\eta-5}{2\eta}v_3}^{+\infty} g(v_1)g(v_2)g(v_3)dv_1dv_2dv_3 \tag{3.8}$$

Example 3.1. Uniform Case

Suppose $v \sim U[-x; x]$. Let's find η. To do this we need to adjust the limits of integration since it has to be the case that

$$\frac{5+\eta}{2\eta}v_2 + \frac{\eta-5}{2\eta}v_3 < x$$

or

$$v_2 < \frac{2\eta}{\eta+5}x - \frac{\eta-5}{\eta+5}v_3$$

which always binds for $v_3 < x$. Substituting these limits into the formula above we get after simplification

$$8x^3\eta = 6\int_{-x}^{x}\int_{v_3}^{\frac{2\eta}{\eta+5}x-\frac{\eta-5}{\eta+5}v_3}\int_{\frac{5+\eta}{2\eta}v_2+\frac{\eta-5}{2\eta}v_3}^{x} dv_1dv_2dv_3 = 2\frac{\eta}{\eta+5}\left(8x^3\right)$$

Implying

$$\eta = \frac{2\eta}{\eta+5}$$

which doesn't have a solution for $\eta \in (0,1)$

Example 3.2. Pareto distribution (with a parameter k and $x_{\min} = 1$)

This distribution has very "fat" tails, so, not surprisingly, there exists non-zero solution, which is, nevertheless rapidly declining[1]

	$k \to 1$	$k = 2$	$k = 3$	$k = 4$
η	0.465	0.259	0.139	0.058

[1]Mathematica was used to solve the necessary equations numerically.

The intuition for these two different examples comes from the equation (3.7). One could notice that for non-zero solution to exist, it's necessary to have "fat" tails of a distribution function, so that the probability of first-best alternative being relatively large is sufficiently high so that a person faces high stakes sufficiently often.

It will be shown in the next part, that equilibria that involve strategic voting can be seen as a cooperation across individuals with different preferences profiles that leads to the shift of probability mass of different preference rankings in such a way that enables them to benefit from such move. This cooperation is incentive compatible in a sense that for all individuals with given preferences it is optimal to pursue the above strategy.

3.2.2 Welfare Analysis

Theorem 3.1. *Outcome of equilibria that involve strategic voting are ex ante welfare improving compared to a sincere voting equilibrium outcome for any distribution function generating utilities* $G(v)$

Proof. To analyze welfare with and without strategic voting, we have to compare the probabilities of different outcomes and utilities at those outcomes, taking into account that utilities from a given alternative are dependent on the fact of whether voter deviated or not from sincere voting. We shall analyze equilibrium with strategic voting when $\gamma > \varepsilon$ (with the other equilibrium being completely symmetrical) and we will consider the situation, where first voter have xyz or xzy preference rankings, while the second one has yxz or yzx preference rankings (other blocks of possible rank preferences are again completely symmetrical).

For the preference rankings above we have the following outcomes:

For a preference profile of (xyz, yxz) only the first voter will vote strategically if the stakes are high enough. So with probability $1-\eta$ there is no misreporting resulting in a tie between $x$and y and with probability η we get (xzy, yxz) ballot profile with x being chosen.

For a preference profile of (xyz, yzx) both voters will vote strategically sometimes. Thus, with probability $(1-\eta)^2$ we have sincere voting with y being chosen. With probability $\eta(1-\eta)$ we get (xyz, yxz) as a ballot profile with a tie between x and y. With probability $\eta(1-\eta)$ we get (xzy, yzx) ballot profile with z being chosen. Finally, with probability η^2 we get (xzy, yxz) ballot profile with x being the winner.

For a preference profile of (xzy, yxz) - there is only sincere voting and x is being chosen.

For a preference profile of (xzy, yzx) - with probability $1-\eta$ there is no misreporting and z is chosen and with probability η we get (xzy, yxz) ballot profile with x as a winner.

Let's denote $\widetilde{a}, \widetilde{b}, \widetilde{c}$ as expectations of first-best, second-best and third-best alternatives conditional on the fact that a voter has voted strategically, $\widehat{a}, \widehat{b}, \widehat{c}$ as expectations of v_1, v_2, v_3 conditional on the fact that a voter has voted sincerely and a, b, c as unconditional expectations of v_1, v_2, v_3. Then,

