Abstract

Mechanism Design with Endogenous Information Acquisition, Endogenous Status and Endogenous Quantities

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In the first chapter, I study optimal auction design in a private value setting where a seller wants to sell a single object to one of several potential buyers who can each covertly acquire information about their valuations prior to participation. A simple but robust finding is that the buyers' incentives to acquire information increase as the reserve price moves toward the mean valuation. Thus, a seller who wants to encourage information acquisition should set the reserve price closer to the mean valuation than the standard reserve price in Myerson (1981). We present conditions under which the seller will prefer that the buyers acquire more information, conditions under which standard auctions with an adjusted reserve price are optimal, and conditions under which the buyers will acquire socially excessive information in standard auctions. These results are obtained in a general setting with rotation-ordered information structures and continuous information acquisition.

In the second chapter, which is joint with Benny Moldovanu and Aner Sela, we study the optimal design of organizations under the assumption that agents in a contest care about their relative position. A judicious definition of status categories can be used by a principal in order to influence the agents' performance. We first consider a pure status case where there are no tangible prizes. The top status category always contains a unique element. For distributions of abilities that have an increasing failure rate (IFR), a proliferation of status classes is optimal, while the optimal partition involves only two categories if the distribution of abilities is sufficiently concave. Moreover, for IFR distributions, a coarse partition with only two status categories achieves at least half of the output obtained in the

optimal partition with a proliferation of classes. Finally, we modify the model to allow for status categories that are endogenously determined by monetary prizes of different sizes. If status is solely derived from monetary rewards, we show that the optimal partition in status classes contains only two categories.

In the third chapter, I evaluate the performance of split-award auctions, a popular practice in which the seller divides one object (contract) into several units (subcontracts) and each bidder can win at most one unit. I provide a justification for its prominence by presenting a model in which split-award can increase revenue (or reduce procurement cost) when bidders are asymmetric and entry is endogenous. I prove that split-award auctions could increase revenue when bidders are asymmetric and entry is endogenous: bidder asymmetry amplifies the importance of endogenous entry. This result highlights the importance of the interaction between bidder asymmetry and endogenous entry in auction design. Numerical simulation shows that the advantage of the split-award auction is more prominent when bidders are more asymmetric and the entry cost is relatively higher.

Mechanism Design with Endogenous Information Acquisition, Endogenous Status and Endogenous Quantities

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Contents

List of Figures

List of Tables

This dissertation is dedicated to my parents.

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Chapter 1

Optimal Auctions with Information Acquisition

1.1 Introduction

1.1.1 Overview

The mechanism design literature studies how a principal can design the rules of a game to achieve certain objectives given that agents will play strategically and may hold private information. A typical assumption in most of the existing literature is that the information held by market participants is exogenous. In many real world situations, however, the agents' information is acquired rather than endowed. For example, when a firm files bankruptcy under Chapter 7 and is offered for sale, potential buyers may not know how much they are willing to pay, and assessing the value of the firm may be costly. Moreover, the selling mechanism proposed by the seller affects the buyers' incentives to collect information about the goods and services being traded. The purpose of this paper is to study how the seller should design the selling mechanism when information acquisition is endogenous.

Specifically, we consider a model where a seller wants to sell an indivisible object to one of several potential buyers (or bidders). Buyers' valuations for the object are unknown ex-ante to both parties, but prior to participation, buyers can privately acquire costly information about their valuations. The buyers can improve the informativeness of their signals, but with an increasing convex cost. The timing of the game is as follows: first, the seller announces the selling mechanism; after observing the mechanism, buyers decide how much information to acquire, and based on the acquired information buyers determine whether to participate; each participating buyer then submits a report about their private information to the seller; and the outcome is realized.

If the seller chooses a mechanism that encourages information acquisition, the efficiency of allocation may increase because buyers with higher valuations will get the object more often, but the buyers' information rent is also higher. In contrast, if the seller chooses a mechanism that discourages information acquisition, the rent left to the buyers will be lower, but the allocation may be less efficient. The seller's task is therefore to choose a selling mechanism that balances these two forces. The optimal trade-off between surplus extraction and incentives to acquire information is the focus of this paper.

In order to study this problem, we adopt Myerson's (1981) symmetric independent private values framework.[1] Myerson shows that, under some regularity conditions, standard auctions with a reserve price are optimal if the buyers' information is exogenous.[2] We refer to the reserve price in Myerson's optimal auctions as the *standard reserve price.* If information is costly, however, the seller faces an additional constraint: the chosen mechanism must provide the buyers with incentive to acquire the level of information that she prefers.

A simple but robust finding of this paper is that a buyer's incentive to acquire information increases as the reserve price moves toward the mean valuation. To see this, consider the simple setting with one buyer and binary information acquisition. The seller first posts

[1] In a private value setting, a buyer's valuation does not depend on the private information of his opponents.

[2] In this paper, we use standard auctions to denote the four commonly used auction formats: first price auctions, Vickery auctions, English auctions, and Dutch auctions.

a price, and then the buyer decides whether to acquire information and whether to buy. If the reserve price is very high or very low, new information is unlikely to change the buyer's purchasing decision. In contrast, if the reserve price is close to the mean valuation, new information is valuable because it helps the buyer make the right decision: buy or not buy. This observation remains valid in a general setting.

It follows naturally from this observation that the optimal reserve price will be closer to the mean valuation than the standard reserve price if the seller wants to encourage information acquisition. We present conditions under which the seller benefits from more information, conditions under which standard auctions with an adjusted reserve price are optimal among the class of selling mechanisms considered in Myerson (1981),[3] and conditions under which bidders have socially excessive incentive to acquire information in standard auctions.

This paper contributes to the mechanism design literature with endogenous information acquisition and is complementary to the existing literature on optimal auctions with information acquisition. Most of the existing literature assumes that the seller can control either the information sources or the timing of the information acquisition (*centralized* information acquisition).[4] In contrast, information acquisition in our analysis is *decentralized*: buyers can choose to acquire information prior to participation. The information structure we study is quite general, and we allow buyers to choose the level of information acquisition continuously.

To illustrate the model, we first study optimal auctions with a single bidder and the Gaussian specification. Here the optimal selling mechanism is to post a (reserve) price. The buyer's true valuation is normally distributed, and is ex-ante unobservable to both parties. The buyer, however, can acquire a noisy signal, which is the sum of the true

[3]There are selling mechanisms more general than those considered here. For example, we do not consider sequential mechanisms, as in Cremer, Spiegel and Zheng (2003), and pre-play communication, as in Gerardi and Yariv (2006).

[4]For example, Bergemann and Pesendorfer (2001), and Eso and Szentes (2006) assume the seller controls the information sources. On the other hand, in Levin and Smith (1994), and Cremer, Spiegel and Zheng (2003), the seller controls the timing of the information acquisition. See next subsection for a detailed discussion.

valuation and a normally distributed error. The buyer can increase the informativeness of his signal by reducing the variance of the error, but with an increasing cost.

Because the buyer pays the information cost but may have to share the gain from more information with the seller, the buyer and the seller may have conflicting interests in information choice: the one preferred by the buyer may be excessive or insufficient to the seller. Moreover, the buyer's information choice is not observable to the seller. Therefore, we can interpret it as a principle-agent problem in which the seller (principal) sets a reserve price to align the buyer's interest with her own. Taking into account the buyer's information decision, the ex-post optimal standard reserve price (or the monopoly price in this case) is not optimal ex-ante to the seller.

Since the buyer always prefers a low reserve price, it may seem, at first glance, that a lower reserve price always gives the buyer a higher incentive to gather information. This intuition is wrong, however, because the buyer's incentive to acquire information depends on his relative gain from information acquisition rather than on his absolute payoff. Indeed, as we pointed out earlier, the marginal value of information to the buyer increases as the reserve price moves towards the mean valuation.

It turns out that the equilibrium reserve price is always adjusted downward compared to the standard reserve price in this simple setting. The reason is the following. If the standard reserve price is higher than the mean valuation, more information will benefit the seller because more information increases the probability of trade. In order to induce the buyer to acquire more information, the seller must adjust the standard reserve price downwards. On the other hand, if the standard reserve price is lower than the mean valuation, more information will hurt the seller because more information reduces the probability of trade. Again, the seller will set the equilibrium reserve price lower than the standard reserve price, but this time to induce the buyer to acquire less information.

To summarize, the simple one-bidder model has two main findings. First, the buyer's incentive to acquire information is higher when the reserve price is closer to the mean valuation. Second, with endogenous information acquisition by the buyer, the seller will set the optimal reserve price lower than the standard reserve price. The analysis of the general model is more subtle and complicated because the optimal selling mechanism no longer admits the simple form of a posted price. But the first observation remains valid The adjustment of reserve price, however, is not as simple as in the case with one bidder. With sufficiently many bidders, the optimal reserve price is adjusted toward the mean valuation compared to the standard reserve price.

In order to generalize the first result to general information structures, we need an information order to rank the informativeness of different signals. Motivated by the observation that two commonly used information technologies, the Gaussian specification and the "truth-or-noise" technology,[5] both generate a family of distributions that is rotation ordered, we adopt the *rotation order* as our information order.[6] This order has an intuitive interpretation: more informative signals lead to more spread out distributions of the buyers' posterior estimates. We show that the marginal value of information to a buyer increases when the reserve price moves toward the mean valuation if and only if the signals are rotation ordered.

In order to generalize the second result, we need to characterize the solution of the seller's optimization problem with many bidders. Since bidders are ex-ante symmetric, we focus, for tractability, on the symmetric equilibrium in which all bidders acquire the same level of information.[7] In addition, for tractability, we replace the information acquisition constraint with the first order condition of the buyers' maximization problem — this is the so-called *first order approach* in the principal-agent literature (Mirrlees (1999), Rogerson (1985)).[8]

[5]The truth-or-noise technology was introduced into the literature by Lewis and Sappington (1994). Under this specification, the signal sometimes perfectly reveals underlying value, but at other times is just noise.

[6]The rotation order was recently introduced by Johnson and Myatt (2006) in order to model how advertising, marketing and product design affect consumers' valuations.

[7]The symmetry restriction is not needed in identifying the rule of adjusting researve price. But we need this restriction to determine the sign of Lagrangian multiplier of the information acquisition constraint. This is an important restriction. Although the function of information cost is assumed to be convex, we cannot exclude the possibility that the seller might still become better off by implementing an asymmetric equilibrium rather than a symmetric one.

[8]A condition, analogous to CDFC in Rogerson (1985), is shown to be sufficient for the first order

For the general model, we first provide sufficient conditions under which the seller benefits from a reduction of the marginal cost of information, i.e., the seller prefers more information. In many cases, more information is beneficial to the seller as long as there are sufficiently many buyers. The reason is that the seller's revenue is related to the second order statistic of buyers' posterior estimates which, for a large number of buyers, increases as buyers acquire more information and make the distribution of their posterior estimates more spread out.

Second, applying Myerson's (1981) technique, we show that the seller should set the optimal reserve price between the mean valuation and the standard reserve price if she benefits from more information acquired by the buyers. Otherwise, she should set the optimal reserve price away from the mean valuation compared to the standard reserve price. As shown in Appendix B, this simple rule for adjusting the reserve price is also robust to an alternative specification in which the information acquisition is discrete.

Third, for the Gaussian specification and the truth-or-noise technology with sufficiently many bidders, standard auctions with an adjusted reserve price are optimal. This result implies revenue equivalence, and can be generalized to other information structures with the property that the gain from information acquisition is higher for bidders with higher posterior estimates.

Finally, the information efficiency of the standard auctions is investigated. We show that the buyers' incentives to acquire equilibrium are socially excessive when the reserve price is lower than the mean valuation. The intuition is that when the reserve price is zero, the equilibrium information choice coincides with the social optimum. As the reserve price increases from zero to the mean valuation, the bidders' incentives to acquire information increase and exceed the social optimum.

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approach to be valid when the support of buyers' posterior estimates is invariant to buyers' information choices. This condition, however, does not hold for the two leading information technologies. Different sufficient conditions are presented for these two information technologies in Appendix B.

1.1.2 Related Literature

This paper is related to the growing literature on information and mechanism design. First of all, our framework extends the principal-agent model with information acquisition to a multi-agent setting. Our analysis is also related to studies on information acquisition in given auction formats. Finally, this paper is close to the existing optimal auction literature where information acquisition is centralized. For a broad survey of the literature on information and mechanism design, see Bergemann and Valimaki (2006a).

The first strand of literature related to this work studies information acquisition in the principal-agent model. Cremer and Khalil (1992) and Cremer, Khalil, and Rochet (1998a, 1998b) introduce endogenous information acquisition into the Baron-Myerson (1982) regulation model. They illustrate how the standard Baron-Myerson contract has to be adjusted in order to give the agent incentives to acquire information.[9] Szalay (2005) extends their framework to a setting with continuous information acquisition, and demonstrates that their findings are robust. These models share with ours a similar information structure and a focus on the interim participation constraint, but their models lack the strategic interaction among bidders that we incorporate.

Another strand studies information acquisition in auctions. Matthews (1984) studies information acquisition in a first price, common value auction, and investigates how the seller's revenue varies with the amount of information bidders acquire, and whether the equilibrium price fully reveals bidders' information. Persico (2000) shows that the incentive to acquire information is stronger in the first price auction than in the second price auction if bidders' valuations are affiliated. Ye (2006) studies information acquisition in two-stage auctions and shows that efficient entry is not guaranteed in the second stage. Compte and Jehiel (2006) make the important observation that bidders have the option to acquire information in the middle of dynamic auctions, and argue that ascending auctions or multi-round auctions perform better than static sealed-bid auctions.[10] In contrast, the current

[9] See also Lewis and Sappington (1993) for a principal-agent model with an ignorant agent.

[10] Other related papers include Tan (1992) and Arozamena and Cantillon (2004), who study investment

paper studies the optimal mechanism that maximizes the seller's revenue, rather than studying the given auction formats.[11]

A final related strand studies mechanism design problems where the seller controls either the information sources or the timing of information acquisition. Since Milgrom and Weber (1982), the seller's disclosure policy in the affiliated value setting has been extensively investigated. Recent studies in the independent private value setting, however, are more closely related to this paper. The information order used in the present paper, the rotation order, was first introduced by Johnson and Myatt (2006). They use it to show that a firm's profits are a U-shaped function of the dispersion of consumers' valuations, so a monopolist will pursue extreme positions, providing either a minimal or maximal amount of information. Eso and Szentes (2006) study optimal auctions in a setting where the seller controls the information sources. They show that the seller will fully reveal her information and can extract all of the benefit from the released information.[12] In these models, the seller, rather than the buyers, makes the information decision.

Several papers study the optimal selling mechanism in a setting where buyers make the information decision, but the seller controls the timing of the information acquisition. These models (hereafter refer to as "entry models") impose the ex-ante participation constraint, so the buyers' information decision is essentially an entry decision. The optimal selling mechanism typically consists of a participation fee followed by a second price auction with no reserve price, with the participation fee being equal to the bidders' expected rent from attending the auction. For example, Levin and Smith (1994) demonstrate that a second price auction with no reserve price and no entry fee maximizes the seller's revenue.[13] Similarly, with an ex-ante participation constraint, Cremer, Spiegel and Zheng

incentives before auctions.

[11] Bergemann and Valimaki (2002) also study information acquisition and mechanism design, but their focus is on efficient mechanisms.

[12] Bergemann and Pesendorfer (2001) characterize the optimal information structure in the optimal auctions, while Ganuza and Penalva (2006) study the seller's optimal disclosure policy when the information is costly.

[13] Ye (2004) extends their results to the setting where bidders can learn additional information after costly entry. Stegeman (1996) studies efficient auctions when the buyers' private information are endowed but the communication between the seller and buyers is costly.

(2003) construct a sequential selling mechanism in which the seller charges a positive entry fee and extracts the full surplus from buyers.

In contrast to these papers, information acquisition in the present paper is decentralized: buyers make the information decision, and can acquire information prior to participation. Thus, we impose the interim rather than the ex-ante participation constraint, which makes our model different from and complementary to the existing literature.[14] The relationship between our model and the existing literature can be partially summarized in the following table.

	given auction formats	mechanism design approach
centralized info acquisition	optimal disclosure in auctions	entry models
decentralized info acquisition	info acquisition in auctions	our model

Table 1.1 Relationship between our model and the existing literature

The remainder of the paper is organized as follows. Section 1.2 introduces the model, Section 1.3 studies optimal auctions with a single bidder and the Gaussian specification, Section 1.4 contains the analysis of optimal auctions with many bidders, and we conclude in Section 1.5. All proofs are relegated to Appendix A, unless otherwise noted. Appendix B contains discussions and extensions omitted in the text.

1.2 The Model

A seller wants to sell a single object to n ex-ante symmetric buyers (or bidders), indexed by $i \in \{1, 2, ..., n\}$.[15] Both the seller and buyers are risk neutral. The buyers' true valuations $\{\omega_i : i = 1, ..., n\}$, unknown ex-ante, are independently drawn from a common distribution F with support $[\underline{\omega}, \overline{\omega}]$. F has a strict positive and differentiable density f. The mean

[14] Cremer, Spiegel and Zheng (2006) also analyze optimal auctions where buyers can acquire information prior to participation, but the seller, rather than the buyer, pays the information cost.

[15] It is straigthforward to extend the analysis to a multi-unit setting where each buyer has a unit demand.

valuation μ is defined as:

$$\mu = \int_{\underline{\omega}}^{\overline{\omega}} \omega_i f(\omega_i)\, d\omega_i.$$

A buyer with valuation ω_i gets utility u_i if he wins the object and pays t_i:

$$u_i = \omega_i - t_i.$$

The seller's valuation for the object is normalized to be zero.

1.2.1 The Information Structure

Buyer i can acquire a costly signal s_i about ω_i, with $s_i \in [\underline{s}, \overline{s}] \subseteq \mathbb{R}$. Signals received by different buyers are independent. Buyer i acquires information by choosing a joint distribution of (s_i, ω_i) from a family of joint distributions $G_{\alpha_i} : \mathbb{R} \times [\underline{\omega}, \overline{\omega}] \rightarrow [0, 1]$, indexed by $\alpha_i \in [\underline{\alpha}, \overline{\alpha}]$. Each fixed α_i corresponds to a statistical experiment, and the signal with higher α_i is more informative in a sense to be defined later. We refer to the joint distribution G_{α_i}, or simply α_i, as the information structure. The cost of performing an experiment α_i is $C(\alpha_i)$, which is assumed to be convex in α_i. A buyer can conduct the experiment $\underline{\alpha}$ at no cost, so $\underline{\alpha}$ is interpreted as the endowed signal.

Let $G_{\alpha_i}(\cdot|\omega_i)$ denote the prior distribution of signal s_i conditional on ω_i, and $G_{\alpha_i}(\cdot|s_i)$ denote the posterior distribution of ω_i conditional on s_i. With a little abuse of notation, $G_{\alpha_i}(\omega_i)$ and $G_{\alpha_i}(s_i)$ are used to denote the marginal distributions of ω_i and s_i, respectively. They are defined in the usual way, that is, $G_{\alpha_i}(\omega_i) = \mathbb{E}_{s_i}[G_{\alpha_i}(\omega_i|s_i)]$ and $G_{\alpha_i}(s_i) = \mathbb{E}_{\omega_i}[G_{\alpha_i}(s_i|\omega_i)]$. Consistency requires that $G_{\alpha_i}(\omega_i) = F(\omega_i)$ for all α_i and i. We use $g_{\alpha_i}(s_i, \omega_i)$, $g_{\alpha_i}(\cdot|\omega_i)$, $g_{\alpha_i}(\cdot|s_i)$, $g_{\alpha_i}(\omega_i)$ and $g_{\alpha_i}(s_i)$ to denote the corresponding densities.

A buyer who observes a signal s_i from experiment α_i will update his prior belief on ω_i according to Bayes' rule:

$$g_{\alpha_i}(\omega_i|s_i) = \frac{g_{\alpha_i}(s_i|\omega_i) f(\omega_i)}{\int_{\underline{\omega}}^{\overline{\omega}} g_{\alpha_i}(s_i|\omega_i) f(\omega_i)\, d\omega_i}$$

Let $v_i(s_i, \alpha_i)$ denote buyer i's revised estimate of ω_i after performing experiment α_i and observing s_i :

$$v_i(s_i, \alpha_i) \equiv \mathbb{E}[\omega_i | s_i, \alpha_i] = \int_{\underline{\omega}}^{\overline{\omega}} \omega_i g_{\alpha_i}(\omega_i | s_i) \, d\omega_i$$

To simplify notation, we use v_i to denote $v_i(s_i, \alpha_i)$, and use v to denote the n-vector $(v_1, ..., v_n)$. Occasionally, we also write v as (v_i, v_{-i}), where $v_{-i} = (v_1, ...v_{i-1}, v_{i+1}, ..., v_n)$. We assume $v_i(s_i, \alpha_i)$ is increasing in s_i, that is, a higher signal leads to a higher posterior estimate given the information choice. Let H_{α_i} denote the distribution of v_i with corresponding density h_{α_i}. Then

$$H_{\alpha_i}(x) \equiv \Pr\{\mathbb{E}[\omega_i | s_i, \alpha_i] \leq x\} = \int_{\underline{s}}^{v_i^{-1}(x, \alpha_i)} g_{\alpha_i}(s_i) \, ds_i.$$

The upper limit of the integral, $v_i^{-1}(x, \alpha_i)$, is well defined since $v_i(s_i, \alpha_i)$ is increasing in s_i. That is, $H_{\alpha_i}(x)$ is the probability that the buyer i's posterior estimate v_i is below x given his information choice α_i. The family of distributions $\{H_{\alpha_i}\}$ have the same mean because

$$\mathbb{E}_{s_i}[v_i(s_i, \alpha_i)] = \mathbb{E}\omega_i = \mu.$$

For bidder i, different information choices $\{\alpha_i\}$ lead to different distributions $\{H_{\alpha_i}\}$. So choosing α_i is equivalent to choosing an H_{α_i} from the family of distributions $\{H_{\alpha_i}\}$. In what follows, we will extensively work with the posterior estimate v_i and its distribution H_{α_i}.

