Sargan Lectures: Weakly Identified Econometric Models

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1. Linear IV model with weak instruments
2. Many Instruments
3. Weak GMM and other structural models

Sargan Lecture 1
Weak Instruments

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Outline

1. Classic asymptotic results
2. Weak IV phenomenon
3. Identification Robust Inference
   - Full parameter inference
   - Sub-vector inference
4. Estimation
5. Summary and Open Questions

Introduction

- Linear regressions with i.i.d. data:

$$Y_i = \beta' X_i + \gamma' W_i + e_i$$

  can be estimated by OLS when we have exogeneity
  - $\mathbb{E}[e_i(X_i'; W_i')'] = 0$
  - sometimes we assume $\mathbb{E}[e_i | X_i, W_i] = 0$
- In observational data exogeneity often does not hold
- **Example 1**: Angrist and Krueger (1991)

$$wage_i = \beta \ education_i + controls + e_i,$$

- Reasons for violations of exogeneity include missing confounding factors, simultaneity, mis-measurement

Introduction

$$Y_i = \beta' X_i + \gamma' W_i + e_i$$

- Common strategy in applied economics - Instrumental Variables
- Instrument Z_i is a variable the produces exogenous variation in the regressor (conditional on controls)
  - Z moves X (relevance)
  - $\mathbb{E}[Ze] = 0$ (exogeneity)
- **Example 1**: Angrist and Krueger (1991)

$$wage_i = \beta \ education_i + controls + e_i,$$

- Instrument is quarter of birth

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Classic IV results

$$Y_i = \beta' X_i + \gamma' W_i + e_i, \text{ with } \mathbb{E}[W_i e_i] = 0$$

- Instruments Z_i (K-dimensional) satisfy:
  - exogeneity: $\mathbb{E}[Z_i e_i] = 0$
  - relevance: $X_i = \pi' Z_i + \delta' W_i + v_i$, $rank(\pi) = dim(X_i)$
- GMM moment condition

$$\mathbb{E}\left[(Y_i - \beta' X_i - \gamma' W_i)(Z_i', W_i')'\right] = 0$$

Classic IV results

$$Y_i = \beta' X_i + \gamma' W_i + e_i, \text{ with } \mathbb{E}[W_i e_i] = 0$$

- Instruments Z_i (K-dimensional)
- In case of small dimensional W_i asymptotic analysis is equivalent to considering partialled out model

$$\tilde{Y}_i = \beta' \tilde{X}_i + \tilde{e}_i$$

  with instrument $\tilde{Z}_i$, where variables with tilde $\tilde{Y} = (I - W(W'W)^{-1}W')Y$ are residuals from regression on W.

Classic IV results

- From now we concentrate on model without controls (controls are partialled out):
$$Y_i = \beta' X_i + e_i,$$
- Instruments Z_i (K-dimensional) satisfy:
  - exogeneity: $\mathbb{E}[Z_i e_i] = 0$
  - relevance: $X_i = \pi' Z_i + v_i$, $rank(\pi) = dim(X_i)$
- GMM estimator:
$$\widehat{\beta}(S) = \arg\min_{\beta} (Y - X\beta)' Z S Z' (Y - X\beta)$$

  where S is some positive-definite weight matrix

Classic IV results

- In just identified case weight matrix is unimportant

$$\widehat{\beta} = (Z'X)^{-1}Z'Y$$

- The logic of consistency and asymptotic gaussianity:

$$\sqrt{n}(\widehat{\beta} - \beta_0) = \left(\frac{1}{n}Z'X\right)^{-1}\frac{1}{\sqrt{n}}Z'e$$

  - Law of Large Numbers $\frac{1}{n}Z'X \to^p \mathbb{E}[Z_iX_i']$
  - CLT: $\frac{1}{\sqrt{n}}Z'e \Rightarrow N(0, \mathbb{E}[e_i^2 Z_i Z_i'])$