$$6\Delta SW = \eta\left(\frac{1}{2}\widetilde{a} - \frac{1}{2}\widetilde{b} + \frac{1}{2}b - \frac{1}{2}a\right) + \eta(1-\eta)\left(\frac{1}{2}\widetilde{c} - \frac{1}{2}\widetilde{a} + \frac{1}{2}\widehat{a} - \frac{1}{2}\widehat{b}\right) + \\ + \eta(1-\eta)\left(\widetilde{c} - \widetilde{b} - \widehat{a} + \widehat{b}\right) + \eta^2\left(\widetilde{a} - \widetilde{b} + \widetilde{c} - \widetilde{a}\right) + \eta\left(\widetilde{c} - \widetilde{b} + a - b\right)$$

$$6\Delta SW = \eta\left(\frac{5}{2}\widetilde{c} - \frac{5}{2}\widetilde{b} - \frac{1}{2}b + \frac{1}{2}a - \frac{1}{2}\widehat{a} + \frac{1}{2}\widehat{b}\right) + \eta^2\left(\frac{1}{2}\widetilde{a} - \frac{1}{2}\widetilde{c} + \frac{1}{2}\widehat{a} - \frac{1}{2}\widehat{b}\right)$$

Note that $a-b=\eta\left(\widetilde{a}-\widetilde{b}\right)+(1-\eta)\left(\widehat{a}-\widehat{b}\right)$ and, thus,

$$\begin{aligned}6\nabla SW &= \eta\left(\frac{5}{2}\widetilde{c}-\frac{5}{2}\widetilde{b}-\frac{1}{2}\widehat{a}+\frac{1}{2}\widehat{b}+\frac{1}{2}\eta\left(\widetilde{a}-\widetilde{b}\right)+\frac{1}{2}(1-\eta)\left(\widehat{a}-\widehat{b}\right)\right)+\\ &+ \eta^2\left(\frac{1}{2}\widetilde{a}-\frac{1}{2}\widetilde{c}+\frac{1}{2}\widehat{a}-\frac{1}{2}\widehat{b}\right)=\eta\left(\frac{5}{2}\widetilde{c}-\frac{5}{2}\widetilde{b}\right)+\eta^2\left(\widetilde{a}-\frac{1}{2}\widetilde{b}-\frac{1}{2}\widetilde{c}\right)\end{aligned}$$

Recall that in the region with strategic voting we have $v_1>\frac{5+\eta}{2\eta}v_2+\frac{\eta-5}{2\eta}v_3$ and so it must be the case that $\widetilde{a}>\frac{5+\eta}{2\eta}\widetilde{b}+\frac{\eta-5}{2\eta}\widetilde{c}$. Hence,

$$\begin{aligned}6\nabla SW &= \eta\left(\frac{5}{2}\widetilde{c}-\frac{5}{2}\widetilde{b}\right)+\eta^2\left(\widetilde{a}-\frac{1}{2}\widetilde{b}-\frac{1}{2}\widetilde{c}\right)>\\ &> \eta\left(\frac{5}{2}\widetilde{c}-\frac{5}{2}\widetilde{b}+\eta\left(\frac{5+\eta}{2\eta}\widetilde{b}+\frac{\eta-5}{2\eta}\widetilde{c}-\frac{1}{2}\widetilde{b}-\frac{1}{2}\widetilde{c}\right)\right)=0\end{aligned}$$

□

Therefore, we have shown that through strategic voting and, thus, better realization of cardinal intensities, voters can achieve higher ex ante total welfare. Whenever strategic voting equilibria exist, they are always ex ante Pareto improving compared to sincere voting.

3.3 Costs and Two Alternative Voting

In this section I will emphasize intuition from the first section but through a different channel, namely through costly voting. In the previous model with three alternatives strategic voting has a cost of increasing probability that the least preferred alternative is going to be the winner. Hence, strategic voting was only used by voters who had high enough cardinal difference between the first best and the second best. In this section I link this intuition to a relatively more studied environment where there are only two alternatives, but voting itself is costly. Again, as is the case in Ledyard (1984), who considers an environment of two alternatives and infinite number of voters, it will turn out that only those voters who have high enough difference in utilities for the alternatives will vote and through better realization of cardinal differences and tilting of an outcome towards an alternative with higher preference intensity, higher total ex ante welfare can be achieved.