1.2.2 Timing

The timing of the game is as follows: the seller first proposes a selling mechanism; then given the mechanism, each buyer decides how much information to acquire; after the signals are realized, each buyer decides whether to participate; each participating buyer submits a report about his private information; and finally, an outcome, consisting of an allocation of the object and payments, is realized. Figure 1.1 summarizes the timing of the game:

seller announces mechanism	buyer i chooses α_i	buyer i observes s_i and decides whether to participate	buyers report private information	outcome realized

Figure 1.1 The timing of the game

The payoff structure, the timing of the game, the information structure $\{G_{\alpha_i}\}$ and distribution F are assumed to be common knowledge.

1.2.3 Mechanisms

In our setting, the buyer's private information is two-dimensional: the information choice α_i and the realized signal s_i. This suggests that the design problem here is multi-dimensional and could potentially be very complicated. However, similar to Biais, Martimont and Rochet (2002) and Szalay (2005), one single variable, the posterior estimate $v_i(\alpha_i, s_i)$, completely captures the dependence of buyer i's valuation on the two-dimensional information. Furthermore, the seller cannot screen the two pieces of information separately. For example, suppose there are two buyers, i and j, with the same posterior estimate $(v_i = v_j)$, but $\alpha_i > \alpha_j$. If the seller wants to favor the buyer with α_i, then buyer j can always report to have α_i. Therefore, the posterior estimate v_i is the only variable that the seller can use to screen different buyers.

Thus, we can invoke the Revelation Principle to focus on the direct revelation mechanisms $\{q_i(v), t_i(v)\}_{i=1}^n$:

$$\begin{aligned} q_i &: [\underline{\omega}, \overline{\omega}]^n \to [0,1], \\ t_i &: [\underline{\omega}, \overline{\omega}]^n \to \mathbb{R}, \end{aligned}$$

where $q_i(v)$ denotes the probability of winning the object for bidder i when the vector of report is v, and $t_i(v)$ denotes bidder i's corresponding payment.

Define

$$\begin{aligned} Q_i(v_i) &= \mathbb{E}_{v_{-i}} q_i(v_i, v_{-i}), \\ T_i(v_i) &= \mathbb{E}_{v_{-i}} t_i(v_i, v_{-i}). \end{aligned}$$

$Q_i(v_i)$ and $T_i(v_i)$ are the expected probability of winning and the expected payment conditional on v_i, respectively. The interim utility of bidder i who has a posterior estimate v_i and reports v_i' is

$$U_i(v_i, v_i') = v_i Q_i(v_i') - T_i(v_i').$$

Define $u_i(v_i) = U_i(v_i, v_i)$, the payoff of bidder i who has a posterior estimate v_i and reports truthfully.

A feasible mechanism has to satisfy the individual rationality constraint (IR):

$$u_i(v_i) = U_i(v_i, v_i) \geq 0, \quad \forall v_i \in [\underline{\omega}, \overline{\omega}], \tag{IR}$$

and the incentive compatibility constraint (IC):

$$U_i(v_i, v_i) \geq U_i(v_i, v_i'), \quad \forall v_i, v_i' \in [\underline{\omega}, \overline{\omega}]. \tag{IC}$$

With endogenous information acquisition, a feasible mechanism also has to satisfy the information acquisition constraint (IA): no bidder has an incentive to deviate from the equilibrium choice α_i^* :

$$\alpha_i^* \in \arg\max_{\alpha_i} \ \mathbb{E}_{v,\alpha_{-i}^*}[u_i(v_i(s_i, \alpha_i))] - C(\alpha_i). \tag{IA}$$

Note that $\mathbb{E}_{v,\alpha_{-i}^*}[u_i(v_i(s_i, \alpha_i))]$ is bidder i's expected payoff by choosing α_i conditional on other bidders choosing α_j^*, $j \neq i$. The subscript α_{-i}^* is to emphasize that the expectation depends on the information choices of i's opponents.

Since bidders are ex-ante symmetric, we focus on the symmetric equilibrium where $\alpha_i^* = \alpha^*$ for all i. The seller chooses mechanism $\{q_i(v), t_i(v)\}_{i=1}^n$ and α^* to maximize her expected sum of payment from all bidders,

$$\pi_s = \mathbb{E}_{v,\alpha^*} \sum_{i=1}^{n} T_i(v_i),$$

subject to (IA), (IC), and (IR).

1.3 Optimal Auctions with One Bidder and Gaussian Specification

We start with a simple model with only one buyer. If the buyer's information is exogenous, Riley and Zeckhauser (1983) show that the optimal selling mechanism is to post a non-negotiable price. With endogenous information, their logic still applies and a posted price is optimal.[16] Therefore, with a single buyer, designing the optimal auction is equivalent to choosing a reserve price.

This section will focus on a special but important information structure: the *Gaussian* specification. We first analyze the buyer's information decision problem, and show that the marginal value of information to the buyer increases as the reserve price moves toward the mean valuation. Then we formulate the mechanism design problem as a principal-agent problem and derive the seller's optimal pricing strategy. We show that the equilibrium reserve price is always lower than the standard reserve price. Finally, the informational efficiency of the optimal auction is investigated.

[16]The one-bidder model is a special case of the general model we study later. As shown in the next section, after incorporating the information acquisition constraint, the seller's objective function will be the Lagrangian specified in (1.11). If there is only one bidder, it reduces to a simple form similar to the one analyzed in Riley and Zechhauser (1983). Therefore, their proof of the optimality of the posted price mechanism still applies here.

1.3.1 Gaussian Specification

The buyer's true valuations ω_i are drawn from a normal distribution with mean μ and precision β :

$$\omega_i \sim N(\mu, 1/\beta).$$

Lowering β has the consequence that the prior distribution becomes more spread out, yielding more potential gains from information acquisition.

The buyer can observe a costly signal s_i:

$$s_i = \omega_i + \varepsilon_i,$$

where the additive error ε_i is independent of ω_i, and $\varepsilon_i \sim N(0, 1/\alpha_i)$. The higher the α_i, the more precise the signal is. Thus, we interpret α_i as the informativeness (precision) of buyer's signal. α_i is assumed to have two parts:

$$\alpha_i = \underline{\alpha} + \gamma_i.$$

The first part, $\underline{\alpha}$, is the endowed signal precision; the incremental term γ_i is the additional precision obtained by investing in information acquisition. For illustration purposes, the cost of information is assumed to be linear in the incremental precision. That is,

$$C(\alpha_i) = c\gamma_i = c(\alpha_i - \underline{\alpha}),$$

where c is the constant marginal cost of one additional unit of precision.

After observing a signal s_i with precision α_i, the buyer updates his belief of ω_i. By the standard normal updating technique, the posterior valuation distribution conditional on the signal s_i will be normal:

$$\omega_i | s_i \sim N\left(\frac{\beta\mu + \alpha_i s_i}{\alpha_i + \beta}, \frac{1}{\alpha_i + \beta}\right).$$

It immediately follows that

$$v_i\left(s_i,\alpha_i\right)\equiv\mathbb{E}\left(\omega_i|s_i,\alpha_i\right)=\frac{\beta\mu+\alpha_i s_i}{\alpha_i+\beta}.$$

Thus, the distribution of posterior estimate v_i, $H_{\alpha_i}\left(v_i\right)$, is normal:

$$v_i\sim N\left(\mu,\sigma^2\left(\alpha_i\right)\right),\text{ where }\sigma\left(\alpha_i\right)=\sqrt{\frac{\alpha_i}{\left(\alpha_i+\beta\right)\beta}}.$$

Note that the variance of v_i is increasing in the information choice α_i. So the distribution H_{α_i} will be more spread out for a more precise signal. The following two graphs capture the relationship between two distributions of the posterior estimate with different signals.

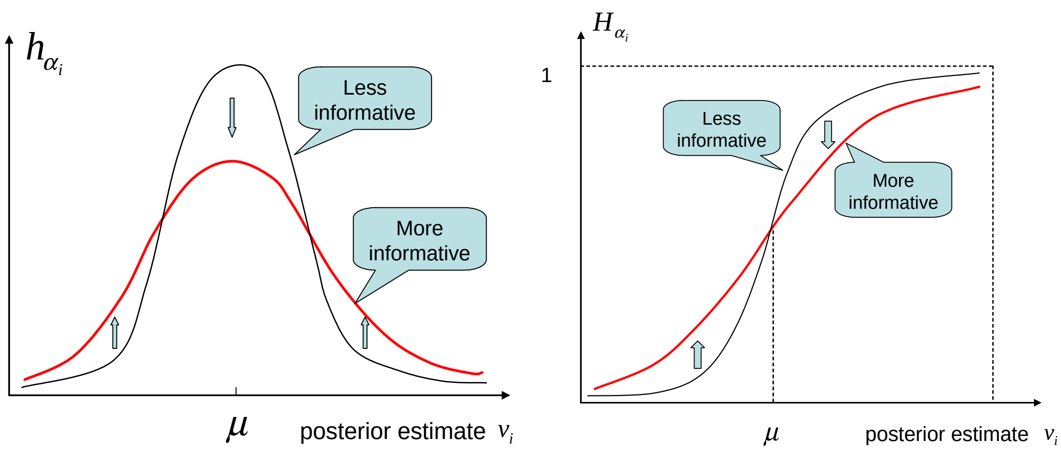

Figure 1.2. PDF and CDF of the posterior estimate with different signals

The left graph in Figure 1.2 shows that the density of the posterior estimate with a more informative signal is more dispersed than the one with a less informative signal. The right graph shows that the distribution with a less informative signal crosses the distribution with a more informative one from below at the mean valuation. In fact, with some algebra, we can show

$$v_i\gtrless\mu\Leftrightarrow\frac{\partial H_{\alpha_i}\left(v_i\right)}{\partial\alpha_i}\lesseqgtr 0. \tag{1.1}$$

This property is critical to our analysis and will be used to motivate the rotation order used in the paper.

1.3.2 The Marginal Value of Information to the Buyer

Given the reserve price r, the buyer chooses α_i to maximize his expected payoff:

$$\begin{aligned} &\max_{\alpha_i} \int_r^{\infty} (v_i - r)\, h_{\alpha_i}(v_i)\, dv_i - c\,(\alpha_i - \underline{\alpha}) \\ = &\max_{\alpha_i} \int_r^{\infty} (1 - H_{\alpha_i}(v_i))\, dv_i - c\,(\alpha_i - \underline{\alpha}). \end{aligned}$$

The first order condition is

$$-\int_r^{\infty} \frac{\partial H_{\alpha_i}(v_i)}{\partial \alpha_i} dv_i = c. \tag{1.2}$$

The left hand side is the marginal value of information (MVI) to the buyer:

$$MVI \equiv -\int_r^{\infty} \frac{\partial H_{\alpha_i}(v_i)}{\partial \alpha_i} dv_i.$$

Thus, in equilibrium the marginal value of information to the buyer is equal to the marginal information cost.

The following proposition shows how the marginal value of information to the buyer varies with respect to the reserve price.

Proposition 1.1 (Marginal Value of Information to the Buyer) *The marginal value of information to the buyer increases as the reserve price r moves toward the mean valuation μ, and achieves maximum at $r = \mu$.*

This finding is crucial in understanding other results obtained in this paper. To understand it better, let us consider a discrete version of the marginal value of information. Suppose there are two signals α_i and α_i', with $\alpha_i' > \alpha_i$. The buyer's gain from having signal α_i' rather than α_i is

$$\Delta VI = \int_r^{\infty} (H_{\alpha_i}(v_i) - H_{\alpha_i'}(v_i)) dv_i. \tag{1.3}$$

Since the two distributions have the same mean, we have

$$\mu = \int_{-\infty}^{\infty} (1 - H_{\alpha_i'}(v_i))dv_i = \int_{-\infty}^{\infty} (1 - H_{\alpha_i}(v_i))\, dv_i.$$

Therefore, we can also write the gain from more information as

$$\Delta VI = \int_{-\infty}^{r} (H_{\alpha_i'}(v_i) - H_{\alpha_i}(v_i))dv_i. \tag{1.4}$$

The following two graphs illustrate the buyer's gain from more information.[17]

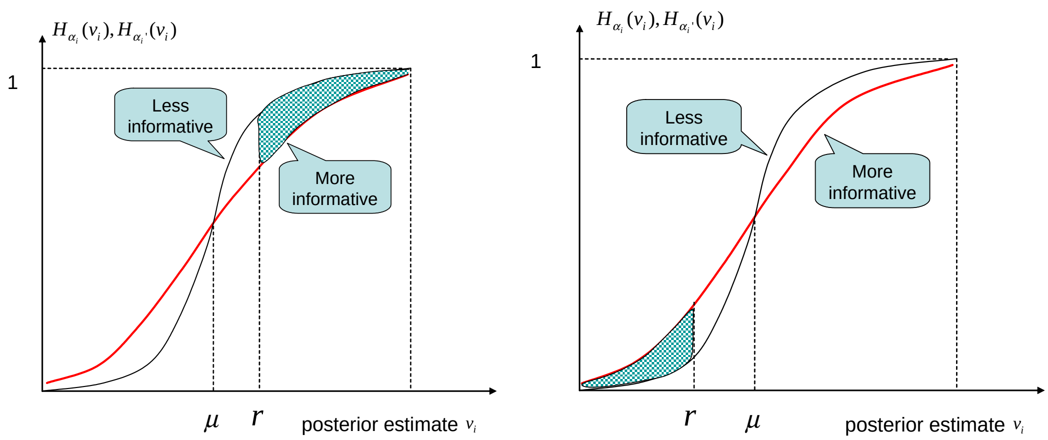

Figure 1.3: Buyer's gain from more information

Left ($r \geq \mu$): buyer's gain from more information (shaded area) decreases as r increases

Right ($r \leq \mu$): buyer's gain from more information (shaded area) increases as r increases

Given the reserve price r, the payoff of the buyer with signal α_i is the area above the distribution H_{α_i} but below one and to the right of reserve price r. When $r \geq \mu$, the buyer's relative gain with signal α_i' rather than α_i is the shaded area in the left graph (see also expression (1.3)). On the other hand, when $r \leq \mu$, according to expression (1.4), the buyer's gain from more information is the shaded area in the right graph. In both cases, the shaded area expands as r moves toward μ and achieves maximum at the mean

[17]I would like to thank Ben Polak for suggesting these two graphs.

valuation.

Another important observation obtained from Figure 1.3 is that the buyer's gain from a more informative signal is always positive. Under mild conditions, the buyer's expected payoff is an increasing concave function of α_i. Hence, the solution to the buyer's maximization problem will be unique, and the buyer's information choice will be decreasing in the information cost c (see Proposition 1.3 below).

1.3.3 The Seller's Pricing Decision

For the seller, she chooses r and equilibrium α^* to maximize her revenue. That is

$$\begin{aligned} & \max_{r,\alpha^*} \; r\left(1 - H_{\alpha^*}(r)\right) \\ s.t. \quad : \quad & \alpha^* \in \arg\max_{\alpha_i} \int_r^\infty (v_i - r)\, h_{\alpha_i}(v_i)\, dv_i - c\left(\alpha_i - \underline{\alpha}\right). \end{aligned}$$

The buyer's (agent) information choice is unobservable to the seller (principal), and the seller sets r to align the buyer's incentive to her own. Thus, we can interpret it as a principal-agent problem. The standard way to solve this problem, the so-called *first order approach,* is to assume that the second order condition of the agent's maximization problem is satisfied, and use the first order condition to replace the incentive constraint. We will assume the second order condition is satisfied for now, and discuss it in detail at the end of this subsection.

Then, we can replace the buyer's optimization problem with the first order condition, and rewrite the seller's optimization problem as[18]

$$\begin{aligned} & \max_{r,\alpha^*} \; r\left(1 - H_{\alpha^*}(r)\right) \\ s.t. \quad : \quad & -\int_r^\infty \frac{\partial H_{\alpha^*}(v_i)}{\partial \alpha^*} dv_i - c = 0. \qquad [\lambda] \end{aligned}$$

Let λ be the Lagrangian multiplier for the constraint. We write the Lagrangian in a way

[18] To simplify notation, in what follows, we will use $\frac{\partial H_{\alpha^*}(v_i)}{\partial \alpha^*}$ to denote $\frac{\partial H_{\alpha_i}(v_i)}{\partial \alpha_i}|_{\alpha_i=\alpha^*}$.

such that a positive value of λ means that the seller benefits from a reduction in the information cost; in other words, the seller prefers a more informed buyer.

Lemma 1.1 *For a fixed reserve price r, the seller's revenue increases in α^* if and only if $r > \mu$, and the seller's revenue decreases in α^* if and only if $r < \mu$.*

Proof. Immediate from the definition of the seller's revenue and property (1.1) of the Gaussian specification. ■

The intuition for this result is straightforward by looking at Figure 1.3. Suppose the buyer's information choice increases from α_i to α_i'. If $r > \mu$ (left figure), then more information increases the probability of trade from $(1 - H_{\alpha_i}(r))$ to $(1 - H_{\alpha_i'}(r))$. More information will therefore benefit the seller. In contrast, if $r < \mu$ (right figure), more information decreases the probability of trade from $(1 - H_{\alpha_i}(r))$ to $(1 - H_{\alpha_i'}(r))$, so more information will hurt the seller.

If we reinterpret our model as a monopoly pricing problem with a continuum of consumers, then this result is similar to one of the main findings in Johnson and Myatt (2006). To see this, we classify all markets into either *niche markets* or *mass markets* following Bergemann and Valimaki (2006b), and Johnson and Myatt (2006):

Definition 1.1 (Niche Market and Mass Market) *A market is said to be a niche (mass) market if the monopoly price is higher (lower) than the mean valuation μ.*

Therefore, the lemma states that the seller would prefer a more informed buyer if she is in a niche market. In contrast, the seller in a mass market will prefer a less informed buyer. This result immediately leads to the key insight in Johnson and Myatt (2006): if information is free, then a seller in the niche (mass) market will provide the maximal (minimal) amount information to consumers to maintain its niche (mass) position.

Before stating our results about the optimal reserve price, we need to define a benchmark: the *standard reserve price* when information is endowed rather than acquired.

Definition 1.2 (Standard Reserve Price) *The standard reserve price r_α is the optimal reserve price when the buyer's signal α is exogenous. That is*

$$r_\alpha \in \arg\max_r r\left(1 - H_\alpha\left(r\right)\right) \Rightarrow r_\alpha - \frac{1 - H_\alpha\left(r_\alpha\right)}{h_\alpha\left(r_\alpha\right)} = 0.$$

In particular, we will denote $r_{\underline{\alpha}}$ as the standard reserve price when no additional information (other than the endowed signal $\underline{\alpha}$) is acquired, and denote $r_{\overline{\alpha}}$ as the standard reserve price when the buyer can observe his true valuation for free. Since normal distributions have an increasing hazard rate, r_α is uniquely defined for each H_α. The seller's optimal pricing rule can thus be stated as follows:

Proposition 1.2 (Optimal Reserve Price) *For a fixed β, there exists a $\widehat{\mu}$ such that*

$$\begin{cases} \mu < r^* < r_{\alpha^*} & \text{if} \quad \mu < \widehat{\mu} \\ r^* = r_{\alpha^*} = \mu & \text{if} \quad \mu = \widehat{\mu} \\ r^* < r_{\alpha^*} < \mu & \text{if} \quad \mu > \widehat{\mu} \end{cases}.$$

Therefore, the optimal reserve price r^* with endogenous information is always (weakly) lower than the standard reserve price r_{α^*}.

In order to understand the seller's optimal pricing strategy, we can decompose the effect of a price increase on the seller's profits in three parts:

$$\frac{d\pi_s}{dr}|_{r=r^*} = \underbrace{1 - H_{\alpha^*}\left(r^*\right)}_{A} + \underbrace{\left[-r^* h_{\alpha^*}\left(r^*\right)\right]}_{B} + \underbrace{\left[-r^* \frac{\partial H_{\alpha^*}\left(r^*\right)}{\partial \alpha^*} \frac{\partial \alpha^*}{\partial r}\right]}_{C}.$$

First, the seller's profits increase given that a trade is made (term A). Second, for a fixed information choice, a price increase will reduce the probability of trade (term B). Third, with endogenous information acquisition, a price increase will affect the buyer's incentive to acquire information, thereby the probability of trade (term C). The first two terms are standard, while the last one is specific to the setting with endogenous information acquisition. If $r^* > \mu$, then an increase in r^* will discourage information acquisition. That is,

$$\frac{\partial \alpha^*}{\partial r} < 0.$$

In addition, by (1.1), for $r^* > \mu$,

$$\frac{\partial H_{\alpha^*}(r^*)}{\partial \alpha^*} < 0.$$

Therefore, term C is negative and the probability of trade decreases. Thus, the seller has less incentive to increase price compared to the case of exogenous information. As a result, $r^* < r_{\alpha^*}$. On the other hand, if $r^* < \mu$, then a price increase will encourage information acquisition, leading to

$$\frac{\partial H_{\alpha^*}(r^*)}{\partial \alpha^*} > 0, \text{ and } \frac{\partial \alpha^*}{\partial r} > 0.$$

Again, term C is negative and the seller is less willing to charge a high price compared to the case of exogenous information. Thus, $r^* < r_{\alpha^*}$. Finally, if $r^* = \mu$, a marginal increase in price does not affect buyer's incentive to acquire information. So $r^* = r_{\alpha^*}$.