- If $\mathbb{E}[Z_iX_i']$ is a full rank matrix (relevance), then $\widehat{\beta}$ is consistent and asymptotically gaussian

$$\sqrt{n}(\widehat{\beta} - \beta_0) \Rightarrow N\left(0, \left(\mathbb{E}[X_iZ_i']\right)^{-1}\mathbb{E}[e_i^2 Z_iZ_i']\left(\mathbb{E}[Z_iX_i']\right)^{-1}\right)$$

Classic IV results

- GMM estimator in an over-identified case:

$$\widehat{\beta}(S) = \arg\min_{\beta}(Y - X\beta)'ZSZ'(Y - X\beta)$$

where S is some positive-definite weight matrix

- Under some conditions (relevance is one of them):

$$\widehat{\beta}(S) = \left(X'ZSZ'X\right)^{-1} X'ZSZ'Y$$

is consistent and asymptotically gaussian

- Optimal choice of weight matrix $S^* = (\mathbb{E}[e_i^2 Z_i Z_i'])^{-1}$

Classic IV results

- GMM estimator (let X be d-dimensional, $d < K$):

$$\widehat{\beta}(S) = \arg\min_{\beta}(Y - X\beta)' Z S Z' (Y - X\beta)$$

- First order condition:

$$\frac{X'Z}{n} S Z'(Y - X\beta) = 0$$

combines instruments to a just identified set $Z_s = Z \cdot S \cdot \frac{(Z'X)}{n}$

- Optimal instrument

$$Z^* = Z \cdot (\mathbb{E}[e_i^2 Z_i Z_i'])^{-1} \cdot \mathbb{E}[Z_i X_i']$$

weights signal and noise optimally

Classic IV results

- GMM estimator: $\widehat{\beta}(S) = \arg\min_\beta (Y - X\beta)' Z S Z' (Y - X\beta)$
- Infeasible optimal choice of weight matrix $S^* = (\mathbb{E}[e_i^2 Z_i Z_i'])^{-1}$
- We need $\hat{\Sigma}(\beta)$ a consistent estimator of asymptotic variance of $Z'(Y - X\beta)$
- Feasible realizations of efficient IV:
  - 2 step efficient GMM
    $\widehat{\beta}_{2GMM} = \arg\min_\beta (Y - X\beta)' Z \left[\hat{\Sigma}(\hat{\beta}_0)\right]^{-1} Z'(Y - X\beta)$
  - Continuously updated GMM
    $\widehat{\beta}_{CUE} = \arg\min_\beta (Y - X\beta)' Z \left[\hat{\Sigma}(\beta)\right]^{-1} Z'(Y - X\beta)$
- Under homoskedasticity $\Sigma(\beta_0) \propto (Z'Z)$
  - 2 Step GMM= TSLS
  - CUE=LIML

Summary

Classical asymptotics is based on two ideas:

1. A properly selected just-identified IV is consistent and asymptotically gaussian because:
   - Numerator is gaussian (exogeneity important here)
   - Denominator is consistent and full rank (relevance!)
2. Signal is strong enough that one can select consistently an optimal instrument

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Examples of Weak IV

- **Example 1**: Angrist and Krueger (1991)

$$wage_i = \beta \; education_i + controls + e_i,$$

- Instrument is quarter of birth
- Empirical sample
  - very large sample ($>$ 300K observations)
  - very low first stage R^2 (between .0001 and .0002)
  - quite tight standard errors for IV estimate
- Bound, Jaeger and Baker (1995) randomly simulated quarter of birth and re-run estimation - nothing in their IV estimates suggested that they are invalid (!)