Suppose there are two alternatives x, y and two voters. Utility from an alternative U_x, U_y is an iid with a distribution function $G()$. The cost of voting is c. Suppose now, that probability of abstention is p. Then, the expected utilities from voting and abstention for the voter that has x alternative as a first best are as follows:

$$U_{xy}^v = \frac{1}{2}(1-p)\left(\frac{1}{2}U_x + \frac{1}{2}U_y\right) + \left(1 - \frac{1}{2}(1-p)\right)U_x - c$$

$$U_{xy}^a = \frac{1}{2}U_x + \frac{1}{2}U_y$$

Thus, a voter will vote whenever $U_{xy}^v - U_{xy}^a$ is positive or when

$$\left(1 - \frac{1}{2}(1-p)\right)\left(\frac{1}{2}U_x - \frac{1}{2}U_y\right) - c > 0$$

So, the probability of abstention can be found from the following equation

$$p = \Pr\left\{U_x - U_y < \frac{4c}{1+p} | U_x > U_y\right\} \tag{3.9}$$

Let's now turn to the normative analysis. The matrix of outcomes when the first voter is of a type xy will look as follows

	xy		yx	
	V	A	V	A
V	x	x	$\tilde{xy}$	x
A	x	$\tilde{xy}$	y	$\tilde{xy}$

where $\widetilde{xy}$ means a tie between x and y.

So, compared to sincere voting with no costs, there is a loss in total welfare when both voters are of the same type xy and they both abstain, but there is a gain when two voters are of different types and one voter abstains, while the other voter votes since in this case the outcome is tilted towards more intensively supported alternative. Moreover, there are costs from just voting. In the end, whether the gains outweigh the losses is determined in equilibrium and depends on the primitives $G()$ and c. We might expect that if the society is relatively homogeneous (i.e. density function $g()$ has high peak and small tails) the gains from costly voting would be small, while if there are "fat" tails in the distribution, introducing some costs can be beneficial to total welfare.

Let's denote by $\widetilde{a}, \widetilde{b}$ the expected utilities for first-ranked alternative and second-ranked alternative when the voter has voted and by $\widehat{a}, \widehat{b}$ when the voter hasn't. Then,

$$\begin{aligned} \Delta SW &= \frac{1}{2}p^2\left(\widehat{a}+\widehat{b}-2\widehat{a}\right)+p(1-p)\left(\widehat{b}+\widetilde{a}-\left(\frac{1}{2}\widehat{a}+\frac{1}{2}\widehat{b}\right)-\left(\frac{1}{2}\widetilde{a}+\frac{1}{2}\widetilde{b}\right)\right)- \\ &- 2c(1-p)=\frac{1}{2}p(1-p)\left(\widetilde{a}-\widetilde{b}\right)+\frac{1}{2}p\left(\widehat{b}-\widehat{a}\right)-2c(1-p) \end{aligned}$$

Now, let's assume that the support of $G()$ is from zero to infinity and let's denote $\frac{4c}{1+p}$ by t. Then, from (3.9)

$$p = 2\int_0^\infty \int_y^{y+t} g(x)g(y)dxdy$$

and

$$\Delta SW = p\int_0^\infty \int_{y+t}^\infty (x-y)\, g(x)g(y)dxdy - \int_0^\infty \int_y^{y+t} (x-y)\, g(x)g(y)dxdy - \frac{1}{2}t(1-p^2)$$

When $c = 0$, $p = 0$ and hence $t = 0$ and so we can analyze ΔSW with respect to t. Note that

$$p_t^{'}|_{t=0} = 2\int_0^\infty g^2(y)dy$$

and, thus,

$$\Delta SW_t'|_{t=0} = 2\int_0^\infty g^2(y)dy \int_0^\infty \int_y^\infty (x-y)\, g(x)g(y)dxdy - \frac{1}{2}$$

So, whenever the difference between the first and second ranked alternatives are expected to be high and whenever the density is not too heavily concentrated near zero, positive costs are going to increase total expected welfare compared to the situation with no costs.

Example 3.3. Uniform density

$$\Delta SW_t'|_{t=0} = 2\int_0^1 dy \int_0^1 \int_y^1 (x-y)\, dxdy - \frac{1}{2} = -\frac{1}{6}$$

Example 3.4. Pareto Distribution

$$\Delta SW_t'|_{t=0} = \frac{4k^2+k-1}{2(2k-1)(2k+1)(k-1)} > 0$$

Confirming our intuition for different distribution functions.

3.4 Mechanism Design Problem

In this section I would study how optimal rules in Bayesian mechanism design with non-transferable utility are connected to voting rules. This setting also appears to be relatively understudied and, hence, is of its own interest.

Consider an environment similar to the one in the previous section, with the following normalization assumption: the utility from a first-best alternative is equal to 1 and from the worst alternative is equal to 0. This normalization follows our research in section 2 and can be loosely thought of as allowing scoring rule to depend on the intensity of the second-best preference. It will be, therefore, interesting, to see if such a relaxation in the scoring rule can help to achieve greater ex ante total welfare. This normalization assumption, as it will turn out, is not without a loss of generality since in a situation when two players have completely reverse orderings of preferences, e.g. XYZ and ZYX, it doesn't matter for efficiency whether X or Z is chosen, whenever alternative Y is not optimal. Hence, it will be easier to satisfy incentive compatibility constraints by choosing X and Z alternatives optimally.[2] In this sense, we can view our results as an upper bound to what can be achieved.