We conclude this subsection by presenting sufficient conditions for the second order condition of the buyer's maximization problem to be satisfied. Under these conditions, the first order approach is valid and the buyer's expected payoff is globally concave in the information choice α_i.

Proposition 1.3 (Validity of the First Order Approach) *If* $r \in [\mu - 2\sigma(\underline{\alpha}), \mu + 2\sigma(\underline{\alpha})]$ *and* $\underline{\alpha} \geq \beta$,*the second order condition of the buyer's maximization problem is satisfied.*

These conditions are stronger than necessary and are not very restrictive. Note that more than 95% of the normal density is within two standard deviations of the mean. Thus, the first condition is to ensure that the probability of trade under r will be higher than 2.5% but lower than 97.5%. In other words, the reserve price r is neither extremely high nor extremely low ensuring that the probability of trade is neither close to 1 nor close to 0. The second condition $\underline{\alpha} \geq \beta$ is to ensure $\alpha_i > \beta$ for all α_i.[19] It requires that signals be

[19]Under this condition, the equilibrium information level is away from zero. Therefore, we can avoid the

informative relative to the prior.

1.3.4 Informational Efficiency

In this subsection, we will investigate the informational efficiency of the single-bidder auction with a reserve price r. Since there is only one bidder, the individual cost of information is the same as the social cost of information. Thus, if the marginal value of information to a buyer in an auction is higher than the social marginal value of information, the equilibrium information acquisition will be socially excessive.

Recall that, at information level α_i, the marginal value of information to the buyer is

$$MVI\left(\alpha_i\right) = -\int_r^{\infty} \frac{\partial H_{\alpha_i}\left(v_i\right)}{\partial \alpha_i} dv_i. \tag{1.5}$$

From the social point of view, the social planner chooses α_i to solve the following maximization problem

$$\max_{\alpha_i} \int_0^{\infty} \left(1 - H_{\alpha_i}\left(v_i\right)\right) dv_i - c\left(\alpha_i - \underline{\alpha}\right).$$

At information level α_i, the marginal value of information to the social planner is

$$MVI^{FB}\left(\alpha_i\right) = -\int_0^{\infty} \frac{\partial H_{\alpha_i}\left(v_i\right)}{\partial \alpha_i} dv_i. \tag{1.6}$$

Therefore, with the Gaussian specification, the difference between the individual and the social marginal value of information is

$$\begin{aligned} \Delta\left(\alpha_i\right) &= MVI^{FB}\left(\alpha_i\right) - MVI\left(\alpha_i\right) \\ &= \frac{1}{2\sqrt{2\pi}} \sqrt{\frac{\beta^3}{\alpha_i^3\left(\alpha_i + \beta\right)}} \sigma^2 \left(\exp\left(-\frac{\mu^2}{2\sigma^2}\right) - \exp\left(-\frac{\left(r - \mu\right)^2}{2\sigma^2}\right)\right). \end{aligned}$$

This proves the following result:

non-concavity of the value of information identified in Radner and Stiglitz (1984).

Proposition 1.4 (Informational Efficiency) *If $r < (>) 2\mu$, information acquisition in auctions with a reserve price r is socially excessive (insufficient).*

Note that when $r = 0$, the individual incentive to acquire information coincides with the social optimum. As r increases, the buyer's incentive to acquire information first increases then decreases after r exceeds μ. For Gaussian specification, the individual incentive to acquire information coincides with the social optimum again when $r = 2\mu$. Therefore, auctions with a single bidder and $r \in (0, 2\mu)$ lead to over-provision of information, while auctions with $r > 2\mu$ lead to under-provision of information.

1.4 Optimal Auctions with Many Bidders

The single-bidder model is simple because the strategic interaction among bidders is absent and because the Gaussian specification is special. This section studies the optimal auctions with many bidders and general information structures, and show that most of the insights from the previous section carry through as long as different signals are *rotation ordered*, a notion we will introduce below. Specifically, we show that: 1) A bidder's incentive to acquire information increases as the reserve price moves toward the mean valuation; 2) In the optimal auction, the seller who wants to encourage information acquisition sets the reserve price closer to the mean valuation than the standard reserve price; 3) Under some conditions, standard auctions with an adjusted reserve price are optimal; 4) The bidders's incentive to acquire information is socially excessive in standard auctions with a reserve price lower than the mean valuation.

One insight that cannot be carried over from the one-bidder case, however, concerns the seller's information preferences. If there are sufficiently many bidders, the seller will encourage information acquisition — even when the standard reserve price is lower than the mean valuation. We show that, in many cases, the seller will prefer that bidders acquire more information, as long as the number of bidders is large.

1.4.1 Information Order

In order to analyze a model with general information structures, we need an information order to compare the informativeness of different signals. As we showed before, the relevant variable for screening is the posterior estimate v_i, and there is one-to-one mapping between the information choice α_i and the distribution H_{α_i} of v_i. Thus, we would like to have an information order that directly ranks H_{α_i}. The rotation order, recently introduced by Johnson and Myatt (2006), meets this requirement.

Definition 1.3 (Rotation Order) *The family of distributions $\{H_{\alpha_i}\}$ is rotation-ordered if, for every α_i, there exists a rotation point $v_{\alpha_i}^{+}$, such that*

$$v_i \gtrless v_{\alpha_i}^{+} \Leftrightarrow \frac{\partial H_{\alpha_i}(v_i)}{\partial \alpha_i} \lessgtr 0. \tag{1.7}$$

Two distributions ordered in terms of rotation cross only once: the distribution with lower α_i crosses the distribution with higher α_i from below. As shown below, the rotation order implies second order stochastic dominance. However, the reverse is not true, because two distributions ordered in terms of second order stochastic dominance can cross each other more than once.

Lemma 1.2 (Rotation Order Implies Second Order Stochastic Dominance) *If a family of distributions $\{H_{\alpha_i}\}$ is rotation-ordered and they all have the same mean, then they are also ordered in terms of second order stochastic dominance.*

Proof. See Theorem 2.A.17 in Shaked and Shanthikumar (1994). ■

Following Blackwell (1951, 1953), we say that one signal is more informative than the other if a decision-maker can achieve a higher expected utility when basing a decision on the realization of the more informative signal. We extend Blackwell's information criterion to our multi-agent setting by applying his criterion to each bidder while fixing other bidders' information choices.

Proposition 1.5 *Suppose that* $\{H_{\alpha_i}\}$ *is rotation-ordered and* $\alpha_i' > \alpha_i''$*. Then under any mechanism* $\{q_i(v), t_i(v)\}$ *that is incentive compatible, bidder* i *achieves a higher expected payoff with signal* α_i' *than signal* α_i''*.*

The above result is intuitive. Because the bidder i's interim payoff $u(v_i)$ is convex in v_i under any incentive compatible mechanism (Rochet (1987)), and because $H_{\alpha_i'}$ second order stochastic dominates $H_{\alpha_i''}$ (by Lemma 1.2), the bidder i's expected payoff is higher under the more risky prospect $H_{\alpha_i'}$. Therefore, if $\{H_\alpha\}$ is rotation-ordered and $\alpha_i' > \alpha_i''$, then signal α_i' is indeed more informative than signal α_i'' because α_i' corresponds to a higher expected payoff for bidder i.

1.4.2 Characterization of the Optimal Auctions

After introducing the rotation order, we are now ready to characterize the optimal auctions. Since the posterior estimate is the only relevant variable that the seller can contract on, by the Revelation Principle, we can restrict attention to the direct revelation mechanisms. The seller's optimization problem is to choose a menu $(q_i(v_i, v_{-i}), t_i(v_i, v_{-i}))$ and a vector of information choices $(\alpha_1^*, \alpha_2^*, ..., \alpha_n^*)$ to maximize her revenue subject to (IC) (IR) and (IA) constraints. We focus on the symmetric equilibrium with $\alpha_1^* = \cdots = \alpha_n^* = \alpha^*$. Before formally stating the seller's optimization problem, we first need to reformulate the three constraints.

It is well-known (Myerson (1981) and Rochet (1987)) that the incentive compatibility constraint (IC) is equivalent to the following two conditions:

$$u_i(v_i) = u_i(\underline{\omega}) + \int_{\underline{\omega}}^{v_i} Q_i(x)\, dx, \tag{1.8}$$

and

$$Q_i(v_i) \text{ is nondecreasing in } v_i. \tag{1.9}$$

With equation (1.8), we can write the individual rationality constraint (IR) simply as

$u_i(\underline{\omega}) \geq 0$.

The information acquisition constraint (IA) requires that α^* be each bidder's best response given that other bidders choose α^*. That is, for all i,

$$\alpha^* \in \arg\max_{\alpha_i} \mathbb{E}_{v_{-i},\alpha^*} \left\{ \int_{\underline{\omega}_{\alpha_i}}^{\overline{\omega}_{\alpha_i}} [1 - H_{\alpha_i}(v_i)]\, q_i(v_i, v_{-i})\, dv_i - C(\alpha_i) \right\}.$$

As before, the subscript α^* of the expectation operator is to remind the readers that the expectation depends on the information choice α^* of bidder i's opponents. The subscript α_i in the lower and upper limits $(\underline{\omega}_{\alpha_i}, \overline{\omega}_{\alpha_i})$ is to emphasize the fact that the support of the posterior estimate may depend on the information choice α_i. The first order condition is

$$-\mathbb{E}_{v,\alpha^*} \left[\frac{\partial H_{\alpha^*}(v_i)}{\partial \alpha^*} \frac{1}{h_{\alpha^*}(v_i)} q_i(v_i, v_{-i}) \right] - C'(\alpha^*) = 0. \tag{1.10}$$

If the first order approach is valid, we can replace bidder i's optimization problem by (1.10). This approach is valid if the second order condition is satisfied, which we will assume for now, and discuss later in detail. In principle, there is a system of n first order conditions: one for each bidder. The restriction to the symmetric equilibrium helps us reduce the system of first order conditions to a single equation (1.10).[20]

Replacing the incentive constraint by equation (1.8) and (1.9), and replacing the (IA) constraint by (1.10), we can transform the seller's optimization problem from the allocation-transfer space into the allocation-utility space. That is,

$$\max_{q_i, u(\underline{\omega}), \alpha^*} \left\{ \mathbb{E}_{v,\alpha^*} \sum_{i=1}^{n} \left[\left(v_i - \frac{1 - H_{\alpha^*}(v_i)}{h_{\alpha^*}(v_i)} \right) q_i(v_i, v_{-i}) \right] - n u_i(\underline{\omega}) \right\}$$

[20] A sufficient condition for the existence of a symmetric equilibrium is that there exists a α^* satisfying both the first order condition and the second order condition of the buyer's maximization problem. If we assume $\lim_{\alpha \to \underline{\alpha}} C'(\alpha) = 0$, and $\lim_{\alpha \to \overline{\alpha}} C'(\alpha) = \kappa$ (where κ is a large positive number), then there must exist a α^* satisfies the first order condition (1.10). If the cost function is sufficiently convex, that is, $C''(\alpha)$ is sufficient large, then the second order condition is satisfied (See Appendix B for more detail). A quadradic cost function $C(\alpha) = \kappa_0 (\alpha - \underline{\alpha})^2$ with large κ_0 meets all the requirements.

subject to

$$0 \leq q_i(v_i, v_{-i}) \leq 1; \sum_{i=1}^{n} q_i(v_i, v_{-i}) \leq 1, \tag{Regularity}$$

$$Q_i(v_i) \text{ is nondecreasing in } v_i, \tag{Monotonicity}$$

$$u_i(\underline{\omega}) \geq 0, \tag{IR}$$

$$\mathbb{E}_{v,\alpha^*}\left[-\frac{\partial H_{\alpha^*}(v_i)}{\partial \alpha^*}\frac{1}{h_{\alpha^*}(v_i)}q_i(v_i, v_{-i})\right] - C'(\alpha^*) = 0. \tag{IA}$$

It is easy to see that the (IR) constraint must be binding. For now we can ignore the regularity constraint and the monotonicity constraint and verify them later. Then the only remaining constraint is the (IA) constraint. Let λ denote the Lagrangian multiplier for the (IA) constraint, and write the Lagrangian for the seller's maximization problem as

$$\mathcal{L} = \mathbb{E}_{v,\alpha^*}\sum_{i=1}^{n}\left[\left(v_i - \frac{1 - H_{\alpha^*}(v_i)}{h_{\alpha^*}(v_i)} - \frac{\lambda}{n}\frac{\partial H_{\alpha^*}(v_i)}{\partial \alpha^*}\frac{1}{h_{\alpha^*}(v_i)}\right)q_i(v_i, v_{-i})\right] - \lambda C'(\alpha^*). \tag{1.11}$$

Then a positive λ implies that the seller's revenue increases as the marginal cost of information decreases. The virtual surplus function $J^*(v_i)$ can be defined as

$$J^*(v_i) = v_i - \frac{1 - H_{\alpha^*}(v_i)}{h_{\alpha^*}(v_i)} - \frac{\lambda}{n}\frac{\partial H_{\alpha^*}(v_i)}{\partial \alpha^*}\frac{1}{h_{\alpha^*}(v_i)}. \tag{1.12}$$

In order to characterize the optimal solution to the seller's optimization problem, we make the following assumptions:

Assumption 1.1 (Rotation Order) *The family of distributions of the posterior estimate, $\{H_{\alpha_i}\}$, is rotation ordered and the rotation point is μ for all α_i.*

Assumption 1.2 (Monotonicity)

$$-\frac{\partial H_{\alpha_i}(v_i)}{\partial \alpha_i}\frac{1}{h_{\alpha_i}(v_i)} \textit{ is nondecreasing in } v_i \textit{ for all } \alpha_i \textit{ and } v_i \in \left[\underline{\omega}_{\alpha_i}, \overline{\omega}_{\alpha_i}\right].$$

Assumption 1.3 (Regularity)

$$v_i - \frac{1 - H_{\alpha_i}(v_i)}{h_{\alpha_i}(v_i)} \text{ is nondecreasing in } v_i \text{ for all } \alpha_i \text{ and } v_i \in \left[\underline{\omega}_{\alpha_i}, \overline{\omega}_{\alpha_i}\right].$$

Assumption 1.1 assumes that the signals are rotation ordered and the rotation point $v^+_{\alpha_i}$ is μ for all α_i. The assumption $v^+_{\alpha_i} = \mu$ is not critical, but it greatly eases our presentation. We will discuss it later. Assumption 1.2 is stronger than the rotation order assumption, and it says that the expected gain from more information is higher for the buyers with higher v_i.[21] Finally, Assumption 1.3 is a regularity assumption.

Both the rotation order assumption and the regularity assumption are mild assumptions. The monotonicity assumption is relatively more restrictive, but two commonly used information technologies in the literature, the Gaussian specification and the truth-or-noise technology, satisfy all three assumptions.

Definition 1.4 (Truth-or-noise Technology) *The buyers' true valuations $\{\omega_i\}$ are independently drawn from a distribution F, and F has an increasing hazard rate. Buyer i can acquire a costly signal s_i about ω_i. With probability $\alpha_i \in [\underline{\alpha}, 1]$, the signal s_i perfectly matches the true valuation ω_i, and with probability $1-\alpha_i$, s_i is a noise independently drawn from F.*

Under the truth-or-noise specification, the signal s_i sometimes perfectly reveals buyer i's valuation ω_i, but is noise otherwise.

Lemma 1.3 (All Assumptions Hold for the Two Leading Examples) *Both the Gaussian specification and the truth-or-noise technology generate a family of distributions $\{H_{\alpha_i}\}$ that satisfies Assumptions 1.1, 1.2, and 1.3.*

[21] Indeed, the monotonicity assumption, together with the mean-preserving property of our information structures, implies rotation order. To see this, first note that $\frac{\partial H_{\alpha_i}(v_i)}{\partial \alpha_i}$ cannot always be positive or negative, otherwise it will imply first order stochastic dominance which violates the fact that the family of distributions $\{H_{\alpha_i}\}$ have the same mean. Therefore, if monotonicity assumption holds, $\frac{\partial H_{\alpha_i}(v_i)}{\partial \alpha_i}$ must change sign from positive to negative only once. That is, $\{H_{\alpha_i}\}$ is rotation ordered.

Note that Assumption 1.1 does not imply that the underlying distribution F is symmetric. For example, for the truth-or-noise technology, the underlying distribution F could be convex or concave, but the rotation point is still μ.

In the rest of this subsection, we first analyze the buyers' information decision and generalize Proposition 1.1. Then we investigate the seller's information preferences, and characterize the relationship between the optimal reserve price and the standard reserve price to generalize Proposition 1.2. Finally, we present conditions under which standard auctions with an adjusted reserve price are optimal.

Let r^* denote the reserve price in the optimal auction. If bidder i is allocated the object with positive probability, then his posterior estimate is at least r^*. That is,

$$q_i(v_i, v_{-i}) > 0 \Rightarrow v_i \geq r^*.$$

With reserve price r^*, the marginal value of information to bidder i under an incentive compatible mechanism $\{q_i(v), t_i(v)\}$ is

$$MVI = -E_{v_{-i},\alpha^*}\left[\int_{r^*}^{\overline{\omega}_{\alpha_i}} \frac{\partial H_{\alpha_i}(v_i)}{\partial \alpha_i} q_i(v_i, v_{-i})\, dv_i\right].$$

Theorem 1.1 (Marginal Value of Information to a Bidder) *The marginal value of information to a bidder increases as r^* moves toward the mean valuation if and only if Assumption 1.1 is satisfied.*

Theorem 1.1 generalizes Proposition 1.1 to a setting with many bidders and an information structure that is rotation-ordered. Therefore, if a seller wants to induce buyers to acquire more information, she should set a reserve price closer to the mean valuation.

But when will the seller want to encourage information acquisition? By definition, the Lagrangian multiplier λ for (IA) constraint is the seller's marginal gain from a deduction in marginal information cost. Therefore, the seller will encourage bidders to acquire more information if $\lambda > 0$. The following proposition provides sufficient conditions for $\lambda > 0$.

Lemma 1.4 *The seller benefits from a reduction of marginal cost ($\lambda > 0$), when either one of the following two sets of conditions is satisfied:*

(1) Assumptions 1.1, 1.2 and 1.3 hold and

$$\mu < \frac{1 - H_{\underline{\alpha}}(\mu)}{h_{\underline{\alpha}}(\mu)}.$$

(2) The Gaussian specification or the truth-or-noise technology, and large n.

The first condition implies $r^* > \mu$ which is sufficient for $\lambda > 0$. Recall that, in the case of one bidder, the seller prefers more information if $r^* > \mu$. An increase in the number of bidders only strengthens the seller's preference for more information. The second condition should be contrasted with Lemma 1.1 in the case of one bidder. The strategic interaction between buyers, which is absent in the one-bidder model, plays a crucial part here.[22] As shown by condition (2) in Lemma 1.4, as long as n is large, the seller will prefer that bidders acquire more information regardless of whether the optimal reserve price is higher or lower than the mean valuation. To see this, note that the seller's revenue is determined by the valuation of the marginal bidder (for example, the second highest bidder) and the reserve price. With many bidders, the valuation of the marginal bidder will be higher than the mean valuation. This valuation is likely to be higher when more information is acquired. In the case with one bidder, however, a seller will prefer a more informed buyer only when the optimal reserve price is higher than the mean valuation (niche market).

Remark. The next two theorems will characterize the optimal selling mechanism contingent on the sign of the endogenous Lagrangian multiplier λ. With Lemma 1.4, we can always restate the theorems by replacing the condition $\lambda > 0$ by the exogenous condition (1) or (2). However, since both condition (1) and (2) are not necessary for $\lambda > 0$, we state our theorems in terms of λ in order to be precise.

Now, we can present a simple rule for adjusting the reserve price in optimal auctions

[22] In the discrete information acquisition setting, an important consequence of the strategic interaction is the possibility of the symmetric mixed strategy equilibrium. See Appendix B for an anlysis of the case of the discrete information acquisition.

with information acquisition:

Theorem 1.2 (Simple Rule for Adjusting the Reserve Price) *Suppose Assumptions 1.1 and 1.3 hold. If $\lambda > 0$, then the optimal reserve price r^* is closer to the mean valuation μ than the standard reserve price r_{α^*}. Specifically,*

$$\begin{cases} \mu \leq r^* < r_{\alpha^*} & \text{if} \quad r_{\alpha^*} > \mu \\ r^* = \mu & \text{if} \quad r_{\alpha^*} = \mu \\ r_{\alpha^*} < r^* \leq \mu & \text{if} \quad r_{\alpha^*} < \mu \end{cases}.$$

If $\lambda < 0$, then $r^* < r_{\alpha^*} < \mu$.