Examples of Weak IV

- Euler equation

$$\mathbb{E}[(\Delta c_{t+1} - \tau - \psi r_{t+1}) Z_t] = 0$$

- Can estimate it two different ways:

$$\Delta c_{t+1} = \tau + \psi r_{t+1} + e_t \text{ using instruments } Z_t$$

$$r_{t+1} = \mu + \gamma \Delta c_{t+1} + v_t \text{ using instruments } Z_t$$

- We have $\psi = 1/\gamma$
- In data the standard confidence sets for ψ obtained in two ways do not intersect
- Problem: Δc_{t+1} is hard to forecast (leads to weak instruments)

Weak IV with 1 regressor and 1 instrument

$$Y_i = \beta' X_i + e_i, \text{ with instrument } Z_i$$

- Estimator does not depend on weight matrix (in just id case)

$$\widehat{\beta} = \frac{\sum_i Z_i Y_i}{\sum_i Z_i X_i}; \quad \sqrt{n}(\widehat{\beta} - \beta_0) = \frac{\frac{1}{\sqrt{n}} \sum_i Z_i e_i}{\frac{1}{n} \sum_i Z_i X_i}$$

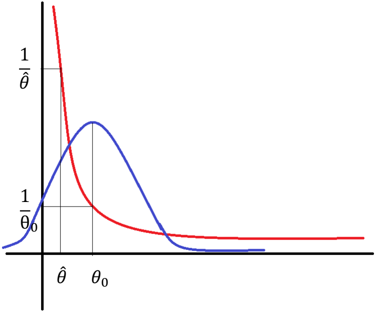
- Exogeneity means $\mathbb{E}[Z_i e_i] = 0$, CLT holds under quite wide set of assumptions: $\frac{1}{\sqrt{n}} \sum_i Z_i e_i \Rightarrow N(0, \Sigma)$
- $\frac{1}{n} \sum_i Z_i X_i \rightarrow^p \mathbb{E}[Z_i X_i]$, by relevance $\mathbb{E}[Z_i X_i] \neq 0$

Weak IV with 1 regressor and 1 instrument

$$\sqrt{n}(\widehat{\beta} - \beta_0) = \frac{\frac{1}{\sqrt{n}} \sum_i Z_i e_i}{\frac{1}{n} \sum_i Z_i X_i}$$

- CLT for denominator $\frac{1}{\sqrt{n}} \sum_i (Z_i X_i - \mathbb{E}[Z_i X_i]) \Rightarrow N(0, \Omega)$
- Denominator $\frac{1}{n} \sum_i Z_i X_i \approx \mathbb{E}[Z_i X_i] + \frac{1}{\sqrt{n}} N(0, \Omega)$
- If noise $\frac{1}{\sqrt{n}} N(0, \Omega)$ is comparable in size to the signal $\mathbb{E}[Z_i X_i]$, the numerator may be close to zero

Weak identification: explanation



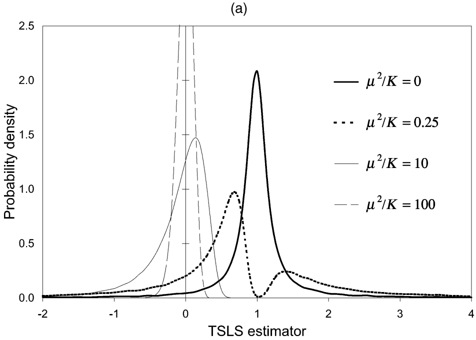
- Here $\theta_0 = \mathbb{E}[Z_i X_i]$ and $\widehat{\theta} = \frac{1}{n} \sum_i Z_i X_i$
- Concentration parameter $\mu^2 = \frac{(\mathbb{E}[Z_i X_i])^2}{Var(Z_i X_i)/n}$, signal-to-noise ratio measures the extent of problem

Weak identification: explanation

$$\mu(\widehat{\beta} - \beta_0) = \sqrt{\frac{\Sigma}{\Omega}} \cdot \frac{\xi_e}{1 + \frac{\xi_x}{\mu}}$$