This environment is close to the one studied by Borgers and Postl (2009). They study the situation, where the performance rankings of two players are completely asymmetrical, e.g. XYZ and ZYX, and find that they can't use the usual approach of a public choice literature (e.g. d'Aspremont and Gerard-Varet (1979) and Guth

[2]If we, however, attach different weights to utilities of agents, such normalization would be without loss of generality. Unfortunately, it turns out that this case is much harder to solve analytically, so we have to focus our attention to normalized equal weights environment.

and Hellwig (1986)) since when they use the usual incentive compatibility constraint without imposing non-negativity constraints on probabilities, these constraints are violated. In my extension of the analysis to the whole possible set of preference orderings I don't encounter this problem and, hence, my results are closer to the traditional public good literature.

Let's denote by $f^X(R_1, R_2, a_1, a_2)$, $f^Y(R_1, R_2, a_1, a_2)$, $f^Z(R_1, R_2, a_1, a_2)$ mechanism rules that correspond to the probabilities that X, Y, Z alternatives are chosen given the preference orderings and utility from a second-best alternative. The ex ante probabilities for a first player that alternative X or Y is chosen are equal to

$$p^1(R_1, a_1) = \frac{1}{6} \sum_{R_2} \int f^X(R_1, R_2, a_1, a_2) g(a_2) da_2$$

$$q^1(R_1, a_1) = \frac{1}{6} \sum_{R_2} \int f^Y(R_1, R_2, a_1, a_2) g(a_2) da_2$$

Suppose a voter has a profile XYZ, then the first player will reveal his preference ranking and utility from the second-best alternative truthfully whenever the following IC conditions are satisfied.

$$p^1(XYZ, a_1) + q^1(XYZ, a_1)a_1 \geq p^1(R_1', a_1') + q^1(R_1', a_1')a_1$$

$$p^1(R_1', a_1') + q^1(R_1', a_1')a_1' \geq p^1(XYZ, a_1) + q^1(XYZ, a_1)a_1'$$

which implies local incentive compatibility constraint

$$p^1_{a_1}(XYZ, a_1) + q^1_{a_1}(XYZ, a_1)a_1 = 0$$

and global incentive compatibility constraints of [3]

$$p^1(XYZ, a_1) + q^1(XYZ, a_1)a_1 \geq \max_{R' \neq XYZ, a_1'} p^1(R_1', a_1') + q^1(R_1', a_1')a_1$$

Now, we can note that utility of a first player is

$$U^1(XYZ, a_1) = p^1(XYZ, a_1) + q^1(XYZ, a_1)a_1$$

and hence, taking derivative with respect to the utility of second-best alternative and using local incentive compatibility constraint, we get the standard result that $U^1_{a_1}(XYZ, a_1) = q^1(XYZ, a_1)$, which implies that probability of getting the first-best alternative for any a_1 can be derived by the following equation

$$p^1(XYZ, a_1) = p^1(XYZ, 1) + q^1(XYZ, 1) - q^1(XYZ, a_1)a_1 - \int_{a_1}^{1} q^1(XYZ, x)dx \tag{3.10}$$

We would now assume uniform distribution since most of analytical results could only be derived for this case. Then we have the following lemma proved in the appendix 2. [4]

Lemma 3.2. *When there exist a Pareto dominating alternative, it is always chosen. When utility of one alternative is always non-smaller than utility of any other*

[3] In this environment, as is the standard case in the literature, local incentive compatibility constraints would imply global ones. However, for the agent not to misreport its preference ranking, it's important to assume a symmetric environment, where every preference profile of the second agent is equally likely. In such an environment, when nothing is known about preferences of the second player, it is never optimal to misreport one's own preference ranking. We show that global IC constraints are satisfied in the end of the paper.