Theorem 1.2 is conceptually a direct consequence of Theorem 1.1, and generalizes Proposition 1.2. It characterizes the relationship between the optimal reserve price in our setting and the standard reserve price in Myerson's optimal auctions. If the seller wants to encourage information acquisition, she has to set the optimal reserve price between the mean valuation and the standard reserve price because the bidders' incentives to acquire information are stronger when the reserve price is closer to the mean valuation.

This result is important in practice when the seller is concerned about bidders' incentives to acquire information. The reserve price is always the most important decision she has to make other than choosing the auction format. Theorem 1.2 identifies a simple rule to adjust the reserve price when endogenous information acquisition is important. The rule is simple and robust in the sense that it holds also in the discrete information acquisition specification (see Appendix B).

Furthermore, the empirical auction literature has attempted to evaluate the optimality of a seller's reserve price policy. Most of these studies assume exogenous information and do not consider the bidders' incentives to acquire information. They use observed bids and the equilibrium bidding behavior to recover the distribution of bidders' valuations, and then compare the actual reserve price with the standard reserve price calculated from the estimated distribution. Our results indicate that, in situations where information acquisition is important, the standard reserve price may not be an appropriate benchmark for comparison. The optimal reserve price in optimal auctions could be higher or lower than the standard reserve price when information is endogenous.

The next result shows that under the stronger Assumption 1.2, standard auctions with an appropriately chosen reserve price are optimal.

Theorem 1.3 (Optimal Auctions) *Suppose Assumptions 1.1, 1.2 and 1.3 hold, and* $\lambda > 0$*. Then standard auctions with the reserve price* r^* *adjusted according to Theorem 1.2 are optimal.*

Assumption 1.2 is critical for the above theorem. It ensures that the bidders with higher posterior estimate gain more from information acquisition. Therefore, if the allocation rule assigns the object to the bidder with the highest posterior estimate (just as standard auctions do), then bidders' expected gain from information acquisition will be maximized, and bidders will have a strong incentive to acquire information. But this is exactly what the seller would like to see when $\lambda > 0$: an increase in information acquisition benefits the seller. An immediate consequence of Theorem 1.3 is the revenue equivalence among all standard auctions, because the allocation rule is the same across all standard auctions. Furthermore, since the bidders' expected gain from information acquisition is the same for all standard auctions, the equilibrium amount of information acquired is the same across standard auctions as well.

The restriction of symmetric equilibrium is important for the above result. If we allow different bidders to acquire different levels of information in equilibrium, then the revenue equivalence fails in general, and different auctions will induce different level of information acquisition. Moreover, this result may not be generalized to discrete information acquisition setting. With discrete information acquisition, in general, bidders will play mixed strategies in equilibrium, which will introduce asymmetry into the interim stage when all bidders have made their information decisions. Because the first price auction and the second price auction are not equivalent when bidders are asymmetric, revenue equivalence fails.

1.4.3 Informational Efficiency

Theorem 1.3 states that standard auctions with an adjusted reserve price are optimal under some conditions. In this subsection we will therefore focus on the informational efficiency of standard auctions to obtain a slightly more general results that apply to optimal auctions.

Since we restrict attention to the symmetric equilibrium in the optimal auctions, we need a symmetric benchmark as well. Thus, we assume that the social optimal information choice α^{FB} is the same for all bidders. That is, α^{FB} solves the following maximization problem for all i :

$$\alpha^{FB} \in \arg\max_{\alpha_i} \int_0^{\overline{\omega}_{\alpha_i}} \left(1 - H^n_{\alpha_i}(v_i)\right) dv_i - nC(\alpha_i).$$

At information level α_i, the marginal value of information to the social planner is

$$MVI^{FB}(\alpha_i) = -n \int_0^{\overline{\omega}_{\alpha_i}} \frac{\partial H_{\alpha_i}(v_i)}{\partial \alpha_i} H^{n-1}_{\alpha_i}(v_i)\, dv_i. \tag{1.13}$$

Recall that, at information level α_i, the marginal value of information to the bidder i is

$$MVI(\alpha_i) = -\int_r^{\overline{\omega}_{\alpha_i}} \frac{\partial H_{\alpha_i}(v_i)}{\partial \alpha_i} H^{n-1}_{\alpha_i}(v_i)\, dv_i. \tag{1.14}$$

Since the social planner has to pay n times the individual information cost, we normalize the social value of information by multiplying $1/n$. The difference between the social and individual gain from acquiring information is

$$\Delta(\alpha_i, n) = \frac{1}{n} MVI^{FB}(\alpha_i) - MVI(\alpha_i) = -\int_0^r \frac{\partial H_{\alpha_i}(v_i)}{\partial \alpha_i} H^{n-1}_{\alpha_i}(v_i)\, dv_i. \tag{1.15}$$

By definition of rotation order, if $r < \mu$, $\Delta(\alpha_i, n) < 0$. That is, information acquisition in auctions with $r < \mu$ is socially excessive. Thus, we have proved the following result.

Proposition 1.6 (Informational Efficiency) *Suppose Assumption 1.1 holds. There ex-*

ists $\delta > 0$ such that bidders have socially excessive incentives to acquire information in standard auctions if and only if $r < \mu + \delta$.

When $r = 0$, the bidders' incentive to acquire information coincides with the social optimum, which can be easily seen from equation (1.15).[23] As r increases, the buyers' incentive to acquire information increases, reaches maximum at $r = \mu$, and declines afterwards. Consequently, there exists a $\delta > 0$, such that the individual incentive to acquire information coincides with the social optimum when $r = \mu + \delta$. Therefore, the bidders' incentive to acquire information is socially excessive when $r \in (0, \mu + \delta)$. For the one-bidder model with the Gaussian specification, $\delta = \mu$, as shown in Proposition 1.4.

1.4.4 Discussion

In our model, the rotation order ranks different information structures by comparing the distributions of the posterior estimate. In contrast, most existing information orders (for example, Lehmann (1988)) impose restrictions on the prior or posterior distributions of underlying states and signals. One can show that a weaker version of Lehmann's order, the MIO-ND order in Athey and Levin (2001), generates a family of distributions $\{H_{\alpha_i}\}$ ordered in terms of second order stochastic dominance. The rotation order also implies second order stochastic dominance, but second order stochastic dominance is not strong enough for our analysis.

Assumption 1.1 restricts the rotation point to be the mean valuation. However, if the rotation point is different from the mean valuation, our results (Theorem 1-3 and Proposition 6-7) still hold as long as we replace μ in the statements of the results by the rotation point. If the rotation order assumption fails, so that two distributions of the posterior estimate cross each other more than once, then some of our results (for instance, Theorem 1) still hold locally around one of the crossing points.

[23] Note that the standard auctions with zero reserve price are efficient mechanisms. Bergemann and Valimaki (2002) show that the individual incentives to acquire information coincide with the social optimum for efficient mechanisms in the private value setting. Thus, information acquisition in standard auctions with zero reserve price is also social optimal.

The first order approach greatly simplifies our analysis and is valid if the second order condition of the bidder's maximization problem is satisfied. In Appendix B, we provides several sets of sufficient conditions for this condition to hold. First, it is satisfied if the cost function is sufficiently convex. Second, if the support of H_{α_i} is invariant with respect to α_i, then a condition analogous to the CDFC condition in the principal-agent literature (Mirrlees (1999), Rogerson (1985)) is sufficient. Third, we present sufficient conditions for the case of the Gaussian specification and the truth-or-noise technology, respectively. See Appendix B for further discussion of these conditions.

As pointed by Bolton and Dewatripont (2005), however, the requirement that the bidders' first-order condition be necessary and sufficient is too strong. All we need is that the replacement of the (IA) constraint by the first order condition can generate necessary conditions for the seller's original maximization problem. Thus, our analysis may remain valid even when the second order condition of the bidders' maximization problem fails.

In order to check whether our results are robust to alternative information specifications, we study discrete information acquisition in Appendix B. To ease comparison to the existing literature, we assume that information acquisition is binary and focus on the symmetric mixed strategy equilibrium. Under some technical assumptions, we show that the simple rule for adjusting the reserve price still holds. This result can also partially alleviate any concerns about the first order approach. With discrete information acquisition, however, standard auctions are no longer optimal because mixed strategy introduces asymmetry into the post-information game: in the bidding stage, bidders are no longer symmetric.

Finally, although our model focuses on the independent private value framework, it can also be immediately applied to a setting with a common component. For example, suppose buyer i's true valuation θ_i has two components:

$$\theta_i = \omega_i + y.$$

The first term ω_i represents the individual idiosyncratic valuation and is unknown ex-ante. Buyer i can acquire costly information about ω_i. The second term y is the common value component, and both the buyers and the seller learn it for free.[24] In this situation, all our analysis still applies as the common component only shifts the distribution but does not affect the buyers' incentives.

1.5 Conclusion

The mechanism design literature studies how carefully designed mechanisms can be used to elicit agents' private information in order to achieve a desired goal. Most of the papers in the literature, however, ignore the influence of the proposed mechanisms on agents' incentives to gather information. In particular, with endogenous information acquisition, the optimal selling mechanism should take into account the bidders' information decision as a response to the proposed mechanism. We show that under some conditions standard auctions with a reserve price remain optimal but the reserve price has to be adjusted in order to incorporate the buyers' incentives to acquire information.

Relative to the existing literature, our model has three distinctive features. First, we study the optimal mechanism that maximizes revenue in the presence of information acquisition. This distinguishes our model from papers studying information acquisition in fixed auction formats. Second, we study private and decentralized information acquisition, thus differing from previous studies on the seller's optimal disclosure policy and various entry models. Finally, the information structure required for our results is more general than most of the existing literature on mechanism design: we require only that the distributions of the posterior estimate be rotation-ordered.

[24] For example, a firm typically has two types of assets: liquid and illiquid. All potentiall buyers of the firm may value liquid assets in the same way, but they may value the illiquid assets differently. The value of liquid assets can be easily learned from financial statements.

1.6 Appendix A

Proof of Proposition 1.1: With some algebra, we can show that the partial derivative of $H_{\alpha_i}(v_i)$ with respect to informativeness, α_i, is given by

$$\frac{\partial H_{\alpha_i}(v_i)}{\partial \alpha_i} = -\frac{(v_i-\mu)}{2\sqrt{2\pi}} \exp\left(-\frac{(v_i-\mu)^2}{2\sigma^2}\right)\sqrt{\frac{\beta^3}{\alpha_i^3(\alpha_i+\beta)}}. \tag{1.16}$$

Insert this into the expression of MVI, we have

$$\begin{aligned} MVI &= \int_r^\infty \frac{(v_i-\mu)}{2\sqrt{2\pi}} \exp\left(-\frac{(v_i-\mu)^2}{2\sigma^2}\right)\sqrt{\frac{\beta^3}{\alpha_i^3(\alpha_i+\beta)}} dv_i \\ &= \frac{1}{2\sqrt{2\pi}}\sqrt{\frac{\beta^3}{\alpha_i^3(\alpha_i+\beta)}}\sigma \int_r^\infty \left(\frac{v_i-\mu}{\sigma}\right)\exp\left(-\frac{(v_i-\mu)^2}{2\sigma^2}\right) dv_i \\ &= \frac{1}{2\sqrt{2\pi}}\sqrt{\frac{\beta^3}{\alpha_i^3(\alpha_i+\beta)}}\sigma^2 \exp\left(-\frac{(r-\mu)^2}{2\sigma^2}\right). \end{aligned}$$

Therefore, as $r \to \mu$, MVI increases. ■

Proof of Proposition 1.2: We can write the Lagrangian of the seller's optimization problem as follows:

$$\begin{aligned} \mathcal{L}(r,\alpha^*) &= r(1-H_{\alpha^*}(r)) + \lambda\left(-\int_r^\infty \frac{\partial H_{\alpha^*}(v_i)}{\partial \alpha^*} dv_i - c\right) \\ &= r(1-H_{\alpha^*}(r)) + \lambda\left(\frac{1}{2\sqrt{2\pi}}\sqrt{\frac{\beta^3}{\alpha^{*3}(\alpha^{*3}+\beta)}}\sigma^2\exp\left(-\frac{(r-\mu)^2}{2\sigma^2}\right) - c\right). \end{aligned}$$

The last equality follows by substituting in expression (1.16). The first order conditions are

$$\frac{\partial \mathcal{L}}{\partial r} = 1 - H_{\alpha^*}(r^*) - rh_{\alpha^*}(r^*) - \lambda\frac{1}{2\sqrt{2\pi}}\sqrt{\frac{\beta^3}{\alpha^{*3}(\alpha^*+\beta)}}\exp\left(-\frac{(r^*-\mu)^2}{2\sigma^2}\right)(r^*-\mu) = 0, \tag{1.17}$$

$$\frac{\partial \mathcal{L}}{\partial \alpha^*} = -r^*\frac{\partial H_{\alpha^*}(r^*)}{\partial \alpha^*} + \lambda\left(-\int_{r^*}^\infty \frac{\partial^2 H_{\alpha^*}(v_i)}{\partial \alpha^{*2}} dv_i\right) = 0. \tag{1.18}$$

The second order condition of the buyer's maximization problem implies that

$$-\int_{r^*}^{\infty} \frac{\partial^2 H_{\alpha^*}(v_i)}{\partial \alpha^{*2}} dv_i < 0,$$

In addition, from (1.16) we can show

$$\frac{\partial H_{\alpha^*}(r^*)}{\partial \alpha^*} \lesseqgtr 0 \Leftrightarrow r^* \gtreqless \mu.$$

Therefore, condition (1.18) implies that

$$r^* \gtreqless \mu \Leftrightarrow \lambda \gtreqless 0. \tag{1.19}$$

Suppose $r_{\alpha^*} > \mu$. Then $r^* < r_{\alpha^*}$. To see this, suppose the opposite is true: $r^* \geq r_{\alpha^*}$. Then $r^* \geq \mu$ and $\lambda > 0$. Therefore, $\frac{\partial \mathcal{L}}{\partial r}|_{r=r^*} < 0$. A contradiction to the optimality of r^*. Next we argue that r^* cannot be less than μ. Suppose $r^* < \mu$ for contradiction. Then $\lambda < 0$ by (1.19). But $r^* < \mu$ and $\lambda < 0$ imply $\frac{\partial L}{\partial r} > 0$, a contradiction. In sum, $r^* \in [\mu, r_{\alpha^*})$. The other two cases can be proved analogously.

Therefore, we only need to prove that for a fixed β, there exists a $\widehat{\mu}$ such that $r^* > \mu$ if and only if $\mu < \widehat{\mu}$. Note that the first order condition for the buyer's maximization problem is

$$\frac{1}{2\sqrt{2\pi}} \sqrt{\frac{\beta}{\alpha (\alpha + \beta)^3}} \exp \left(-\frac{(r-\mu)^2}{2\sigma^2} \right) - c = 0$$

With some algebra, we can show

$$\frac{\partial \alpha}{\partial r} \begin{cases} > 0 & \text{if} \quad r < \mu \\ = 0 & \text{if} \quad r = \mu \\ < 0 & \text{if} \quad r > \mu \end{cases} \quad \text{and} \quad \frac{\partial^2 \alpha}{\partial r \partial \mu} > 0.$$

Note that we can also write the necessary first order condition the seller's maximization problem as

$$\frac{d\pi_s}{dr}|_{r=r^*} = 1 - H_{\alpha^*}(r^*) - r^* h_{\alpha^*}(r^*) - r^* \frac{\partial H_\alpha(r^*)}{\partial \alpha^*} \frac{\partial \alpha^*}{\partial r} = 0$$

Define

$$\Gamma(r^*, \mu) = 1 - H_{\alpha^*}(r^*) - r^* h_{\alpha^*}(r^*) - r^* \frac{\partial H_{\alpha^*}(r^*)}{\partial \alpha^*} \frac{\partial \alpha^*}{\partial r}.$$

Then

$$\frac{\partial \Gamma(r^*, \mu)}{\partial \mu} = \frac{\partial H_{\alpha^*}(r^*)}{\partial \alpha^*} \frac{\partial \alpha^*}{\partial r^*} + r^* \frac{\partial h_{\alpha^*}(r^*)}{\partial \alpha^*} \frac{\partial \alpha^*}{\partial r} + r \frac{\partial^2 H_{\alpha^*}(r)}{\partial \alpha^{*2}} \left(\frac{\partial \alpha^*}{\partial r^*}\right)^2 - r^* \frac{\partial H_{\alpha^*}(r^*)}{\partial \alpha^*} \frac{\partial^2 \alpha^*}{\partial r^* \partial \mu}$$

and

$$\frac{\partial \Gamma(r^*, \mu)}{\partial r^*} = -2h_{\alpha^*}(r^*) - 2\frac{\partial H_{\alpha^*}(r^*)}{\partial \alpha^*} \frac{\partial \alpha^*}{\partial r^*} - r^* \frac{\partial h_{\alpha^*}(r^*)}{\partial \alpha^*} \frac{\partial \alpha^*}{\partial r^*} - r^* \frac{\partial^2 H_{\alpha^*}(r^*)}{\partial \alpha^{*2}} \left(\frac{\partial \alpha^*}{\partial r^*}\right)^2 + r^* \frac{\partial H_{\alpha^*}(r^*)}{\partial \alpha^*} \frac{\partial^2 \alpha^*}{\partial r^* \partial \mu}$$

Therefore, if $r^* > \mu$

$$\frac{\partial \Gamma(r^*, \mu)}{\partial \mu} > 0, \text{ and } \frac{\partial \Gamma(r^*, \mu)}{\partial r} < 0$$

Furthermore,

$$\frac{\partial \Gamma(r^*, \mu)}{\partial \mu} < -\frac{\partial \Gamma(r^*, \mu)}{\partial r}$$

Therefore, for $r^* > \mu$,

$$\frac{dr^*}{d\mu} = -\frac{\frac{\partial \Gamma(r^*,\mu)}{\partial \mu}}{\frac{\partial \Gamma(r^*,\mu)}{\partial r}} \in (0,1).$$

Furthermore, for $r^* = \mu$,

$$\frac{\partial \Gamma(r^*, \mu)}{\partial \mu} = 0, \ \frac{\partial \Gamma(r^*, \mu)}{\partial r} < 0, \text{ and } \frac{dr^*}{d\mu} = 0.$$

Note that $r^*(\mu) > \mu$ for $\mu < 0$. Therefore, there must exists a $\widehat{\mu}$ such that

$$r^*(\widehat{\mu}) = \widehat{\mu}.$$

Moreover, because $\frac{dr^*}{d\mu} \in (0,1)$ for $r^* > \mu$, and $\frac{dr^*}{d\mu} = 0$ for $r^* = \mu$, $\widehat{\mu}$ is unique and $\mu < \widehat{\mu} \Leftrightarrow r^* > \mu$. ■

Proof of Proposition 1.3: The first order condition for the buyer's optimization problem is

$$-\int_r^\infty \frac{\partial H_{\alpha_i}(v_i)}{\partial \alpha_i} dv_i - c = 0.$$

The second order condition is

$$-\int_r^\infty \frac{\partial^2 H_{\alpha_i}(v)}{\partial \alpha_i^2} dv_i < 0.$$

With some algebra, we can show

$$\frac{\partial^2 H_{\alpha_i}(v_i)}{\partial \alpha_i^2} = \frac{4\alpha_i + 3\beta}{2\alpha_i(\alpha_i+\beta)} \frac{(v_i - \mu)}{2\sqrt{2\pi}} \sqrt{\frac{\beta^3}{\alpha_i^3(\alpha_i+\beta)}} \exp\left(-\frac{(v_i-\mu)^2}{2\sigma^2}\right)\left(1 - \frac{\alpha_i+\beta}{4\alpha_i+3\beta}\frac{\beta^2}{\alpha_i}(v_i-\mu)^2\right).$$

Therefore, we can rewrite the second order condition as

$$\int_r^\infty \frac{(v_i-\mu)}{2\sqrt{2\pi}} \exp\left(-\frac{(v_i-\mu)^2}{2\sigma^2}\right)\left(1 - \frac{\alpha_i+\beta}{4\alpha_i+3\beta}\frac{\beta^2}{\alpha_i}(v_i-\mu)^2\right) dv_i > 0.$$

By a change of variable with $y = \frac{v_i - \mu}{\sigma}$, we can obtain

$$\int_x^\infty y \exp\left(-\frac{1}{2}y^2\right)\left(1 - ky^2\right) dy > 0,$$

where

$$k = \frac{\alpha_i+\beta}{4\alpha_i+3\beta}\frac{\beta^2}{\alpha_i}\sigma^2 = \frac{\beta}{4\alpha_i+3\beta}, \quad \text{and} \quad x = \frac{r-\mu}{\sigma} = (r-\mu)/\sqrt{\frac{\alpha_i}{(\alpha_i+\beta)\beta}}.$$

The above inequality can be simplified into

$$-e^{-\frac{1}{2}x^2}\left(kx^2+2k-1\right)>0 \Leftrightarrow k<\frac{1}{2+x^2}.$$

Substitute the expression of k and x and we can obtain

$$\frac{4\alpha_i+\beta}{\alpha_i+\beta}\frac{\alpha_i}{\beta^2}>(r-\mu)^2. \tag{1.20}$$