- $\xi_e = \frac{1}{\sqrt{n\Sigma}} \sum_i Z_i e_i \Rightarrow N(0,1)$
- $\xi_x = \frac{1}{\sqrt{n\Omega}} \sum_i (Z_i X_i - \mathbb{E}[Z_i X_i]) \Rightarrow N(0,1)$
- ξ_e and ξ_x are correlated (endogeneity of X_i), this creates a bias
- If the instrument is irrelevant ($\mu = 0$), then $\widehat{\beta} - \beta_0 = \sqrt{\frac{\Sigma}{\Omega}} \cdot \frac{\xi_e}{\xi_x}$ is asymptotically centered at $\widehat{\beta}_{OLS}$
- Concentration parameter μ^2 serves as effective sample size

Weak identification: explanation



- Simulations from Stock et al (2002). Homoskedastic just identified ($K = 1$) case, $\frac{\Sigma}{\Omega} = 1$, correlation of errors is 0.99

Weak identification: pre-test

- Concentration parameter $\mu^2 = \frac{(\mathbb{E}[Z_i X_i])^2}{Var(Z_i X_i)/n}$, signal-to-noise ratio measures the extent of problem
- If μ^2 is small, $\widehat{\beta}$ is biased towards OLS, t-statistics tests/confidence sets are not reliable
- First stage regression $X_i = \pi Z_i + v_i$
- First stage F-statistics for testing $H_0 : \pi = 0$ is a good proxy for μ^2 in just-identified case with 1 endogenous regressor
- Empirical rule of thumb: $F > 10$ gives reliable inference
- This rule is not satisfactory outside of 1-regressor, 1-instrument settings

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AR test

$$Y_i = \beta' X_i + e_i$$

- Assumption is exogeneity: $\mathbb{E}[Z_i e_i] = 0$
- Want to test $H_0 : \beta = \beta_0$ in a reliable way
- Idea: if β_0 is true parameter value then $e_i = Y_i - \beta_0' X_i$ is uncorrelated with Z_i
- Under minor assumptions: $\frac{1}{\sqrt{n}} \sum_i Z_i (Y_i - \beta_0' X_i) \Rightarrow N(0, \Sigma)$
- We can construct a consistent estimator of Σ

$$AR(\beta_0) = \frac{1}{n}(Y - \beta_0 X)' Z \widehat{\Sigma}^{-1} Z'(Y - \beta_0 X) \Rightarrow \chi^2_K$$

AR test

- Intuition of AR test in a just identified case
- $\theta_1 = \mathbb{E}[Z_i Y_i]$ and $\widehat{\theta}_1 = \frac{1}{n}\sum_i Z_i Y_i$ and $\theta_2 = \mathbb{E}[Z_i X_i]$ and $\widehat{\theta}_2 = \frac{1}{n}\sum_i Z_i X_i$
- $\sqrt{n}(\widehat{\theta}_1 - \theta_1) \Rightarrow N(0, \Sigma)$ and $\sqrt{n}(\widehat{\theta}_2 - \theta_2) \Rightarrow N(0, \Omega)$
- Parameter of interest $\beta = \frac{\theta_1}{\theta_2}$ and $\widehat{\beta} = \frac{\widehat{\theta}_1}{\widehat{\theta}_2}$
- By delta-method $\sqrt{n}(\widehat{\beta} - \beta) \Rightarrow N(0, \Sigma_\beta)$
- Classic asymptotics test for $H_0 : \beta = \beta_0$ using t-statistics : $\sqrt{\frac{n}{\Sigma_\beta}}(\widehat{\beta} - \beta_0) \Rightarrow N(0, 1)$
- AR reformulates a hypothesis as $H_0 : \theta_1 = \beta_0\theta_2$ and tests this hypothesis:

$$AR(\beta_0) = \frac{\left(\widehat{\theta}_1 - \beta_0\widehat{\theta}_2\right)^2}{Var(\widehat{\theta}_1 - \beta_0\widehat{\theta}_2)}$$