[4] For the non-uniform distribution this is true under certain monotonicity assumptions. Since our focus here will be on the uniform case and since this simplifies exposition a lot while not changing any results, we use this result from the very beginning.

alternative, it is always chosen unless second-best alternative has the value 0 *or* 1 *for at least one of the players. Formally,*

$$f^X(XYZ, R_2, a_1, a_2) = 1 \text{ if } R_2 = XYZ,\ XZY \text{ or } ZXY$$

$$f^X(XYZ, R_2, a_1, a_2) = 1 \text{ if } R_2 = YZX$$

It follows from the lemma that for uniform distribution

$$\int_0^1 p^1(XYZ, a_1)\, da_1 = \frac{1}{2} + \frac{1}{6}\int_0^1\int_0^1 f^X(XYZ, YXZ, a_1, a_2)\, da_1 da_2 + \\ + \frac{1}{6}\int_0^1\int_0^1 f^X(XYZ, ZYX, a_1, a_2)\, da_1 da_2 \tag{3.11}$$

$$\int_0^1 q^1(XYZ, a_1)\, da_1 = \frac{1}{6} + \frac{1}{6}\int_0^1\int_0^1 f^Y(XYZ, YXZ, a_1, a_2)\, da_1 da_2 + \\ + \frac{1}{6}\int_0^1\int_0^1 f^Y(XYZ, ZYX, a_1, a_2)\, da_1 da_2 \tag{3.12}$$

Using the IC condition (3.10) we can show that

$$\begin{aligned}
\int_0^1 p^1\left(XYZ,a_1\right)da_1 &= p^1(XYZ,1)+q^1(XYZ,1)-\int_0^1 q^1\left(XYZ,a_1\right)a_1 da_1 - \\
-\int_0^1\int_{a_1}^1 q^1\left(XYZ,x\right)dx da_1 &= p^1(XYZ,1)+q^1(XYZ,1)-\int_0^1 q^1\left(XYZ,a_1\right)a_1 da_1 - \quad (3.13) \\
-\int_0^1\int_0^x q^1(XYZ,x)da_1 dx &= p^1(XYZ,1)+q^1(XYZ,1)-\int_0^1 q^1(XYZ,a_1)\left[2a_1\right]da_1
\end{aligned}$$

We can prove the following theorem.

Theorem 3.2. *For the uniform distribution the optimal mechanism rules are characterized by*

$$f^Y(XYZ,YXZ,a_1,a_2)=1 \text{ iff } a_1 \geq a_2$$

$$f^Y(XYZ,ZYX,a_1,a_2)=1 \text{ iff } a_1+a_2 \geq 1.157$$

Thus, inefficiency occurs only when the preference ranks of two players are opposite to each other and when the sum of utilities of the second-best alternative is between 1 *and* 1.157*.*

Proof. Using lemma 2 the equation (3.13) simplifies to

$$\int_0^1 p^1\left(XYZ,a_1\right)da_1 = p^1(XYZ,1)+q^1(XYZ,1)-\frac{1}{6}-$$

$$-\frac{1}{3}\int_0^1\int_0^1 a_1 f^Y\left(XYZ,YXZ,a_1,a_2\right)da_1 da_2-\frac{1}{3}\int_0^1\int_0^1 a_1 f^Y\left(XYZ,ZYX,a_1,a_2\right)da_2 da_1$$

Hence, using (3.11) and (3.12), we derive

$$\frac{2}{3}+\frac{1}{6}\int_0^1\int_0^1 f^A(XYZ,YXZ,a_1,a_2)da_1da_2+\frac{1}{6}\int_0^1\int_0^1 f^A(XYZ,ZYX,a_1,a_2)da_1da_2 = \\ = p^1(XYZ,1)+q^1(XYZ,1) \\ -\frac{1}{3}\int_0^1\int_0^1 a_1\left[f^B(XYZ,YXZ,a_1,a_2)+f^B(XYZ,ZYX,a_1,a_2)\right]da_1da_2 \quad (3.14)$$

Due to symmetry reasons and dropping constant terms, we can concentrate on maximizing the following part of total ex ante welfare

$$EU\left(XYZ,YXZ,a_1,a_2\right)+EU\left(XYZ,ZYX,a_1,a_2\right)= \\ =\frac{1}{6}\int_0^1\int_0^1 f^X\left(XYZ,YXZ,a_1,a_2\right)\left(1+a_2\right)da_1da_2+ \\ +\frac{1}{6}\int_0^1\int_0^1 f^Y\left(XYZ,YXZ,a_1,a_2\right)\left(1+a_1\right)da_1da_2+ \\ +\frac{1}{6}\int_0^1\int_0^1 f^Y(XYZ,ZYX,a_1,a_2)(a_1+a_2-1)da_1da_2$$

The relevant constraints for maximizing this part of the total welfare are incentive compatibility constraint for a first player when her rank is XYZ (3.14) and two similar constraints for a second player, when his ranks are YXZ and ZYX. Let's assume that Lagrange multipliers corresponding to these constraints are: $\lambda,\ \mu_1,\ \mu_2$.