Now if $r\in[\mu-2\sigma(\underline{\alpha}),\mu+2\sigma(\underline{\alpha})]$, then $r\in[\mu-2\sigma,\mu+2\sigma]$ because $\sigma(\alpha_i)\geq\sigma(\underline{\alpha})$ for all α_i. Therefore, a sufficient condition for (1.20) is

$$\frac{4\alpha_i+\beta}{\alpha_i+\beta}\frac{\alpha_i}{\beta^2}>4\sigma^2,$$

or equivalently,

$$\alpha_i>\frac{3}{4}\beta.$$

Since $\alpha_i>\underline{\alpha}$ for all i, the second order condition is satisfied when $\underline{\alpha}>\beta$. That is, the first order approach is valid when

$$r\in[\mu-2\sigma(\underline{\alpha}),\mu+2\sigma(\underline{\alpha})] \text{ and } \underline{\alpha}>\beta.$$

Thus, we conclude the proof. ■

Proof of Proposition 1.5: Under mechanism $\{q_i(v_i,v_{-i}),t_i(v_i,v_{-i})\}$, a bidder's expected payoffs (information rent) with information structure α_i' and α_i'' are, respectively[25]

$$\begin{aligned}\mathbb{E}u\left(v_i;\alpha_i'\right) &= \mathbb{E}_{v_{-i}}\left[\int_{\underline{\omega}}^{\overline{\omega}}\left(1-H_{\alpha_i'}\left(v_i\right)\right)q_i\left(v_i,v_{-i}\right)dv_i\right],\\ \mathbb{E}u\left(v_i;\alpha_i''\right) &= \mathbb{E}_{v_{-i}}\left[\int_{\underline{\omega}}^{\overline{\omega}}\left(1-H_{\alpha_i''}\left(v_i\right)\right)q_i\left(v_i,v_{-i}\right)dv_i\right].\end{aligned}$$

Therefore,

$$\begin{aligned}&\mathbb{E}u\left(v_i;\alpha_i'\right)-\mathbb{E}u\left(v_i;\alpha_i''\right)\\ =\ &\mathbb{E}_{v_{-i}}\left[\int_{\underline{\omega}}^{\overline{\omega}}\left(H_{\alpha_i''}\left(v_i\right)-H_{\alpha_i'}\left(v_i\right)\left(v_i\right)\right)q_i\left(v_i,v_{-i}\right)dv_i\right]\\ =\ &-\mathbb{E}_{v_{-i}}\left[\int_{\underline{\omega}}^{\overline{\omega}}\left(\int_{\underline{\omega}}^{\omega_i}q_i\left(x,v_{-i}\right)dx\right)\left(h_{\alpha_i''}\left(v_i\right)-h_{\alpha_i'}\left(v_i\right)\right)dv_i\right]\quad\text{(integration by part)}\\ =\ &-\int_{\underline{\omega}}^{\overline{\omega}}\left(\int_{\underline{\omega}}^{\omega_i}Q_i\left(x\right)dx\right)\left(h_{\alpha_i''}\left(v_i\right)-h_{\alpha_i'}\left(v_i\right)\right)dv_i\quad\left(\text{where }Q_i\left(x\right)=\mathbb{E}_{v_{-i}}\left[q_i\left(x,v_{-i}\right)\right]\right).\end{aligned}$$

Since $Q_i(x)$ is nondecreasing in x, $\int_{\underline{\omega}}^{\omega_i} Q_i(x)\,dx$ is convex. By Lemma 1.2 , $H_{\alpha_i'}$ SOSD $H_{\alpha_i''}$ and have the same mean. Therefore, $\mathbb{E}u(v_i;\alpha_i') - \mathbb{E}u(v_i;\alpha_i'') > 0$.■

Proof of Lemma 1.3: For the Gaussian specification, we know from the text that

$$H_{\alpha_i}\left(v_i\right)=\int_{-\infty}^{v_i}\frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{\left(x-\mu\right)^2}{2\sigma^2}\right)dx\quad\text{where }\sigma^2=\frac{\alpha_i}{\left(\alpha_i+\beta\right)\beta}.$$

Since H_α is normal, it has an increasing hazard rate and the regularity assumption is satisfied. Recall equation (1.16)

$$\frac{\partial H_{\alpha_i}\left(v_i\right)}{\partial\alpha_i}=-\frac{\left(v_i-\mu\right)}{2\sqrt{2\pi}}\exp\left(-\frac{\left(v_i-\mu\right)^2}{2\sigma^2}\right)\sqrt{\frac{\beta^3}{\alpha_i^3\left(\alpha_i+\beta\right)}}.$$

In addition,

$$\frac{\partial H_{\alpha_i}\left(v_i\right)}{\partial\alpha_i}\frac{1}{h_{\alpha_i}\left(v_i\right)}=-\frac{\beta\left(v_i-\mu\right)}{2\alpha_i\left(\alpha_i+\beta\right)}.$$

[25] If the support of the posterior estimate varies with respect to signal informativeness, we need to redefine the distribution as follows. Suppose under information structure α_i, the support is $\left[\underline{\omega}_{\alpha_i},\overline{\omega}_{\alpha_i}\right]$. Then define $H_{\alpha_i}(v_i)=0$ if $v_i\in\left[\underline{\omega},\underline{\omega}_{\alpha_i}\right]$ and $H_{\alpha_i}(v_i)=1$ if $v_i\in\left[\overline{\omega}_{\alpha_i},\overline{\omega}\right]$.

It is easy to see that the other two assumptions are satisfied as well.

For the truth-or-noise technology, a buyer who observes a realization s_i with precision α_i will revise his posterior estimate as follows:

$$v_i\left(s_i,\alpha_i\right)=\mathbb{E}\left(\omega_i|s_i,\alpha_i\right)=\alpha_i s_i+\left(1-\alpha_i\right)\mu.$$

The distribution and density of the posterior estimate are, respectively

$$H_{\alpha_i}\left(v_i\right)=F\left(\frac{v_i-\left(1-\alpha_i\right)\mu}{\alpha_i}\right);\quad h_{\alpha_i}\left(v_i\right)=\frac{1}{\alpha_i}f\left(\frac{v_i-\left(1-\alpha_i\right)\mu}{\alpha_i}\right).$$

Simple calculations lead to

$$\frac{\partial H_{\alpha_i}\left(v_i\right)}{\partial\alpha_i} = f\left(\frac{v_i-\left(1-\alpha_i\right)\mu}{\alpha_i}\right)\frac{\left(\mu-v_i\right)}{\alpha_i^2}, \tag{1.21}$$

$$\frac{\partial H_{\alpha_i}\left(v_i\right)}{\partial\alpha_i}\frac{1}{h_{\alpha_i}\left(v_i\right)} = -\frac{v_i-\mu}{\alpha_i}, \tag{1.22}$$

$$\frac{h_{\alpha_i}\left(v_i\right)}{1-H_{\alpha_i}\left(v_i\right)} = \frac{1}{\alpha_i}\frac{f\left(\frac{v_i-(1-\alpha_i)\mu}{\alpha_i}\right)}{1-F\left(\frac{v_i-(1-\alpha_i)\mu}{\alpha_i}\right)}. \tag{1.23}$$

Equation (1.21) shows that the family of distributions $\{H_{\alpha_i}\left(\cdot\right)\}$ is rotation-ordered with rotation point equal to μ. Equation (1.22) shows that

$$\frac{\partial H_{\alpha_i}\left(v_i\right)}{\partial\alpha_i}/h_{\alpha_i}\left(v_i\right)$$

is decreasing in v_i. Finally, $H_{\alpha_i}\left(\cdot\right)$ has an increasing hazard rate, because, by assumption, the underlying distribution $F\left(\cdot\right)$ has an increasing hazard rate. Therefore, the family of distributions $\{H_{\alpha_i}\left(\cdot\right)\}$ generated by the "truth-or-noise" technology satisfies all assumptions.

■

Proof of Theorem 1.1: Given mechanism $\{q,t\}$ and reserve price r^*, buyer i chooses α_i

to maximize his expected payoff:

$$\max_{\alpha_i} \mathbb{E}_{v_{-i},\alpha^*} \left\{ \int_{r^*}^{\overline{\omega}_{\alpha_i}} \left[1 - H_{\alpha_i}(v_i)\right] q_i(v_i, v_{-i})\, dv_i - C(\alpha_i) \right\}.$$

Therefore, the marginal value of information to buyer i is

$$MVI = -\int_{r^*}^{\overline{\omega}_{\alpha_i}} \frac{\partial H_{\alpha_i}(v_i)}{\partial \alpha_i} Q_{i,\alpha^*}(v_i)\, dv_i.$$

Therefore,

$$\frac{\partial [MVI]}{\partial r^*} = \frac{\partial H_{\alpha_i}(r^*)}{\partial \alpha_i} Q_{i,\alpha^*}(r^*). \tag{1.24}$$

Sufficiency: if different signals are rotation-ordered, the above equation shows that MVI is increasing in r^* if $r^* < \mu$ and is decreasing in r^* if $r^* > \mu$. In other words, MVI is increasing as r^* moves toward the mean valuation.

Necessity: suppose signals are not rotation-ordered, then two distributions must cross at least twice. Without loss of generality, suppose one of crossing points is lower than μ. Then we can find a $r^* < \mu$ such that

$$\frac{\partial H_{\alpha_i}(r^*)}{\partial \alpha_i} < 0.$$

Then by equation (1.24), MVI decreases as r^* moves towards μ. Therefore, the rotation order is also necessary. ■

Proof of Lemma 1.4: Let α^* denote the equilibrium information choice of bidders in the symmetric equilibrium. We prove the lemma by establishing the following two claims.

Claim 1: The seller's revenue in standard auctions with reserve price r is increasing in α^* if (1) $r \geq \mu$; or (2) Gaussian specification or truth-or-noise information technology, and n is large.

Proof: Let $V_{k,n}$ denote the k-th order statistic from n random variables independently drawn from H_{α^*}. The seller's payoff in standard auctions with reserve price r is:

$$\begin{aligned}
\pi_s(\alpha^*, r) &= r \Pr(V_{n-1,n} < r \leq V_{n,n}) + \mathbb{E}[V_{n-1,n} | V_{n-1,n} \geq r] \Pr(V_{n-1,n} \geq r) \\
&= r [H_{n-1,n}(r) - H_{n,n}(r)] + \int_r^{\overline{\omega}_{\alpha^*}} v_i h_{n-1,n}(v_i) \, dv_i \\
&= r [1 - H_{\alpha^*}(r)^n] + \int_r^{\overline{\omega}_{\alpha^*}} \left[1 - n H_{\alpha^*}(v_i)^{n-1} + (n-1) H_{\alpha^*}(v_i)^n\right] dv_i.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\frac{\partial \pi_s(\alpha^*, r)}{\partial \alpha^*} \\
= \; & -rnH_{\alpha^*}(r)^{n-1} \frac{\partial H_{\alpha^*}(r)}{\partial \alpha^*} - \int_r^{\overline{\omega}_{\alpha^*}} \left[n(n-1) H_{\alpha^*}(v_i)^{n-2} - (n-1) n H_{\alpha^*}(v_i)^{n-1}\right] \frac{\partial H_{\alpha^*}(v_i)}{\partial \alpha^*} dv_i \\
& + [1 - n + (n-1)] \frac{\partial \overline{\omega}_{\alpha^*}}{\partial \alpha^*} \\
= \; & -rnH_{\alpha^*}(r)^{n-1} \frac{\partial H_{\alpha^*}(r)}{\partial \alpha^*} - n(n-1) \int_r^{\overline{\omega}_{\alpha^*}} H_{\alpha^*}(v_i)^{n-2} [1 - H_{\alpha^*}(v_i)] \frac{\partial H_{\alpha^*}(v_i)}{\partial \alpha^*} dv_i.
\end{aligned}$$

Case 1: $r \geq \mu$. Since $r \geq \mu$ and $\frac{\partial H_{\alpha^*}(v_i)}{\partial \alpha^*} \leq 0$ for all $v_i \geq \mu$, $\frac{\partial \pi_s(\alpha^*, r)}{\partial \alpha^*} > 0$.That is, seller's revenue is increasing in α^*.

Case 2: By the analyis of case 1, we only need to prove the case where $r < \mu$. For Gaussian specification, we have

$$\frac{\partial H_{\alpha^*}(v_i)}{\partial \alpha^*} = -\frac{\beta (v_i - \mu)}{2\alpha^* (\alpha^* + \beta)} h_{\alpha^*}(v_i).$$

Therefore,

$$\begin{aligned}
&\frac{\partial \pi_s(\alpha^*, r)}{\partial \alpha^*} \\
= \;& -nrH_{\alpha^*}(r)^{n-1}\frac{\partial H_{\alpha^*}(r)}{\partial \alpha^*} - n(n-1)\int_r^{\infty} H_{\alpha^*}(v_i)^{n-2}\left[1 - H_{\alpha^*}(v_i)\right]\frac{\partial H_{\alpha^*}(v_i)}{\partial \alpha^*}dv_i \\
= \;& nrH_{\alpha^*}(r)^{n-1}\frac{\beta(r-\mu)}{2\alpha^*(\alpha^*+\beta)}h_{\alpha^*}(r) + n(n-1)\int_r^{\infty}\frac{H_{\alpha^*}(v_i)^{n-2}\left[1-H_{\alpha^*}(v_i)\right]\beta(v_i-\mu)}{2\alpha^*(\alpha^*+\beta)}h_{\alpha^*}(v_i)\,dv_i \\
= \;& \frac{n\beta}{2\alpha^*(\alpha^*+\beta)}\left\{\begin{array}{c} r(r-\mu)H_{\alpha^*}(r)^{n-1}h_{\alpha^*}(r) + \left[(1-H_{\alpha^*}(v_i))(v_i-\mu)H_{\alpha^*}(v_i)^{n-1}\right]_r^{\infty} \\ -\int_r^{\infty}(1-H_{\alpha^*}(v_i) - h_{\alpha^*}(v_i)(v_i-\mu))H_{\alpha^*}(v_i)^{n-1}dv_i \end{array}\right\} \\
= \;& \frac{n\beta}{2\alpha^*(\alpha^*+\beta)}\left\{\begin{array}{c} (r-\mu)H_{\alpha^*}(r)^{n-1}h_{\alpha^*}(r)\left(r - \frac{1-H_{\alpha^*}(r)}{h_{\alpha^*}(r)}\right) \\ +\int_r^{\infty}\left(v_i - \frac{1-H_{\alpha^*}(v_i)}{h_{\alpha^*}(v_i)}\right)H_{\alpha^*}(v_i)^{n-1}h_{\alpha^*}(v_i)\,dv_i - \frac{1}{n}\mu\left(1-H_{\alpha^*}(r)^n\right) \end{array}\right\}
\end{aligned}$$

If $r \leq r_{\alpha^*}$, then $r - \frac{1-H_{\alpha^*}(r)}{h_{\alpha^*}(r)} \leq 0$. Therefore, a sufficient condition for $\frac{\partial \pi_s(\alpha^*,r)}{\partial \alpha^*} > 0$ is

$$\int_{-\infty}^{\infty}\left(v_i - \frac{1-H_{\alpha^*}(v_i)}{h_{\alpha^*}(v_i)}\right)nH_{\alpha^*}(v_i)^{n-1}h_{\alpha^*}(v_i)\,dv_i \geq \mu.$$

This condition says that the second order statistic n independent random variables drawn from distribution $H_{\alpha^*}(\cdot)$ is higher than μ. When n is large, it holds in general.

If $r > r_{\alpha^*}$, then $r - \frac{1-H_{\alpha^*}(r)}{h_{\alpha^*}(r)} > 0$.

$$\begin{aligned}
\frac{\partial \pi_s(\alpha^*, r)}{\partial \alpha^*} &= \frac{\beta}{2\alpha^*(\alpha^*+\beta)}\left\{\begin{array}{c} -n\int_r^{\mu} H_{\alpha^*}(r)^{n-1}h_{\alpha^*}(r)\left(r - \frac{1-H_{\alpha^*}(r)}{h_{\alpha^*}(r)}\right)dv_i \\ +n\int_r^{\infty}\left(v_i - \frac{1-H_{\alpha^*}(v_i)}{h_{\alpha^*}(v_i)}\right)H_{\alpha^*}(v_i)^{n-1}h_{\alpha^*}(v_i)\,dv_i - \mu\left(1-H_{\alpha^*}(r)^n\right) \end{array}\right\} \\
&> \frac{\beta}{2\alpha^*(\alpha^*+\beta)}\left[\int_{\mu}^{\infty}\left(v_i - \frac{1-H_{\alpha^*}(v_i)}{h_{\alpha^*}(v_i)}\right)d\left[H_{\alpha^*}^n(v_i)\right] - \mu\right].
\end{aligned}$$

As n is large, the seller's revenue with n bidders and reserve price μ will be higher than μ. Therefore, $\frac{\partial \pi_s(\alpha^*,r)}{\partial \alpha^*} > 0$.

Similarly, for truth-or-noise technology, we have

$$\frac{\partial H_{\alpha^*}(v_i)}{\partial \alpha^*} = -\frac{v_i - \mu}{\alpha^*}h_{\alpha^*}(v_i).$$

Therefore,

$$\begin{aligned}
& \frac{\partial \pi_s(\alpha^*, r)}{\partial \alpha^*} \\
= & -nrH_{\alpha^*}(r)^{n-1}\frac{\partial H_{\alpha^*}(r)}{\partial \alpha^*} - n(n-1)\int_r^{\overline{\omega}_{\alpha^*}} H_{\alpha^*}(v_i)^{n-2}\left[1 - H_{\alpha^*}(v_i)\right]\frac{\partial H_{\alpha^*}(v_i)}{\partial \alpha^*}dv_i \\
= & \; nrH_{\alpha^*}(r)^{n-1}\frac{(r-\mu)}{\alpha^*}h_{\alpha^*}(r) + n(n-1)\int_r^{\overline{\omega}_{\alpha^*}} \frac{H_{\alpha^*}(v_i)^{n-2}\left[1 - H_{\alpha^*}(v_i)\right](v_i - \mu)}{\alpha^*}h_{\alpha^*}(v_i)\,dv_i \\
= & \left\{ \begin{array}{c} \frac{n}{\alpha^*}(r-\mu)H_{\alpha^*}(r)^{n-1}h_{\alpha^*}(r)\left(r - \frac{1-H_{\alpha^*}(r)}{h_{\alpha^*}(r)}\right) \\ +\frac{n}{\alpha^*}\int_r^{\overline{\omega}_{\alpha^*}}\left(v_i - \frac{1-H_{\alpha^*}(v_i)}{h_{\alpha^*}(v_i)}\right)H_{\alpha^*}(v_i)^{n-1}h_{\alpha^*}(v_i)\,dv_i - \frac{\beta}{2\alpha^*(\alpha^*+\beta)}\mu\left(1 - H_{\alpha^*}(r)^n\right) \end{array} \right\}
\end{aligned}$$

If $r \leq r_{\alpha^*}$, then $r - \frac{1-H_{\alpha^*}(r)}{h_{\alpha^*}(r)} \leq 0$. Thus, a sufficient condition for $\frac{\partial \pi_s(\alpha^*, r)}{\partial \alpha^*} > 0$ is

$$\int_{\underline{\omega}}^{\overline{\omega}}\left(v_i - \frac{1-H_{\alpha^*}(v_i)}{h_{\alpha^*}(v_i)}\right)nH_{\alpha^*}(v_i)^{n-1}h_{\alpha^*}(v_i)\,dv_i \geq \mu.$$

This condition holds as long as n is large.

If $r > r_{\alpha^*}$, then $r - \frac{1-H_{\alpha^*}(r)}{h_{\alpha^*}(r)} > 0$. Therefore,

$$\begin{aligned}
\frac{\partial \pi_s(\alpha^*, r)}{\partial \alpha^*} &= \frac{1}{\alpha^*}\left\{ \begin{array}{c} -n\int_r^{\mu} H_{\alpha^*}(r)^{n-1}h_{\alpha^*}(r)\left(r - \frac{1-H_{\alpha^*}(r)}{h_{\alpha^*}(r)}\right)dv_i \\ +n\int_r^{\overline{\omega}_{\alpha^*}}\left(v_i - \frac{1-H_{\alpha^*}(v_i)}{h_{\alpha^*}(v_i)}\right)H_{\alpha^*}(v_i)^{n-1}h_{\alpha^*}(v_i)\,dv_i - \mu\left(1 - H_{\alpha^*}(r)^n\right) \end{array} \right\} \\
&> \frac{1}{\alpha^*}\left[\int_{\mu}^{\overline{\omega}_{\alpha^*}}\left(v_i - \frac{1-H_{\alpha^*}(v_i)}{h_{\alpha^*}(v_i)}\right)d\left[H_{\alpha^*}^n(v_i)\right] - \mu\right]
\end{aligned}$$

Again, as n is large, the seller's revenue with n bidders and reserve price μ is higher than μ. So $\frac{\partial \pi_s(\alpha^*, r)}{\partial \alpha^*} > 0$.

Claim 2: If the seller's revenue is increasing in α^* in standard auctions with reserve price r, then $\lambda > 0$.