AR test

- One can test $H_0 : \beta = \beta_0$ identification robust way accepting $AR(\beta_0) < \chi^2_{k,1-\alpha}$
- Optimal set in a class of robust test with some invariance property for a just-identified case
- Applicable in several regressors and or several instruments cases
- Confidence set can be produced as a set of β_0 accepted by AR test
- Confidence set can be infinite (good feature)
- Confidence set can be empty (has power against mis-specification)

Robust inference in over-identified case

$$Y_i = \beta' X_i + e_i$$

- Over-identified case: instrument Z_i is K-dimensional and $K > d$
- AR test is still robust towards weak identification, but has low power when identification is strong
- When instruments are strong, we can combine them in the efficient way, and do AR test using only efficient instrument
- This is the idea of LM test. Infeasible version (with weight π):

$$LM^*(\beta_0) = \frac{1}{n}(Y - \beta_0 X)'(Z\pi)\left(\pi'\widehat{\Sigma}\pi\right)^{-1}(Z\pi)'(Y - \beta_0 X) \Rightarrow \chi^2_d$$

LM test

- The optimal combination is $\pi = \left(\mathbb{E}[e_i^2 Z_i Z_i']\right)^{-1} \mathbb{E}[Z_i X_i]$
- When signal-to-noise ratio is low, it is hard to estimate the signal $\mathbb{E}[Z_i X_i]$ well, getting variance is a challenge as well
- We give up on variance - find an optimal combination for homoskedatsic case only
- Idea: use OLS for combining $\widehat{\pi} = (Z'Z)^{-1} Z'X$
- Problem: $\widehat{\pi}$ is very volatile under weak IV and is correlated with $Z'e$.
- Naive LM test does not have correct size under weak IV:

$$LM_{naive}(\beta_0) = \frac{1}{n}(Y - \beta_0 X)'(Z\widehat{\pi})\left(\widehat{\pi}'\widehat{\Sigma}\widehat{\pi}\right)^{-1}(Z\widehat{\pi})'(Y - \beta_0 X)$$

LM test

- Kleibergen (2002): create a new estimator $\tilde{\pi}$
  - consistent for π if identification is strong
  - asymptotically independent from $Z'e$ if identification is weak

$$KLM(\beta_0) = \frac{1}{n}(Y - \beta_0 X)'(Z\tilde{\pi})\left(\tilde{\pi}'\widehat{\Sigma}\tilde{\pi}\right)^{-1}(Z\tilde{\pi})'(Y - \beta_0 X)$$

- $\tilde{\pi} = \widehat{\pi} - \widehat{cov}(\widehat{\pi}, Z'e)\left(\widehat{Var}(Z'e)\right)^{-1} Z'e$
- KLM test for $H_0 : \beta = \beta_0$
  - has correct size irrespective of identification strength
  - asymptotically efficient if identification is strong

Conditional inference

- Under strong identification there are asymptotically uniformly most powerful tests for $H_0 : \beta = \beta_0$ (Wald, LM, LR)
- Under weak identification, there is no asymptotically uniformly most powerful (Andrews et al 2006)
- KLM may have very low power under weak identification (Moreira et al 2024)
- Can we use any other test statistics for $H_0 : \beta = \beta_0$, like Wald or LR?
- Problem: asymptotic distributions of Wald or LR depend on π
- Solution: Moreira (2003) - conditional inference

Conditional inference

- Problem: You want to test $H_0 : \beta = \beta_0$
  - distribution of test statistics $\mathcal{S}$ depends on (nuisance) parameter π
  - critical values should depend on π
  - you cannot estimate π with good enough precision
- Solution (Moreira, 2003): there is (asymptotically) sufficient statistics $\mathcal{T}$ for π
  - distribution of data (or any statistics) conditionally on $\mathcal{T}$ does not depend on π
  - create critical values depending on $\mathcal{T}$ (random critical values!!!) to control conditional size

$$\mathbb{P}\{\mathcal{S} > cv_\alpha(t) | \mathcal{T} = t\} = \alpha$$

- Original idea: use simulations conditioning on samples with given value of $\mathcal{T}$
- For some statistics like LR, no simulations are needed- there are analytic way of calculating conditional p-value, (called CLR test)