Incorporating all three IC constraints, the relevant part of a Lagrangian becomes[5]

$$6L = \int_0^1 \int_0^1 \left[\begin{array}{c} f^Y(XYZ, YXZ, a_1, a_2)(1 + a_1 - 2\lambda a_1 - \mu_1) + \\ +f^Y(XYZ, ZYX, a_1, a_2)(a_1 + a_2 - 1 - 2\lambda a_1 - 2\mu_2 a_2 + \mu_2) + \\ +f^X(XYZ, YXZ, a_1, a_2)(1 + a_2 - \lambda - 2\mu_1 a_2) + \\ f^X(XYZ, ZYX, a_1, a_2)(-\lambda + \mu_2) \end{array} \right] da_1 da_2 \tag{3.15}$$

Assuming symmetry in mechanism rules for different players and voting profiles, $\lambda = \mu_1 = \mu_2$ and guessing that optimal $\lambda < 0.5$ we get the following optimal mechanism rules

$$f^Y(XYZ, YXZ, a_1, a_2) = 1 \text{ iff } a_1 \geq a_2 \tag{3.16}$$

$$f^Y(XYZ, ZYX, a_1, a_2) = 1 \text{ iff } a_1 + a_2 \geq \frac{1-\lambda}{1-2\lambda} > 1 \tag{3.17}$$

Thus, we can achieve full efficiency in the case of (XYZ, YXZ) profile, but there is some inefficiency in a (XYZ, ZYX) profile. As was pointed out by Borgers and Postl (2009) such a rule for (XYZ, ZYX) pair resembles the optimal rules in the public good provision literature. Indeed, there are certain similarities between the two models as in our model probability of getting a first-best option may serve essentially as money, while second-best alternative can be thought of a public good, preference for which is varied and has to be revealed. The fact, that the optimal rule in our paper resembles standard optimal rules, while in Borgers and Postl (2009) the rule is different, appears to be a testament to their very specific environment, in which following the standard techniques of the provision of public goods literature results in violation of non-negativity probability constraints.

[5] Technically for $a_1 = 1$, we should also add λf^X and λf^y into the Lagrangian, but this doesn't change any results we are about to present.

As was pointed out earlier, for a profile of preferences (XYZ, ZYX) it does not matter for efficiency whether X or Z is chosen when $a_1 + a_2 \leq 1$, which means that there is greater freedom in choosing $f^X(XYZ, ZYX, a_1, a_2)$ and $f^Z(XYZ, ZYX, a_1, a_2)$ to satisfy incentive compatibility constraints. Indeed, because of this, the only term which mattered for incentive compatibility was only *expected* probability of choosing first alternative conditional on a (XYZ, ZYX) profile. This can be seen from plucking back the optimal mechanism rules into a non-integrated incentive compatibility constraint.

For $a_1 > \frac{\lambda}{1-2\lambda}$, we derive

$$\int_0^1 f^X\left(XYZ, ZYX, a_1, a_2\right) da_2 = \frac{\lambda}{(1-2\lambda)} + a_1 - a_1^2 \tag{3.18}$$

and for $a_1 < \frac{\lambda}{1-2\lambda}$ we get

$$\int_0^1 f^X(XYZ, ZYX, a_1, a_2) da_2 = \frac{\lambda}{(1-2\lambda)} - \frac{1}{2}\frac{\lambda^2}{(1-2\lambda)^2} + a_1 - \frac{1}{2}a_1^2 \tag{3.19}$$

So, there are multiple mechanism rules that are optimal in this context since the only thing that can be pinned down is the expected probability of choosing A or C alternative in a (XYZ, ZYX) case given a_1. To find λ, we have to use the condition that integrated probabilities should add up to one. Integrating (3.18) and (3.19) on the corresponding intervals of a_1 it follows from the above that

$$\int_0^1 \int_0^1 f^X\left(XYZ, ZYX, a_1, a_2\right) da_1 da_2 = \frac{\lambda}{(1-2\lambda)} - \frac{1}{3}\frac{\lambda^3}{(1-2\lambda)^3} + \frac{1}{6} \tag{3.20}$$

and similarly, for the second player

$$\int_0^1 \int_0^1 f^Z(XYZ, ZYX, a_1, a_2) da_1 da_2 = \frac{\lambda}{(1-2\lambda)} - \frac{1}{3}\frac{\lambda^3}{(1-2\lambda)^3} + \frac{1}{6} \tag{3.21}$$

and incorporating mechanism rule (3.17) for $f^Y(XYZ, ZYX, a_1, a_2)$ we get

$$\int_0^1 \int_0^1 f^Y(XYZ, ZYX, a_1, a_2) da_1 da_2 = \frac{1}{2}\left(1 - \frac{\lambda}{1-2\lambda}\right)^2 \tag{3.22}$$