Proof: Recall the seller's maximization problem is

$$\max_{q_i, u(\underline{\omega}), \alpha^*} \left\{ \mathbb{E}_{v,\alpha^*} \sum_{i=1}^{n} \left[\left(v_i - \frac{1 - H_{\alpha^*}(v_i)}{h_{\alpha^*}(v_i)} \right) q_i(v_i, v_{-i}) \right] - nu(\underline{\omega}) \right\}$$

$$s.t. \quad : \quad 0 \leq q_i(v_i, v_{-i}) \leq 1; \sum_{i=1}^{n} q_i(v_i, v_{-i}) \leq 1, \qquad \text{(Regularity)}$$

$$: \quad Q_i(v_i) \text{ is nondecreasing in } v_i, \qquad \text{(Monotonicity)}$$

$$: \quad u(\underline{\omega}) \geq 0, \qquad \text{(IR)}$$

$$: \quad -\mathbb{E}_{v,\alpha^*} \left[\frac{\partial H_{\alpha^*}(v_i)}{\partial \alpha^*} \frac{1}{h_{\alpha^*}(v_i)} q_i(v_i, v_{-i}) \right] - C'(\alpha^*) = 0. \qquad \text{(IA)}$$

Note that the expectation term is independent of $u(\underline{\omega})$, and $u(\underline{\omega})$ is nonnegative, so the seller must set $u(\underline{\omega}) = 0$ to maximize revenue. Ignore the regularity constraint and monotonicity constraint for the moment.

We adopt the same strategy of Rogerson (1985) by weakening the equality (IA) constraint to the following inequality constraint.

$$-\mathbb{E}_{v,\alpha^*} \left[\frac{\partial H_{\alpha^*}(v_i)}{\partial \alpha^*} \frac{1}{h_{\alpha^*}(v_i)} q_i(v_i, v_{-i}) \right] - C'(\alpha^*) \geq 0.$$

With the inequality constraint, the corresponding Lagrangian multiplier δ is always nonnegative. If we can show that $\delta > 0$ at the optimal solution of the relaxed program, then the constraint is binding in equilibrium. Then, the optimal solution of relaxed program is also an optimal solution of the original program, and $\lambda > 0$.

Write the Lagrangian for the relaxed program as

$$\begin{aligned} L &= \mathbb{E}_{v,\alpha^*} \sum_{i=1}^{n} \left[\left(v_i - \frac{1 - H_{\alpha^*}(v_i)}{h_{\alpha^*}(v_i)} \right) q_i(v_i, v_{-i}) \right] + \delta \left[-\mathbb{E}_v \left[\frac{\partial H_{\alpha^*}(v_i)}{\partial \alpha^*} \frac{1}{h_{\alpha^*}(v_i)} q_i(v_i, v_{-i}) \right] - C'(\alpha^*) \right] \\ &= \mathbb{E}_{v,\alpha^*} \sum_{i=1}^{n} \left[\left(v_i - \frac{1 - H_{\alpha^*}(v_i)}{h_{\alpha^*}(v_i)} - \frac{\delta}{n} \frac{\partial H_{\alpha^*}(v_i)}{\partial \alpha^*} \frac{1}{h_{\alpha^*}(v_i)} \right) q_i(v_i, v_{-i}) \right] - \delta C'(\alpha^*) \end{aligned}$$

The necessary first order condition is

$$0=\frac{\partial L}{\partial\alpha^*}=\left\{\begin{array}{c}\frac{\partial\left\{\mathbb{E}_{v,\alpha^*}\sum_{i=1}^n\left[\left(v_i-\frac{1-H_{\alpha^*}(v_i)}{h_{\alpha^*}(v_i)}\right)q_i(v_i,v_{-i})\right]\right\}}{\partial\alpha^*}\\+\delta\frac{\partial\left[-\mathbb{E}_{v,\alpha^*}\left[\frac{\partial H_{\alpha^*}(v_i)}{\partial\alpha^*}\frac{1}{h_{\alpha^*}(v_i)}q_i(v_i,v_{-i})\right]-C'(\alpha^*)\right]}{\partial\alpha^*}\end{array}\right\}. \tag{1.25}$$

Since $\delta\geq 0$, Theorem 1.3 shows that a second price auction is optimal. Therefore, we can restrict attention to second price auctions.

The first term in the big bracket of (1.25) is positive by the assumption of Claim 2. In order to show $\delta>0$, we need to show that the second term is negative. Note that in a second price auction with reserve price r

$$-\mathbb{E}_{v,\alpha^*}\left[\frac{\partial H_{\alpha^*}(v_i)}{\partial\alpha^*}\frac{1}{h_{\alpha^*}(v_i)}q_i(v_i,v_{-i})\right]-C'(\alpha^*)=\int_r^{\overline{\omega}_{\alpha^*}}(1-H_{\alpha^*}(v_i))\,H_{\alpha^*}(v_i)^{n-1}\,dv_i-C'(\alpha^*)\,.$$

Thus,

$$\begin{aligned}&\frac{\partial\left[-\mathbb{E}_{v,\alpha^*}\left[\frac{\partial H_{\alpha^*}(v_i)}{\partial\alpha^*}\frac{1}{h_{\alpha^*}(v_i)}q_i(v_i,v_{-i})\right]-C'(\alpha^*)\right]}{\partial\alpha^*}\\=\;&\underbrace{-\int_r^{\overline{\omega}_{\alpha^*}}\frac{\partial^2H_{\alpha^*}(v_i)}{\partial\alpha^{*2}}H_{\alpha^*}(v_i)^{n-1}\,dv_i+\frac{\partial^2H_{\alpha^*}(\overline{\omega}_{\alpha^*})}{\partial\alpha^{*2}}\frac{\partial\overline{\omega}_{\alpha^*}}{\partial\alpha^*}-C''(\alpha^*)}_{A}\\&\underbrace{-\int_r^{\overline{\omega}_{\alpha^*}}\left(\frac{\partial H_{\alpha^*}(v_i)}{\partial\alpha^*}\right)^2(n-1)\,H_{\alpha^*}(v_i)^{n-2}\,dv_i}_{B}\end{aligned}$$

Since α^* maximize a bidder's expected payoff, the local second order condition of the bidder's maximization problem holds. As a result, term A is negative. Since term B is also negative, the partial derivative is negative.

By condition (1.25), it immediately follows $\delta>0$ at the optimal solution (α^*,q^*). The relaxed program is the same as the original program, and the maximum of the relaxed program can be achieved by the original program. So $\lambda=\delta>0$ if the seller's revenue is increasing in α_i in the optimal auctions.

Note that a sufficient condition for $r_{\alpha^*}>\mu$ is $r_{\underline{\alpha}}>\mu$. To see this, by definition of $r_{\underline{\alpha}}$

and Assumption 1.3, $r_{\underline{\alpha}} > \mu$ implies

$$\mu - \frac{1 - H_{\underline{\alpha}}(\mu)}{h_{\underline{\alpha}}(\mu)} < 0.$$

By Assumption 1.1,

$$H_{\underline{\alpha}}(\mu) = H_{\alpha_i}(\mu) \text{ and } h_{\underline{\alpha}}(\mu) \geq h_{\alpha_i}(\mu) \quad \text{for all } \alpha_i \geq \underline{\alpha}$$

It follows that,

$$\mu - \frac{1 - H_{\alpha_i}(\mu)}{h_{\alpha_i}(\mu)} \leq \mu - \frac{1 - H_{\underline{\alpha}}(\mu)}{h_{\underline{\alpha}}(\mu)} < 0 \quad \text{for all } \alpha_i \geq \underline{\alpha}$$

Thus, $r_{\alpha_i} > \mu$. Therefore, a sufficient condition for $r_{\alpha^*} > \mu$ is

$$\mu < \frac{1 - H_{\underline{\alpha}}(\mu)}{h_{\underline{\alpha}}(\mu)}.$$

Finally, from Theorem 1.2, for $\lambda > 0$, $r_{\alpha^*} > \mu \Leftrightarrow r^* \geq \mu$.

The Lemma now follows from the results of Claim 1 and Claim 2. ■

Proof of Theorem 1.2: Recall the virtual surplus function is

$$J^*(v_i) = v_i - \frac{1 - H_{\alpha^*}(v_i)}{h_{\alpha^*}(v_i)} - \frac{\lambda}{n}\frac{\partial H_{\alpha^*}(v_i)}{\partial \alpha^*}\frac{1}{h_{\alpha^*}(v_i)}.$$

The optimal reserve price r^* has to satisfy

$$q_i(v_i, v_{-i}) > 0 \Rightarrow v_i \geq r^*,$$

and

$$r^* \leq \min\{r : J^*(v_i) \geq 0 \text{ for all } v_i \geq r\}. \tag{1.26}$$

The last condition says that the seller will sell the object as long as the marginal revenue is nonnegative.

Case 1: $\lambda > 0$ and $r_{\alpha^*} > \mu$. First we show $r^* < r_{\alpha^*}$. By definition of r_{α^*},

$$r_{\alpha^*} - \frac{1 - H_{\alpha^*}(v_i)}{h_{\alpha^*}(v_i)} = 0.$$

Then for all $v_i \geq r_{\alpha^*} > \mu$,

$$J^*(v_i) = v_i - \frac{1 - H_{\alpha^*}(v_i)}{h_{\alpha^*}(v_i)} - \frac{\lambda}{n}\frac{\partial H_{\alpha^*}(v_i)}{\partial \alpha^*}\frac{1}{h_{\alpha^*}(v_i)} = -\frac{\lambda}{n}\frac{\partial H_{\alpha^*}(v_i)}{\partial \alpha^*}\frac{1}{h_{\alpha^*}(v_i)} > 0.$$

The last inequality follows from the fact that $\{H_{\alpha^*}\}$ is rotation-ordered. Therefore, there exists $\varepsilon > 0$, such that

$$J^*(r_{\alpha^*} - \varepsilon) \geq 0.$$

Therefore, by (1.26), the optimal reserve price $r^* < r_{\alpha^*}$.

Next, we show $r^* \geq \mu$. Suppose $r^* < \mu$ by contradiction. Then

$$J^*(r^*) = r^* - \frac{1 - H_{\alpha^*}(r^*)}{h_{\alpha^*}(r^*)} - \frac{\lambda}{n}\frac{\partial H_{\alpha^*}(r^*)}{\partial \alpha^*}\frac{1}{h_{\alpha^*}(r^*)} < -\frac{\lambda}{n}\frac{\partial H_{\alpha^*}(r^*)}{\partial \alpha^*}\frac{1}{h_{\alpha^*}(r^*)} < 0.$$

The first inequality follows because $r^* < r_{\alpha^*}$, and the second inequality follows from the rotation order. This contradicts the fact the $J^*(r^*) \geq 0$. Thus, we have shown $\mu \leq r^* < r_{\alpha^*}$

Case 2: $\lambda > 0$ and $r_{\alpha^*} = \mu$. Then for all $v_i > \mu$,

$$J^*(v_i) = v_i - \frac{1 - H_{\alpha^*}(v_i)}{h_{\alpha^*}(v_i)} - \frac{\lambda}{n}\frac{\partial H_{\alpha^*}(v_i)}{\partial \alpha^*}\frac{1}{h_{\alpha^*}(v_i)} \geq -\frac{\lambda}{n}\frac{\partial H_{\alpha^*}(v_i)}{\partial \alpha^*}\frac{1}{h_{\alpha^*}(v_i)} > 0.$$

Therefore, r^* cannot be higher than μ. On the other hand, for all $v_i < \mu$

$$J^*(v_i) = v_i - \frac{1 - H_{\alpha^*}(v_i)}{h_{\alpha^*}(v_i)} - \frac{\lambda}{n}\frac{\partial H_{\alpha^*}(v_i)}{\partial \alpha^*}\frac{1}{h_{\alpha^*}(v_i)} \leq -\frac{\lambda}{n}\frac{\partial H_{\alpha^*}(v_i)}{\partial \alpha^*}\frac{1}{h_{\alpha^*}(v_i)} < 0.$$

Therefore, r^* cannot be lower than μ. Therefore, $r^* = r_{\alpha^*} = \mu$.

Case 3: $\lambda > 0$ and $r_{\alpha^*} < \mu$. Note that for all $v_i < r_{\alpha^*}$,

$$J^*(v_i) = v_i - \frac{1 - H_{\alpha^*}(v_i)}{h_{\alpha^*}(v_i)} - \frac{\lambda}{n}\frac{\partial H_{\alpha^*}(v_i)}{\partial \alpha^*}\frac{1}{h_{\alpha^*}(v_i)} \leq -\frac{\lambda}{n}\frac{\partial H_{\alpha^*}(v_i)}{\partial \alpha^*}\frac{1}{h_{\alpha^*}(v_i)} < 0.$$

Therefore, $r^* > r_{\alpha^*}$. Furthermore, for all $v_i > \mu$,

$$J^*(v_i) = v_i - \frac{1 - H_{\alpha^*}(v_i)}{h_{\alpha^*}(v_i)} - \frac{\lambda}{n}\frac{\partial H_{\alpha^*}(v_i)}{\partial \alpha^*}\frac{1}{h_{\alpha^*}(v_i)} \geq -\frac{\lambda}{n}\frac{\partial H_{\alpha^*}(v_i)}{\partial \alpha^*}\frac{1}{h_{\alpha^*}(v_i)} > 0.$$

Thus, $r^* \leq \mu$. As a result, $r_{\alpha^*} < r \leq \mu$.

Case 4: $\lambda < 0$ and $r_{\alpha^*} < \mu$. Note that for all $v_i \in [r_{\alpha^*}, \mu]$

$$J^*(v_i) = v_i - \frac{1 - H_{\alpha^*}(v_i)}{h_{\alpha^*}(v_i)} - \frac{\lambda}{n}\frac{\partial H_{\alpha^*}(v_i)}{\partial \alpha^*}\frac{1}{h_{\alpha^*}(v_i)} \geq -\frac{\lambda}{n}\frac{\partial H_{\alpha^*}(v_i)}{\partial \alpha^*}\frac{1}{h_{\alpha^*}(v_i)} > 0.$$

In addition, r^* cannot be higher than μ, otherwise $\lambda > 0$. Therefore, $r^* < r_{\alpha^*} < \mu$.

Since $r_{\alpha^*} \geq \mu$ implies $\lambda \geq 0$, the above four cases include all possible cases, and our proof is complete.■

Proof of Theorem 1.3: Under Assumption 1.2 and 1.3,

$$J^*(v_i) = v_i - \frac{1 - H_{\alpha^*}(v_i)}{h_{\alpha^*}(v_i)} - \frac{\lambda}{n}\frac{\partial H_{\alpha^*}(v_i)}{\partial \alpha^*}\frac{1}{h_{\alpha^*}(v_i)}$$

is increasing in v_i. In this case, we can define the reserve price as

$$r^* = \inf\{r : J^*(r) \geq 0\}.$$

Therefore the optimal auctions will assign the object to the bidder with highest posterior estimate provided his estimate is higher than r^*. So standard auctions with reserve price r^* are optimal.■

1.7 Appendix B

1.7.1 Sufficient Conditions for Validity of the First Order Approach

Here we will provide several sets of sufficient conditions to ensure the validity of the first order approach. Recall that bidder i chooses α_i to maximize his payoff given other bidders choose $\alpha_j\,(j\neq i)$. Bidder i's payoff under mechanism $\{q,t\}$ is,

$$\pi_b(\alpha_i)=\mathbb{E}_{v_{-i}}\left\{\int_{\underline{\omega}_{\alpha_i}}^{\overline{\omega}_{\alpha_i}}\left[1-H_{\alpha_i}(v_i)\right]q_i(v_i,v_{-i})\,dv_i-C(\alpha_i)\right\}.$$

The first partial derivative is

$$\frac{\partial\pi_b}{\partial\alpha_i}=\mathbb{E}_{v_{-i}}\left\{-q_i\left(\underline{\omega}_{\alpha_i},v_{-i}\right)\frac{\partial\underline{\omega}_{\alpha_i}}{\partial\alpha_i}-\int_{\underline{\omega}_{\alpha_i}}^{\overline{\omega}_{\alpha_i}}\frac{\partial H_{\alpha_i}(v_i)}{\partial\alpha_i}q_i(v_i,v_{-i})\,dv_i\right\}-C'(\alpha_i),$$

and the second partial derivative is

$$\frac{\partial^2\pi_b}{\partial\alpha_i^2}=-\mathbb{E}_{v_{-i}}\left\{\begin{array}{c}\frac{\partial q_i\left(\underline{\omega}_{\alpha_i},v_{-i}\right)}{\partial v_i}\left(\frac{\partial\underline{\omega}_{\alpha_i}}{\partial\alpha_i}\right)^2+q_i\left(\underline{\omega}_{\alpha_i},v_{-i}\right)\frac{\partial^2\underline{\omega}_{\alpha_i}}{\partial\alpha_i^2}-\frac{\partial H_{\alpha_i}\left(\underline{\omega}_{\alpha_i}\right)}{\partial\alpha_i}q_i\left(\underline{\omega}_{\alpha_i},v_{-i}\right)\frac{\partial\underline{\omega}_{\alpha_i}}{\partial\alpha_i}\\+\frac{\partial H_{\alpha_i}\left(\overline{\omega}_{\alpha_i}\right)}{\partial\alpha_i}q_i\left(\overline{\omega}_{\alpha_i},v_{-i}\right)\frac{\partial\overline{\omega}_{\alpha_i}}{\partial\alpha_i}+\int_{\underline{\omega}_{\alpha_i}}^{\overline{\omega}_{\alpha_i}}\frac{\partial^2H_{\alpha_i}(v_i)}{\partial\alpha_i^2}q_i(v_i,v_{-i})\,dv_i\end{array}\right\}-C''(\alpha_i).$$

The first order approach is valid if

$$\frac{\partial^2\pi_b}{\partial\alpha_i^2}<0. \tag{1.27}$$

It is easy to see that the above condition holds if the cost function is sufficient convex.[26]

If the support of the posterior estimate is independent of information choice α_i, all terms except the last one in the expectation is zero. Therefore, if the last term in the expectation is positive, together with the convex cost function, the first order approach is valid. A sufficient condition for the last term to be nonnegative is

$$\frac{\partial^2H_{\alpha_i}(v_i)}{\partial\alpha_i^2}\geq 0\text{ for all }v_i. \tag{1.28}$$

[26]Persico (2000) makes such a assumption in his example of information acquisition.

Condition (1.28) says the distribution of the posterior estimate is convex in the bidder's information choice. This condition is analogous to the CDFC (convexity of the distribution function condition) in the principal-agent literature, which requires that the distribution function of output be convex in the action the agent takes (Mirrlees 1999, Rogerson 1985).[27]

For a general information structure, it is difficult to verify whether condition (1.28) is satisfied. For specific information technologies, however, we are able to provide sufficient conditions to guarantee the validity of the first order approach.

Proposition 1.7 *For the truth-or-noise technology, if* $C''(\alpha_i)\alpha_i \geq f(\overline{\omega})(\overline{\omega}-\mu)^2$ *for all* α_i*, the second order condition is satisfied either (1)* $F(x)$ *is convex, or (2)* $F(x) = x^b\,(b>0)$ *with support* $[0,1]$*. For the Gaussian specification, the second order condition is satisfied if, for all* α_i*,*

$$\sqrt{\frac{\beta^3}{\alpha_i^3(\alpha_i+\beta)^5}} < 2\sqrt{2\pi}C''(\alpha_i).$$

Proof: For the truth-or-noise technology,

$$\begin{aligned}
\frac{\partial^2\pi_b}{\partial\alpha_i^2} &= -\mathbb{E}_{v_{-i}}\left\{\begin{array}{c} \frac{\partial q_i\left(\underline{\omega}_{\alpha_i},v_{-i}\right)}{\partial v_i}\left(\frac{\partial\underline{\omega}_{\alpha_i}}{\partial\alpha_i}\right)^2 + q_i\left(\underline{\omega}_{\alpha_i},v_{-i}\right)\frac{\partial^2\underline{\omega}_{\alpha_i}}{\partial\alpha_i^2} - \frac{\partial H_{\alpha_i}\left(\underline{\omega}_{\alpha_i}\right)}{\partial\alpha_i}q_i\left(\underline{\omega}_{\alpha_i},v_{-i}\right)\frac{\partial\underline{\omega}_{\alpha_i}}{\partial\alpha_i} \\ +\frac{\partial H_{\alpha_i}\left(\overline{\omega}_{\alpha_i}\right)}{\partial\alpha_i}q_i\left(\overline{\omega}_{\alpha_i},v_{-i}\right)\frac{\partial\overline{\omega}_{\alpha_i}}{\partial\alpha_i} + \int_{\underline{\omega}_{\alpha_i}}^{\overline{\omega}_{\alpha_i}}\frac{\partial^2 H_{\alpha_i}(v_i)}{\partial\alpha_i^2}q_i\left(v_i,v_{-i}\right)dv_i \end{array}\right\} - C''(\alpha_i) \\
&< \left\{\begin{array}{c} -\frac{\partial H_{\alpha_i}\left(\overline{\omega}_{\alpha_i}\right)}{\partial\alpha_i}Q_i\left(\overline{\omega}_{\alpha_i}\right)\frac{\partial\overline{\omega}_{\alpha_i}}{\partial\alpha_i} + \frac{\partial H_{\alpha_i}\left(\underline{\omega}_{\alpha_i}\right)}{\partial\alpha_i}Q_i\left(\underline{\omega}_{\alpha_i}\right)\frac{\partial\underline{\omega}_{\alpha_i}}{\partial\alpha_i} \\ -C''(\alpha_i) \end{array}\right\} - \int_{\underline{\omega}_{\alpha_i}}^{\overline{\omega}_{\alpha_i}}\frac{\partial^2 H_{\alpha_i}(v_i)}{\partial\alpha_i^2}Q_i\left(v_i,v_{-i}\right)dv_i \\
&= \left\{\begin{array}{c} f(\overline{\omega})\frac{(\overline{\omega}-\mu)^2}{\alpha_i}Q_i\left(\overline{\omega}_{\alpha_i}\right) - f(\underline{\omega})\frac{(\mu-\underline{\omega})^2}{\alpha_i}Q_i\left(\underline{\omega}_{\alpha_i}\right) \\ -C''(\alpha_i) \end{array}\right\} - \int_{\underline{\omega}_{\alpha_i}}^{\overline{\omega}_{\alpha_i}}\frac{\partial^2 H_{\alpha_i}(v_i)}{\partial\alpha_i^2}Q_i\left(v_i,v_{-i}\right)dv_i \\
&\leq f(\overline{\omega})\frac{(\overline{\omega}-\mu)^2}{\alpha_i} - C''(\alpha_i) - \int_{\underline{\omega}_{\alpha_i}}^{\overline{\omega}_{\alpha_i}}\frac{\partial^2 H_{\alpha_i}(v_i)}{\partial\alpha_i^2}Q_i\left(v_i,v_{-i}\right)dv_i
\end{aligned}$$

[27] See also Jewitt (1988).