Summary

- If you want to either test $H_0 : \beta = \beta_0$ or construct a confidence set for β
    - you should NOT use a pre-test for weak identification
    - you should use identification robust test (or invert it for a confidence set)
- Arguments against pre-test
    - first stage F works only for 1 regressor-1 instrument case, or over-identified homoskedastic case
    - even in simplest case it may create selection bias (Angrist and Kolesar, 2022)
- Arguments for identification robust testing
    - Control size irrespective of the identification strength
    - Asymptotically efficient if identification is strong (KLM, CLR, conditional Wald)

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Projection method

$$Y_i = \beta_1 X_{1,i} + \beta_2 X_{2,i} + e_i$$

- Instruments Z_i are at least $d_1 + d_2$-dimensional
- β_1 is the parameter of interest, β_2 is nuisance
- How to test $H_0 : \beta_1 = \beta_{1,0}$? Confidence set for β_1 only?
- Projection method:
  - Hypothesis $H_0 : \beta_1 = \beta_{1,0}$ is accepted if $\tilde{H}_0 : \beta = (\beta_{1,0}, \beta_{2,0})$ is accepted for some $\beta_{2,0}$ (search among all potential $\beta_{2,0}$)
  - Create a joint confidence set for β, project it on β_1 space
  - Test (confidence set) for β should be done in identification robust way

Projection method and power loss

- Assume that test is done by using KLM statistics
- Projection method test for $H_0 : \beta_1 = \beta_{1,0}$ accepts if

$$\min_{\beta_{2,0}} KLM(\beta_{1,0}, \beta_{2,0}) = KLM(\beta_{1,0}, \widehat{\beta}_2) \leq \chi^2_d$$

  where $d = d_1 + d_2$
- If all parameters were strongly identified, and we knew that, we would adjust degrees of freedom and used $\chi^2_{d_1}$
- Indeed,

$$\min_{\beta_{2,0}} KLM(\beta_{1,0}, \beta_{2,0}) \leq KLM(\beta_{1,0}, \beta_{2,0}) \sim \chi^2_d$$

  so, projection test use conservative critical values (is not as powerful as it could have been)

Improvements over projection method

- Want to test $H_0 : \beta_1 = \beta_{1,0}$
  - If nuisance parameter is strongly identified then we can adjust critical values for the degrees of freedom
  - If nuisance parameter is weakly identified then distribution of most tests are harder to assess, and they depend on nuisance parameters
  - Projection method controls size (but typically not similar)
- Need to 'pre-test' whether nuisance parameter is strong or weak
- No perfect pre-test exists, but anything that improves power of two-step procedure over projection method is good.
- There are proposals of this type: Chaudhuri and Zivot (2011), Andrews (2018), Guggenberger et al (2012)

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Estimation

- When identification is weak, there is no consistent estimator exists
- Under weak identification in just identified case, TSLS does not have any moments, is median biased (towards OLS)
- Under weak identification in over-identified case, TSLS is biased towards OLS
- LIML and TSLS behaves differently under weak identification
- Porter and Hirano (2015) if instruments can be arbitrary weak, no asymptotically mean-unbiased, no asymptotically median-unbiased estimator exists
- Andrews and Armstrong (2017) if the sign of first stage is known, one can create an asymptotically unbiased estimator

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Summary

- Classical results in IV rely on relevance in two ways: (i) denominator stabilize at a full rank matrix; (ii) consistency of optimal combination of instrument
- Weak identification distort both
- Inferences: AR test solves problem (i) by re-formulating hypothesis
- Inferences: (ii) is 'solved' by KLM, CLR tests
- Estimation is hard, because of not well-defined criteria of quality

Open questions

- (Very hard) Estimation - what can be or reported as an estimator? Or should it be reported at all?
- (Hard) Seems that some pre-test for identification strength is needed to improve performance for a sub-vector tests