Summing up (3.20) (3.21) and (3.22) (one can think of this equation as summing up vectors of probabilities of choosing A alternative, rows of probabilities of choosing C alternative and summing up probabilities of choosing B alternative), we get

$$1 = \frac{2\lambda}{(1-2\lambda)} - \frac{2}{3}\frac{\lambda^3}{(1-2\lambda)^3} + \frac{5}{6} - \frac{\lambda}{1-2\lambda} + \frac{1}{2}\frac{\lambda^2}{(1-2\lambda)^2}$$

and solving the above equation we get that $\frac{\lambda}{1-2\lambda}$ is equal to roughly 0.157. So, the inefficiency for the uniform case is relatively small. [6] □

3.5 Conclusion

Linking the two parts of this research together, we would like to point out that whenever the opportunities to reveal information about preferences are limited as in the section on voting, strategic voting becomes an instrument through which

[6] In the appendix I show that with suggested $\int_0^1 f^A(XYZ, ZYX, 1, a_2) da_2$ and $\int_0^1 f^C(XYZ, ZYX, 1, a_2) da_2$ the relevant probabilities always lie in the range from zero to one. Also I show that global IC conditions are satisfied.

different cardinal intensities are realized. In such environments where opportunities to reveal cardinal intensities are scarce agents implicitly cooperate in such a way that for any distribution function generating utility, welfare in equilibrium with strategic voting is always higher than in an equilibrium with sincere voting.

We have also seen that employing mechanism design can substantially increase ex ante welfare with only minor losses compared to the first best. Also, the optimal rules appear to be following the standard optimal rules of the public good provision literature.

We would also like to highlight that, while in this paper we assumed that only very limited information is known about other player types, it is an open question of how information structure and precision of information influences strategic voting and total welfare. The environment of Gibbard-Satterthwaite corresponds to common knowledge of the whole preference profile, which leads to large amount of strategic voting. At the same time, in an environment, where individuals do not know anything about preferences of others, the optimal strategy will often be to vote sincerely. In this respect, it is important to investigate how social welfare depends on information structure. A conjecture that one may make is that as precision of information increases, first, there are wider possibilities of realization of preference intensities, but later (as the precision gets very high), strategic voting is used too often, which leads to bad voting outcomes. Thus, there might exist a possibility of non-monotonic relation between social welfare and precision of information. We believe that studying benefits of the strategic voting and corresponding mechanisms as a function of information structure is an exciting area for further research.

3.6 Appendix 1. Non-negativity of probabilities and global IC conditions.

Let's show that with suggested mechanism rules of section 4, the probabilities, of getting X or Z, $\int_0^1 f^X(XYZ, ZYX, 1, a_2)da_2$ and $\int_0^1 f^Z(XYZ, ZYX, 1, a_2)da_2$, always lie in the range from zero to one.To do this it is sufficient to check that

$$0 \leq \int_0^1 f^X(XYZ, ZYX, a_1, a_2)da_2 + \int_0^1 f^Y(XYZ, ZYX, a_1, a_2)da_2 \leq 1$$

For $a_1 < \frac{\lambda}{1-2\lambda}$, we derive

$$\int_0^1 f^X(XYZ, ZYX, a_1, a_2)da_2 + \int_0^1 f^Y(XYZ, ZYX, a_1, a_2)da_2 = \\ = \frac{\lambda}{(1-2\lambda)} - \frac{1}{2}\frac{\lambda^2}{(1-2\lambda)^2} + a_1 - \frac{1}{2}a_1^2$$

which is minimal at $a_1 = 0$ and equal to $\frac{\lambda}{(1-2\lambda)} - \frac{1}{2}\frac{\lambda^2}{(1-2\lambda)^2} > 0$ and maximal at $a_1 = \frac{\lambda}{1-2\lambda}$ and equal to $\frac{2\lambda}{(1-2\lambda)} - \frac{\lambda^2}{(1-2\lambda)^2} < 1$.