If $C''(\alpha_i)\alpha_i \geq f(\overline{\omega})(\overline{\omega}-\mu)^2$, then

$$
\begin{aligned}
\frac{\partial^2 \pi_b}{\partial \alpha_i^2} &< -\int_{\underline{\omega}_{\alpha_i}}^{\overline{\omega}_{\alpha_i}} \frac{\partial^2 H_{\alpha_i}(v_i)}{\partial \alpha_i^2} Q_i(v_i)\, dv_i \\
&= -\int_{\underline{\omega}_{\alpha_i}}^{\overline{\omega}_{\alpha_i}} \left\{ f'\left(\frac{v_i-(1-\alpha_i)\mu}{\alpha_i}\right)\frac{(\mu-v_i)^2}{\alpha_i^4} - f\left(\frac{v_i-(1-\alpha_i)\mu}{\alpha_i}\right)\frac{2(\mu-v_i)}{\alpha_i^3}\right\} Q_i(v_i)\, dv_i \\
&= -\mathbb{E}_{v_{-i}} \int_{\underline{\omega}}^{\overline{\omega}} \left\{ f'(s_i)\frac{(s_i-\mu)^2}{\alpha_i} + f(s_i)\frac{2(s_i-\mu)}{\alpha_i}\right\} Q_i(\alpha_i s_i + (1-\alpha_i)\mu)\, ds_i
\end{aligned}
$$

If $F(\cdot)$ is convex, then

$$
\begin{aligned}
\frac{\partial^2 \pi_b}{\partial \alpha_i^2} &< -\frac{2}{\alpha_i}\int_{\underline{\omega}}^{\overline{\omega}} (s_i-\mu) f(s_i) Q_i[\alpha_i s_i + (1-\alpha_i)\mu]\, ds_i \\
&< -\frac{2}{\alpha_i}\int_{\underline{\omega}}^{\mu} (s_i-\mu) f(s_i) Q_i(\mu)\, ds_i - \frac{2}{\alpha_i}\int_{\mu}^{\overline{\omega}} (s_i-\mu) f(s_i) Q_i(\mu)\, ds_i \\
&= -\frac{2}{\alpha_i} Q_i(\mu) \int_{\underline{\omega}}^{\overline{\omega}} (s_i-\mu) f(s_i)\, ds_i \\
&= 0.
\end{aligned}
$$

But the convexity of $F(\cdot)$ is not necessary. Suppose $F(x) = x^b$ $(0 < b \leq 1)$ with support $[0,1]$, then

$$
\begin{aligned}
\frac{\partial^2 \pi_b}{\partial \alpha_i^2} &< -\int_{\underline{\omega}}^{\overline{\omega}} \left\{ f'(s_i)\frac{(s_i-\mu)^2}{\alpha_i} + f(s_i)\frac{2(s_i-\mu)}{\alpha_i}\right\} Q_i(\alpha_i s_i + (1-\alpha_i)\mu)\, ds_i \\
&= -\frac{1}{\alpha_i}\int_0^1 \left\{ b(b-1)s^{b-2}(s-\mu)^2 + 2bs^{b-1}(s-\mu)\right\} Q_i(\alpha_i s + (1-\alpha_i)\mu)\, ds \\
&= -\frac{1}{\alpha_i}\int_0^1 [(b+1)s + (1-b)\mu]\, bs^{b-2}(s-\mu) Q_i(\alpha_i s + (1-\alpha_i)\mu)\, ds \\
&< -\frac{1}{\alpha_i} Q_i(\mu) \int_0^1 ((b+1)s + (1-b)\mu)\, bs^{b-2}(s-\mu)\, ds \\
&= -\frac{1}{\alpha_i} Q_i(\mu)(b+1)\int_0^1 bs^{b-1}(s-\mu)\, ds - \frac{1}{\alpha_i} Q_i(\mu)(1-b)\mu b \int_0^1 s^{b-2}(s-\mu)\, ds \\
&= -\frac{1}{\alpha_i} Q_i(\mu) \frac{b}{(1+b)^2} \\
&< 0.
\end{aligned}
$$

For the Gaussian specification, the second derivative is

$$\frac{\partial^2 \pi_b}{\partial \alpha_i^2} = -\mathbb{E}_{v_{-i}} \left\{ \int_{-\infty}^{\infty} \frac{\partial^2 H_{\alpha_i}(v_i)}{\partial \alpha_i^2} q_i(v_i, v_{-i})\, dv_i \right\} - C''(\alpha_i).$$

With some algebra, we can obtain

$$\frac{\partial^2 H_{\alpha_i}(v_i)}{\partial \alpha_i^2} = \frac{4\alpha_i + 3\beta}{2\alpha_i(\alpha_i+\beta)} \frac{(v_i - \mu)}{2\sqrt{2\pi}} \sqrt{\frac{\beta^3}{\alpha_i^3(\alpha_i+\beta)}} \exp\left(-\frac{(v_i-\mu)^2}{2\sigma^2}\right)\left(1 - \frac{\alpha_i+\beta}{4\alpha_i+3\beta}\frac{\beta^2}{\alpha_i}(v_i-\mu)^2\right).$$

So we can write the second derivative as

$$\begin{aligned}
\frac{\partial^2 \pi_b}{\partial \alpha_i^2} &= \begin{pmatrix} -\mathbb{E}_{v_{-i}} \left\{ \int_{-\infty}^{\infty} \frac{4\alpha_i+3\beta}{2\alpha_i(\alpha_i+\beta)} \frac{(v_i-\mu)}{2\sqrt{2\pi}} \sqrt{\frac{\beta^3}{\alpha_i(\alpha_i+\beta)}} \exp\left(-\frac{(v_i-\mu)^2}{2\sigma^2}\right) q_i(v_i, v_{-i})\, dv_i \right\} \\ +\mathbb{E}_{v_{-i}} \left\{ \int_{-\infty}^{\infty} \frac{\beta^2}{4\alpha_i^2} \frac{(v_i-\mu)^3}{\sqrt{2\pi}} \sqrt{\frac{\beta^3}{\alpha_i^3(\alpha_i+\beta)}} \exp\left(-\frac{(v_i-\mu)^2}{2\sigma^2}\right) q_i(v_i, v_{-i})\, dv_i \right\} \end{pmatrix} - C''(\alpha_i) \\
&= \begin{pmatrix} -\mathbb{E}_{v_{-i}} \left\{ \frac{4\alpha_i+3\beta}{2\alpha_i(\alpha_i+\beta)} \int_{-\infty}^{\infty} \left(-\frac{\partial H_{\alpha_i}(v_i)}{\partial \alpha_i}\right) q_i(v_i, v_{-i})\, dv_i \right\} \\ +\mathbb{E}_{v_{-i}} \left\{ \int_{-\infty}^{\infty} \frac{\beta^2}{4\alpha_i^2} \frac{(v_i-\mu)^3}{\sqrt{2\pi}\sigma} \frac{\beta}{\alpha_i(\alpha_i+\beta)} \exp\left(-\frac{(v_i-\mu)^2}{2\sigma^2}\right) q_i(v_i, v_{-i})\, dv_i \right\} \end{pmatrix} - C''(\alpha_i).
\end{aligned}$$

By Proposition 1.5, bidders always prefer higher α_i, which implies

$$\mathbb{E}_{v_{-i}} \left\{ \int_{-\infty}^{\infty} \left(-\frac{\partial H_{\alpha_i}(v_i)}{\partial \alpha_i}\right) q_i(v_i, v_{-i})\, dv_i \right\} > 0.$$

Thus, a sufficient condition for the second order condition is

$$\begin{aligned}
\mathbb{E}_{v_{-i}} \left\{ \int_{-\infty}^{\infty} \frac{\beta^2}{4\alpha_i^2} \frac{\beta}{\alpha_i(\alpha_i+\beta)} \frac{(v_i-\mu)^3}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(v_i-\mu)^2}{2\sigma^2}\right) q_i(v_i, v_{-i})\, dv_i \right\} &< C''(\alpha_i) \Leftrightarrow \\
\frac{\beta^3}{4\alpha_i^3(\alpha_i+\beta)} \int_{-\infty}^{\infty} \frac{(v_i-\mu)^3}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(v_i-\mu)^2}{2\sigma^2}\right) Q_i(v_i)\, dv_i &< C''(\alpha_i).
\end{aligned}$$

A sufficient condition for the above inequality is,

$$\begin{aligned}
\frac{\beta^3}{4\alpha_i^3(\alpha_i+\beta)} \int_{\mu}^{\infty} \frac{(v_i-\mu)^3}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(v_i-\mu)^2}{2\sigma^2}\right) dv_i &< C''(\alpha_i) \Leftrightarrow \\
\frac{1}{2\sqrt{2\pi}} \sqrt{\frac{\beta^3}{\alpha_i^3(\alpha_i+\beta)^5}} &< C''(\alpha_i).
\end{aligned}$$

Note that if β/α_i is small, the above sufficient condition is easy to be satisfied. Therefore, if $\underline{\alpha}/\beta$ is sufficiently large, the second order condition is satisfied. ■

For the truth-or-noise technology, the condition, $C''(\alpha_i)\,\alpha_i \geq f(\overline{\omega})\,(\overline{\omega}-\mu)^2$, is to ensure that the relative gain from information acquisition is not too high so that bidders will not pursue extreme information choice $\overline{\alpha}$. The convex distribution means that if bidders acquire information they have better chance to get extreme values; the information acquisition is productive. But the convexity of F is not necessary. For example, $F(x) = x^b$ may not be convex but the second order condition is still satisfied.

For the Gaussian specification, if β is small and $\underline{\alpha}$ is large relative to β, then the second order condition is satisfied. This is quite intuitive. Small β implies the prior distribution is quite spread out, so the potential gain from information acquisition is high. If $\underline{\alpha}$ is large relative to β, then signal will be informative, which again implies information acquisition is profitable.

1.7.2 Discrete Information Acquisition

If the investment in information is lumpy, information acquisition may be discrete. A study on discrete information acquisition can help check the robustness of our results and partially relieve the concern about the first order approach. In order to be comparable to the existing literature, information acquisition is assumed to be binary. If a buyer acquires information, he observes his true valuation; otherwise, he does not know his type. We will refer to bidders acquiring information as informed bidders, and bidders not acquiring information as uninformed bidders. Since bidders are risk neutral, the expected valuation of uninformed bidders is the mean valuation μ. That is

$$v_i = \begin{cases} \omega_i & \text{if} \quad \text{bidder } i \text{ becomes informed} \\ \mu & \text{if} \quad \text{bidder } i \text{ stays uninformed} \end{cases}.$$

We focus on the symmetric equilibrium and the direct revelation mechanisms $\{q_i(v), t_i(v)\}_{i=1}^n$.

The timing is the same as before. The seller first announces the mechanism. Each bidder chooses to become informed with probability $p \in [0,1]$. The seller compares revenue under different p's, and chooses $\{q_i(v), t_i(v)\}$ to implement the optimal p.

If information cost c is very low, then the optimal $p = 1$, and the standard Myerson auction is optimal. However, if c is very high, then the optimal $p = 0$, and the posted price $r = \mu$ is optimal. Thus, there exists a cost region $c \in [\underline{c}, \overline{c}]$ such that bidders play mixed strategies with $p \in (0,1)$.[28] We will focus on this case.

If the seller chooses to implement p, the resulting distribution of the bidders' valuation is

$$H(v_i) = pF(v_i) + (1-p) \cdot 1_{\{v_i \geq \mu\}}.$$

Because a fraction of $(1-p)$ bidders choose to stay uninformed with μ as their expected valuation, there is a mass point at the mean valuation.

Since $\{q_i(v), t_i(v)\}$ is incentive compatible, we have

$$u(v_i) = u(\underline{\omega}) + \int_{\underline{\omega}}^{v_i} Q_i(s)\, ds.$$

[28] With some algebra, one can actually identify the cost thresholds $\underline{c}$ and $\overline{c}$.

The expected payment for bidder i is

$$\begin{aligned}
& \mathbb{E}_{v_i}\left[T_i\left(v_i\right)\right] \\
= \; & \mathbb{E}_{v_i}\left[v_i Q_i\left(v_i\right)-u\left(\underline{\omega}\right)-\int_{\underline{\omega}}^{v_i} Q_i\left(s\right) ds\right] \\
= \; & p\int_{\underline{\omega}}^{\overline{\omega}}\left[v_i Q_i\left(v_i\right)-\int_{\underline{\omega}}^{v_i} Q_i\left(s\right) ds\right] f\left(v_i\right) dv_i+\left(1-p\right)\left[\mu Q_i\left(\mu\right)-\int_{\underline{\omega}}^{\mu} Q_i\left(s\right) ds\right]-u\left(\underline{\omega}\right) \\
= \; & p\int_{\underline{\omega}}^{\overline{\omega}}\left[v_i-\frac{1-F\left(v_i\right)}{f\left(v_i\right)}\right] Q_i\left(v_i\right) f\left(v_i\right) dv_i+\left(1-p\right)\left[\mu Q_i\left(\mu\right)-\int_{\underline{\omega}}^{\mu} Q_i\left(s\right) ds\right]-u\left(\underline{\omega}\right) \\
= \; & \int_{\underline{\omega}}^{\overline{\omega}}\left[pv_i-p\frac{1-F\left(v_i\right)}{f\left(v_i\right)}-\left(1-p\right)\frac{1_{\{v_i\leq\mu\}}}{f\left(v_i\right)}\right] Q_i\left(v_i\right) f\left(v_i\right) dv_i+\left(1-p\right)\mu Q_i\left(\mu\right)-u\left(\underline{\omega}\right) \\
= \; & \int_{\underline{\omega}}^{\mu}\left[v_i-\frac{1-pF\left(v_i\right)}{pf\left(v_i\right)}\right] Q_i\left(v_i\right) g\left(v_i\right) dv_i+\int_{\mu}^{\overline{\omega}}\left[v_i-\frac{1-F\left(v_i\right)}{f\left(v_i\right)}\right] Q_i\left(v_i\right) g\left(v_i\right) dv_i \\
& +\left(1-p\right)\mu Q_i\left(\mu\right)-u\left(\underline{\omega}\right) \\
= \; & \mathbb{E}_{\theta}\left[J\left(v_i\right) q_i\left(v_i, v_{-i}\right)\right]-u\left(\underline{\omega}\right),
\end{aligned}$$

where

$$J\left(v_i\right)=\begin{cases} v_i-\frac{1-pF\left(v_i\right)}{pf\left(v_i\right)} & \text{if} \quad v_i<\mu \\ \mu & \text{if} \quad v_i=\mu \\ v_i-\frac{1-F\left(v_i\right)}{f\left(v_i\right)} & \text{if} \quad v_i>\mu \end{cases}.$$

Therefore, the seller's revenue is

$$\mathbb{E}\left[\pi_s\right]=\sum_{i=1}^{n}\mathbb{E}_v\left[J\left(v_i\right) q_i\left(v_i, v_{-i}\right)\right]-nu\left(\underline{\omega}\right).$$

The information acquisition constraint in this setting is:

$$\mathbb{E}_F\left[u\left(v_i\right)\right]-u\left(\mu\right)\geq c, \; \forall v_i\in\left[\underline{\omega},\overline{\omega}\right],$$

where the expectation is taken with respect to $F\left(\cdot\right).$ We can calculate the payoff of both informed and uninformed bidders and rewrite the information acquisition constraint as

$$\int_{\underline{\omega}}^{\overline{\omega}}\left[\frac{1-F\left(v_i\right)}{f\left(v_i\right)}-\frac{1_{\{v_i\leq\mu\}}}{f\left(v_i\right)}\right] Q_i\left(v_i\right) f\left(v_i\right) dv_i\geq c.$$

Again, the (IR) constraint is binding: $u(\underline{\omega}) = 0$. With the reformulated (IC) and (IA) constraints, we can rewrite the seller's optimization problem as

$$\begin{aligned} &\max_{q_i(v)} \sum_{i=1}^{n} \mathbb{E}_v \left[J(v_i) \, q_i(v_i, v_{-i}) \right] \\ s.t. \quad (1) \quad &: \quad 0 \le q_i(v_i, v_{-i}) \le 1; \sum_{i=1}^{n} q_i(v_i, v_{-i}) \le 1 \\ (2) \quad &: \quad Q_i(v_i) \text{ is nondecreasing in } v_i \\ (3) \quad &: \quad \int_{\underline{\omega}}^{\overline{\omega}} \left[\frac{1 - F(v_i)}{f(v_i)} - \frac{1_{\{v_i \le \mu\}}}{f(v_i)} \right] Q_i(v_i) \, f(v_i) \, dv_i \ge c. \end{aligned} \tag{1.29}$$

Let λ denote the Lagrangian multiplier for the information acquisition constraint (1.29). With some algebra, we can simplify the Lagrangian into

$$L = \sum_{i=1}^{n} \mathbb{E}_v \left[J^*(v_i) \, q_i(v_i, v_{-i}) \right] - \lambda c,$$

where the modified virtual surplus function

$$J^*(v_i) = \begin{cases} v_i - \frac{1 - pF(v_i)}{pf(v_i)} - \frac{\lambda}{n} \frac{F(v_i)}{pf(v_i)} & \text{if} \quad v_i < \mu \\ \mu & \text{if} \quad v_i = \mu \\ v_i - \frac{1 - F(v_i)}{f(v_i)} + \frac{\lambda}{n} \frac{1 - F(v_i)}{pf(v_i)} & \text{if} \quad v_i > \mu \end{cases} . \tag{1.30}$$

One can show that as n is large, $\lambda > 0$. That is, the seller's revenue is higher when more bidders become informed.

The following proposition shows that the optimal auction adjusts the reserve price toward mean valuation to provide bidders with incentive to acquire information.

Proposition 1.8 *If F has an increasing hazard rate, $c \in [\underline{c}, \overline{c}]$, and n is large, then the optimal reserve price is adjusted toward the mean valuation μ. That is, (1) if $r_{\overline{\alpha}} > \mu$, then the optimal reserve price $\mu \le r^* < r_{\overline{\alpha}}$; (2) if $r_{\overline{\alpha}} = \mu$, then $r^* = \mu$; (3) if $r_{\overline{\alpha}} < \mu$, then the optimal reserve price $r_{\overline{\alpha}} < r^* \le \mu$.*

Proof: Given $J^*(\cdot)$ is single-crossing zero, the optimal reserve price is the smallest r

such that $J^*(r) \geq 0$. Consider the case $r_{\overline{\alpha}} > \mu$. Suppose $r^* \notin [\mu, r_{\overline{\alpha}}]$. Then either $r^* < \mu$ or $r^* > r_{\overline{\alpha}}$. If $r^* < \mu$,

$$r^* - \frac{1 - F(r^*)}{pf(r^*)} - \frac{\lambda}{n} \frac{F(r^*)}{pf(r^*)} < 0,$$

a contradiction to the assumption that $J(r^*) \geq 0$. If $r^* > r_{\overline{\alpha}}$, notice that

$$r_{\overline{\alpha}} - \frac{1 - F(r_{\overline{\alpha}})}{f(r_{\overline{\alpha}})} + \frac{\lambda}{n} \frac{1 - F(r_{\overline{\alpha}})}{pf(r_{\overline{\alpha}})} > 0,$$

contradicting to the fact that r^* is smallest r such that $J^*(r) \geq 0$. The other two cases can be proved analogously. ■

Proposition 1.8 shows that the simple rule for adjusting the reserve price is robust to the discrete information acquisition specification.

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Chapter 2

Contests for Status

2.1 Introduction

One of the earliest designed society structures was that of Solon's (ca. 638 BC - 558 BC) *timokratia,* an oligarchy with a sliding scale of status determined by precisely defined ranges of measured output (fruit, grain, oil, etc.). Solon divided the entire population of Attica into four status classes,[1] and attached various, more or less tangible rights, to each class. Higher classes had more rights but were also expected to contribute more to the state.