For $a_1 > \frac{\lambda}{1-2\lambda}$, we get

$$\int_0^1 f^X(XYZ, ZYX, a_1, a_2)da_2 + \int_0^1 f^Y(XYZ, ZYX, a_1, a_2)da_2 = \\ = \frac{\lambda}{(1-2\lambda)} + a_1 - a_1^2 + \left(a_1 - \frac{\lambda}{(1-2\lambda)}\right) = 2a_1 - a_1^2$$

which is minimal at $a_1 = \frac{1-\lambda}{1-2\lambda} - 1$ and equal to $\frac{2\lambda}{(1-2\lambda)} - \frac{\lambda^2}{(1-2\lambda)^2} > 0$ and maximal at $a_1 = 1$ and equal to 1. So, we can conclude $\int_0^1 f^Z(XYZ, ZYX, a_1, a_2)da_2$ is always in the range from zero to one.

We also want briefly to show that misreporting one's preference ranking is never optimal. Using the mechanism rules we derive that utility is given by

$$U_1\left(XYZ, a_1\right) = \frac{2}{3} + \frac{1}{6}a_1 + \frac{1}{6}a_1^2 + \frac{1}{6}\frac{\lambda}{1-2\lambda}\left(1-a_1\right)$$

for $a_1 > \frac{\lambda}{1-2\lambda}$, and

$$U_1\left(XYZ, a_1\right) = \frac{2}{3} + \frac{1}{6}a_1 + \frac{1}{12}a_1^2 + \frac{1}{6}\frac{\lambda}{1-2\lambda}\left(1 - \frac{1}{2}\frac{\lambda}{1-2\lambda}\right)$$

for $a_1 < \frac{\lambda}{1-2\lambda}$

We also find that misreporting one's type as XZY is maximized when second-best (mis)report of $a_1^{'} = 0$ for any a_1, which results in the total utility of

$$U_1^{XZY,0}\left(XYZ, a_1\right) = \frac{2}{3} + \frac{1}{6}\frac{\lambda}{1-2\lambda}\left(1-a_1\right)\left(1 - \frac{1}{2}\frac{\lambda}{1-2\lambda}\right) + \frac{1}{6}a_1$$

which is always smaller than the above two utilities.

Similarly, misreporting one's type to YXZ, utility is maximized with the subsequent (mis)report of $a_1^{'} = 1$ for any a_1, resulting in the total utility of

$$U_1^{YXZ,1}\left(XYZ, a_1\right) = \frac{1}{2} + \frac{1}{3}a_1 + \frac{1}{6}\frac{\lambda}{1-2\lambda}a_1 - \frac{1}{6}\frac{\lambda}{1-2\lambda}$$

which is also always smaller than truthful reporting.

Observing the mechanism rules, it becomes clear that all other misreports are always worse than sincere reporting.

3.7 Appendix 2. Proof of lemma 2.

In this section we show that the conjecture used in section 4 is true for the uniform case. Under uniform case equation (3.13), which is an integrated incentive compatibility constraint for a first player with a profile XYZ, takes the following form

$$\frac{1}{6}\sum_{R_2}\int_0^1\int_0^1 \left[f^X\left(XYZ,R_2,a_1,a_2\right)+a_1 f^Y\left(XYZ,R_2,a_1,a_2\right)\right]da_1da_2 = \\ = p^1(XYZ,1)+q^1(XYZ,1)$$

Incorporating the fact that $f^z\left(\right)=1-f^x\left(\right)-f^y\left(\right)$ and ignoring constant terms, we can represent the total welfare conditional on the first player having preference profile XYZ by (abusing notation we drop XYZ and a_1, a_2 from mechanism rules and we indicate the report of a preference of the second player by a lower index) the following term

$$\begin{aligned} 6W &= \int_0^1\int_0^1 \left[2f^X_{XYZ}+\left(a_1+a_2\right)f^Y_{XYZ}+\left(2-a_2\right)f^X_{XZY}+\left(a_1-a_2\right)f^Y_{XZY}\right]da_1da_2 + \\ &+ \int_0^1\int_0^1 \left[\left(1+a_2\right)f^X_{YXZ}+\left(1+a_1\right)f^Y_{YXZ}+\left(1-a_2\right)f^X_{YZX}+\left(1+a_1-a_2\right)f^Y_{YZX}\right]da_1da_2 \\ &+ \int_0^1\int_0^1 \left[a_2f^X_{ZXY}+\left(a_1-1\right)f^Y_{ZXY}+\left(a_1+a_2-1\right)f^Y_{ZYX}\right]da_1da_2 \end{aligned}$$

Thus, introducing Lagrange multipliers, λ_1 for the integrated incentive compatibility constraint of a type XYZ of the first player and $\mu_1, \mu_2 ... \mu_6$ for every type of the second player, we can write down the Lagrangian as we did in section 4. Writing down and comparing the coefficients for every $f_{R_2}^X$ and $f_{R_2}^Y$ we get the results of the conjecture.

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