The kings and queens of feudal states awarded titles of nobility such as *duke (or duchess), marquis, earl, count, viscount, baron, baronet* in return for special services to the crown. Initially there was a strong link between such titles and tangible assets, such as land and serfs, but this link weakened over time.[2]

Today's large corporations (such as large banks) have, besides a single *president*, several *executive vice presidents,* tens of *senior vice-presidents,* and several hundred "mere" *vice-presidents.* The New York Metropolitan Museum of Art offers eight different donor cat-

[1] These were the Pentakosiomedimnoi, the Hippeis, the Zeugitai and the Thetes.

[2] Even today's citizens of the United Kingdom are eligible for more than 50 orders and decorations, awarded for special services to the "queen". These are structured in a strict precedence system, and play an important role in public life. The police currently investigates allegations that close associates of prime minister Blair facilitated the award of honors in exchange for large monetary contributions to the Labor party.

egories[3] for corporate members (such as "*Chairman's Circle*" for donations above $100,000, "*Director's Circle*" for donations between $60,000 and $100,000, and so on) and 10 similar categories for private members.

The common denominator to the above examples is that agents care about social status, and that a self-interested principal is usually able to divert (or "manipulate") this concern to an avenue that is beneficial to himself/herself. The general importance of status concerns for explaining behavior has been long recognized by sociologists and economists.[4] Recent happiness research shows how wage rank affects workers' well-being,[5] and experimental studies pointed out that social status may play a role also in market exchanges.[6] Nevertheless, the literature focusing on the direct implications of status concerns for the design of societies and organizations is relatively thin. William Goode (1979), a leading sociologist, offers a broad study of "prestige" as an instrument of social control. He notes that "individuals and groups give and withhold prestige and approval as a way of rewarding or punishing others."

In this paper we closely follow Goode's perspective, and we study the optimal design of organizations under the assumption that agents care about their relative position. We show how a judicious definition of the number and size of status classes based on performance rank can be used by a principal in order to maximize the agents' output in a contest situation. Our results offer both explanations for commonly observed phenomena (such as having a unique individual at the top) and suggestions for the design of the level structure in a hierarchy. As it will become clear below, major factors affecting the structure of the optimal partition in status categories are: 1) the distribution of abilities in the population, and 2) the relative weight of the monetary component in the determination of status. If outstanding talent is relatively rare or if differences in wealth are crucial for status perceptions,

[3]See Amihai Glazer and Kai A. Konrad (1996) for some empirical evidence and a theoretical model that focuses on conspicuous giving.

[4]See Max Weber (1978), James S. Coleman (1990), Thorstein Veblen (1934), James S. Duesenberry (1949), Milton Friedman and Leonard J. Savage (1948), and Milton Friedman (1953) for some early contributions. Robert H. Frank (1985) offers an entertaining account of some of the issues.

[5]See Gorton Brown, Janathan Gardner, Andrew Oswald and Jian Qian (2004).

[6]See Sheryl Ball, Catherine Eckel, Philip J. Grossman and William Zame (2001).

we find an optimal structure that distinguishes the top performer while lumping together everyone else, irrespective of their performance. This insight yields a novel potential explanation for the well-documented recent increase in the gap between CEO compensation and the compensation of other workers (or even other executives) within the firm. In contrast, if talent is relatively abundant and if status is not too tightly linked to wealth, we find an optimal structure where status categories proliferate and where relatively small differences in performance are rewarded with different status prizes. In those cases, status can serve as a potent substitute for money in order to drive performance.[7]

The tournament literature has shown how prizes based on rank-orders of performance can be effectively used to provide incentives (see Edward Lazear and Sherwin Rosen, 1981, Jerry Green and Nancy Stokey, 1983, and Barry Nalebuff and Joseph Stiglitz, 1983). Charles O'Reilly, Brian Main and Graef Crystal (1988) have emphasized the important role of status in executive compensation, and Donald Hambrick and Albert Cannela (1993) use relative standing as the main factor for explaining departures rates of executives of acquired firms. Michael Bognanno (2001) studies the empirical relation between the number of executive board members and the CEO's compensation in "corporate tournaments".

Benny Moldovanu and Aner Sela (2001, 2006) developed a convenient contest model that can easily accommodate several prizes of different size. Using their methodology, it is a natural step to analyze the incentive effect of "status prizes," and the interplay between such prizes and tangible ones.

In our present model, several agents who are privately informed about their abilities engage in a contest, and are then partitioned into status categories (or classes) according to their performance. A status category consists of all contestants who have performances in a specified quantile, e.g., the top status class may consist of the individual with the highest output, the second class of individuals with the next three highest outputs, and so on... Each individual cares about the number of contestants in classes above and below him. We

[7]For example, this seems to be the case in institutions devoted to scientific research and in many other not-for-profit organizations.

choose a convenient functional formulation that captures well the "zero-sum game" nature of concerns for relative position: if an individual gets higher (lower) status, one or more individuals must get lower (higher) status.

A designer (or principal) determines the number of status classes and their size in order to maximize total output. Since the contest equilibrium only depends on the structure of status classes, and not directly on the designer's goal, our type of analysis can, in principle, be performed for a variety of other goals.

We first analyze the "pure status" case where there are no other tangible prizes to motivate the contestants. We then extend our model to investigate a setting where the designer awards monetary prizes, and where status is purely derived from the differences in monetary compensation, i.e., having a higher monetary prize *per se* implies higher status.[8] These two models represent opposite extremes, and reality is often somewhere in the middle. In most cases, we think that individuals in organizations are, at least partly, motivated by status concerns, but that status is not solely derived from the monetary payoffs attached to various activities.[9]

Since status is a "zero-sum game", it seems, at first glance, that shifts in the allocation of status among agents should not affect total output. The missing factor in this argument is the heterogeneity in abilities. Since higher ability will be, in equilibrium, associated with higher performance, modifications of classes at different levels in the hierarchy may have quite different effects. In particular, because the expected benefit associated with a move upwards in the ranks (which is given by the expected increase in status minus the expected cost of producing an output that is sufficient for the upward move) depends on the bounds of the quantile defining the status class, a manipulation of these bounds affects behavior, and hence total output.

Our results relate the structural features of the optimal partition in status categories

[8] See Arthur J. Robson (1992) for another model where status is defined by wealth.

[9] For example, Chaim Fershtmann and Yoram Weiss (1993) relate status to the length of the education necessary for a specific occupation (their motto is Adam Smith's nicely circular: "Honour makes a great part of the reward of all honourable professions").

to properties of the distribution of abilities in the society:

1) We show that, for any distribution of abilities, the top category in any optimal partition must contain a single agent.[10] This agrees well with the ubiquitous structure of many human (or animal) organizations and social structures, and brings to mind familiar roles such as "queen", "alpha-male", "CEO", etc....

2) Given a partition in status classes, adding a new element to an arbitrary class may, in fact, reduce output. But, we show that the adoption of a policy that resembles "hiring at the lowest level" (see George Baker, Michael Gibbs, and Bengt Holmstrom, 1994) always makes an increase in the number of (ex-ante symmetric) contestants beneficial to the principal.

3) We then identify the main factors leading either to a proliferation of status classes (where each individual is "in a class of his/her own") or to coarse partitions where it is optimal to have a wider range of performances bunched together in the same category. A proliferation of status classes is optimal if the distribution of abilities has an increasing failure (or hazard) rate. This finding points in the same direction as the well known empirical fact that job titles do proliferate, but only in organizations with a relatively professional work-force (see James N. Baron and William T. Bielby, 1986). In contrast, a coarse partition with only two status classes (where all individuals except one belong to the lower class) is optimal if the distribution of abilities is sufficiently concave.

4) If the distribution of abilities has an increasing failure rate, we show that the optimal partition in the class of partitions with only two status categories achieves at least half the performance of the overall optimal partition. Thus, whenever there are transaction costs attached to finer partitions, the coarsest possible non-trivial partition may be ultimately optimal.[11] This is related to an argument made by Preston McAfee (2002) in the context of "coarse matching" of two populations.

5) Finally, we introduce monetary prizes and consider status purely induced by these

[10] This is of course reminiscent of the optimal taxation literature, pioneered by Mirrlees (1971), which has a unique tax rate for the wealthiest individual.

[11] Think about the Econometric society, say, which has two status classes: members and fellows.

prizes. In order to add realism, we assume that the designer is budget constrained, and that agents can choose not to compete if the monetary prize is not enough to compensate them for a potential low status. In this framework, we show that the optimal structure is to have exactly two status classes: the top class consisting of the single most productive agent, while the lower class containing all other agents that get paid just enough to keep them in the contest. Since, as illustrated above, there are many real-life examples where status classes proliferate, our results suggest that in those situations status cannot be solely and entirely induced by monetary wealth.[12]. In contrast, the growing gap between CEO compensation and the compensation of other agents within the firm can be explained by an increase in the status value conferred by the monetary component.

Technically, our results are obtained by combining insights derived from the general analysis of contests with multiple prizes developed by Moldovanu and Sela (2001, 2006) with a novel application of statistical results about stochastic monotonicity properties of *normalized spacings* (i.e., differences) of order statistics (Richard Barlow and Frank Proschan, 1966). For large and interesting classes of distribution functions it is possible to say, for example, whether normalized spacings become stochastically more (less) compressed when we climb higher in the ability range, and we show that such features determine the structure of the optimal partition in status classes.

While many authors put "status" directly into the utility function,[13] the paper most closely related to ours is Pradeep Dubey and John Geanakoplos (2005). These authors study optimal grading of exams in situations where students care about relative ranks. We have borrowed from that paper the present specification of utility functions. Our determination of status categories based on relative effort rank corresponds to what Dubey

[12]On this topic see also Robert H. Frank (1999).

[13]Fershtman and Weiss (1993) construct a general equilibrium model where both status and wealth are determined endogenously. In Gary S. Becker, Kevin M. Murphy and Ivan Werning's (2005) model, status is bought in a market. They assume that there are at least as many status classes as individuals and that status is a complement to other consumption goods. Ed Hopkins and Tatiana Kornienko (2004) study the effect of an exogenous change of income distribution in a model where agents care about their rank in the distribution of consumption. Rick Harbaugh and Tatiana Kornienko (2001) draw a parallel between a decision model that assumes a concern for local status and prospect theory.

and Geanakoplos call in their respective context “grading on a curve”. There are many substantial differences between their model, technique and results and ours. For their main results, Dubey and Geanakoplos focus on absolute grading, assuming that there is complete information, that students are either homogenous or have discrete types, that effort choice is binary, and that the relation between effort and output is stochastic. Moreover, the designer’s goal is to have all students choose the higher effort level out of the two possible ones. Their main finding is that status-conscious students may be better motivated to work hard by a professor who uses coarse grading (e.g., A,B,C,D rather than 100, 99,...). This should be contrasted with our main result about the optimality of the finest partition for a very large and ubiquitous class of distributions.

Another related paper is Rayo (2003). He analyzes the monopolistic design and pricing of positional goods that consumers use to signal their types. A main result is that a monopolist will restrict the variety of positional goods in order to extract surplus from consumers. In his model, a consumer’s utility depends on the average type of consumers paying the same price. Thus (as in our model), utility from being in a certain class is manipulable by the designer. But, in Rayo’s model, utility depends on the characteristics of consumers in the same class, whereas in our model utility depends on the number of agents in superior and inferior classes. Moreover, in Rayo’s model there is a continuum of consumers who interact only indirectly (through the influence of perceptions on utility) - it is this feature which allows the usage of tools from the literature on monopolistic non-linear pricing. In contrast, we have here a finite number of agents who directly and strategically compete for a scarce resource (i.e., places in superior status classes) and therefore we need to use tools from the literature on contest design/statistics. In spite of these differences, several of Rayo’s results resemble ours: the highest possible type should never be pooled with others; if a “virtual valuation function” is monotonic, full separation is optimal, whereas some pooling (which corresponds to coarseness in our model) is optimal if this condition is not satisfied.

Postlewaite (1998) presents an excellent discussion on the advantages/disadvantages of the "direct" modeling approach versus the one where a concern for relative ranking is only implicit, or "instrumental" for other goals that are made explicit (see also Cole et al., 1992). In a nutshell, Postelwaite's argument against a direct approach is that, by adjusting utility functions at will, one can explain every phenomenon. For our purposes, the debate about the right way to model status concerns is only of secondary importance. Our main focus is on the optimal design of status classes (from an incentive point of view) given that agents care, for some direct or instrumental reason, about relative position. We view the assumed utility function as a simplification, and we ask the reader to judge the outcome by Hardy's dictum whereby good science must, at least, provide some "decent" distance between assumptions and results.

The rest of the paper is organized as follows: Section 2.2 presents the contest model with status concerns, and some useful facts about order statistics. In Section 2.3 we derive results that connect the form of the optimal partition in status categories to various properties of the distribution of ability in the population. We first show that, by always adding new entrants to the lowest status category, the designer can ensure that his payoff is monotonically increasing in the number of contestants. Thus, potential contestants need not be excluded from competing. We next show that the top status category in any optimal partition must contain a unique element. For distribution of abilities that have an increasing hazard rate, each status category in an optimal partition will contain a unique element — thus, in this case a proliferation of status classes is optimal. We also show that the optimal partition involves only two categories if the distribution of abilities is sufficiently concave. Finally, we study the properties of optimal partitions with only two status categories. In Section 2.4 we modify the model to allow for status categories that are endogenously determined by monetary prizes of different sizes. If status is solely derived from monetary rewards, we show that the optimal partition contains only two categories, with the top category being a singleton. Section 2.5 concludes. Several proofs and examples are relegated to an Appendix.

2.2 The Model

We consider a contest with n players where each player j makes an effort e_j. For simplicity, we postulate a deterministic relation between effort and output, and assume these to be equal. Efforts are submitted simultaneously. An effort e_j causes a cost denoted by e_j/a_j, where $a_j > 0$ is an ability parameter.

The ability (or *type*) of contestant j is private information to j. Abilities are drawn independently of each other from the interval $[0, 1]$ according to a distribution function F that is common knowledge. We assume that F has a continuous density $f = dF > 0$.

Contestants are ranked according to efforts. Let $\{(0, r_1], (r_1, r_2], ...(r_{i-1}, r_i], ..., (r_{k-1}, n]$ $\}$ be a partition of the integers in the interval $(0, n]$ in $k \geq 1$ *status categories*, where $r_{i-1} < r_i$. Define also for convenience: $r_0 \equiv 0$ and $r_k \equiv n$. Given such a partition and the ordered list of efforts, contestants are divided into the k categories: a player is included in category i, if his effort is between the r_{i-1}-th and r_i-th highest ones.

Each player cares about the number of players in categories both below and above him, and we assume that the "pure status" prize of being in status category i is given by

$$v_i = r_{i-1} - (n - r_i).$$

Thus, a contestant is happier when he has more [less] people below [above] him. Note this formulation well captures the *zero-sum* nature of status: for any partition in status categories, the total value derived from status is given by :

$$\sum_{i=1}^{k}(r_i - r_{i-1})v_i = \sum_{i=1}^{k}(r_i - r_{i-1})(r_i + r_{i-1} - n) = 0$$

To summarize, the timing of the game is as follows: The designer chooses a partition $\{r_i\}_{i=0}^{k}$ and commits to it. Each contestant then gets privately informed about his/her ability. The contestants simultaneously choose effort level according to their ability types. Finally, agents are partitioned into different status categories according to their efforts and the chosen partition.

We assume that each player maximizes the value of the expected status prize minus the expected effort cost, and that the designer maximizes the value of expected total effort by adjusting the partition in status classes.

We use the following notation: 1) $A_{k,n}$ denotes k-th order statistic out of n independent variables independently distributed according to F (note that $A_{n,n}$ is the highest order statistic, and so on..); 2) $F_{k,n}$ denotes the distribution of $A_{k,n}$, and $f_{k,n}$ denotes its density; 3) $E(k,n)$ denotes the expected value of $A_{k,n}$. (Note that $E(n,n)$ is the expectation of the maximum, or highest order statistic, and so on..)

2.3 The Optimal Partition in Status Categories

This section contains our main results about the structure of the optimal partition in status categories. We focus on a symmetric equilibrium where all agents use the same, strictly monotonic equilibrium effort function β. In such an equilibrium, the output rank of player j will be the same as his ability rank among the n contestants.

Let $P_i(a)$ be the probability of a player with ability a being ranked in category i, i.e., her ability is between the r_i-th and r_{i-1}-th highest. These probabilities involve the order statistics of the distribution of abilities in the population. Applying the revelation principle, agent j with ability a chooses to behave as an agent with ability s to solve the following optimization problem:

$$\max_s \sum_{i=1}^{k} P_i\left(s\right)\left[r_{i-1}-\left(n-r_i\right)\right]-\frac{\beta\left(s\right)}{a}$$

In equilibrium, the above maximization problem must be solved by $s=a$. The calculation of equilibrium effort functions and total expected effort yields:

Theorem 2.1 *Assume that contestants are partitioned in k status categories according to* *the family* $\{r_i\}_{i=0}^{k}$*. Then, total expected effort in a symmetric equilibrium is given by*

$$E_{total}^{(k)} = \sum_{i=1}^{k-1}(r_{i+1} - r_{i-1})(n - r_i)E(r_i, n)$$

Proof. See Appendix. ■

Given the above result, we can now formulate the designer's problem: she needs to determine the number of contestants (m) and status categories (k), and the size of each category ($r_i, i = 1, .., k-1$). Explicitly, we obtain the following discrete optimization problem:

$$\max_{m,k,\{r_i\}_{i=0}^{k}} [\sum_{i=1}^{k-1}(r_{i+1} - r_{i-1})(m - r_i)E(r_i, m)]$$

$$\begin{aligned} \text{subject to} &\quad : \\ \text{i) } 2 &\leq m \leq n \\ \text{ii) } 2 &\leq k \leq m \\ \text{iii) } 0 &= r_0 < r_1 < \ \dots \ < r_{k-1} < r_k = m \end{aligned}$$

2.3.1 The Optimal Number of Contestants

In many relevant situations, the number of agents will be exogenously determined by various economic considerations within the group, and can therefore be considered fixed for our purposes. But, it is also of interest to understand whether the designer has incentives to restrict entry that directly stem from the status considerations.[14] We determine here the optimal number of contestants by analyzing the effect of changing the number of contestants (i.e., by entry or hiring) on total expected effort. Given the zero-sum nature of status, the answer is not clear-cut, and it depends on the designer's reaction to entry (i.e., on how the size and number of status categories change). The following example illustrates the possibility that a wrong post-entry adjustment policy may cause total effort to actually go

[14] Taylor (1995) and Fullerton and McAfee (1999) provide models of research tournaments where restricting entry may be beneficial for the designer.

down.

Example 2.2 *Let* $F(x) = x^{1/w}$ *,* $w > 1$*, and consider only partitions with two categories. Total effort is given by*

$$E_n = n(n-r)E(r,n) = n(n-r)\frac{n!\,(w+r-1)!}{(r-1)!\,(n+w)!}$$

where r *is the division point. If we add an additional contestant to the higher category (that is, we do not change the value of* r*), we obtain for* ω *high enough:*

$$E_{n+1} - E_n = \frac{(w+r-1)!n!}{(r-1)!\,(n+w)!}\left[\frac{(n+1)^2(n+1-r)}{(n+1+w)} - n(n-r)\right] < 0$$

That is, for sufficiently high w*, total effort decreases in the number of players.*

We show below that a designer who optimally reacts to additional entry can always ensure that total effort increases. In particular, in the proof, we identify a very simple strategy (without the need of a complex re-optimization!) ensuring that total effort does not decrease: faced with more contestants, the designer can just increase the size of the lowest status category. For an intuition, consider for simplicity a partition with only two status categories. Then the number of "status prizes" is equal to the number of contestants in the top category, and each prize is worth n, the difference in payoffs between the high and low categories. If another agent is added, the value of each status prize becomes $n+1$, independently of which status category is expanded. But, if the expansion is in the lower category, the number of status prizes remains fixed, while an expansion of the higher category also leads to an increase in the number of prizes. Such an increase has an adverse effect on the effort of high ability types, and this may offset the positive effect of having higher prizes. Thus, only by expanding the lower category, the designer increases the value of status prizes without simultaneously increasing their number.

Theorem 2.3 *Total effort in an optimal partition increases in the number of contestants.*

Proof. See Appendix. ■

2.3.2 The Optimal Partition

Given the above result, the designer has no incentives to restrict entry in the contest, and we thus assume below that all n potential contestants are included.[15]

Since the distribution of abilities determines the expected values of the various order statistics appearing in the designer's maximization problem, the optimal number of status categories and the optimal size of each category generally depend on this distribution. Our first main result identifies a robust and general feature that holds for any distribution:

Theorem 2.4 *In any optimal partition, the top status category contains a unique element.*

Proof. Suppose, by contradiction, that the k-th (top) category contains more than one element. Then, divide this category into two sub-categories, and denote by r_d the dividing point: $r_{k-1} < r_d < n$. Using the formula in Theorem 2.1, the difference in expected effort between the new and the old partitions is given by:

$$\begin{aligned} E_{total}^{(k+1)} - E_{total}^{(k)} &= (n - r_{k-1})(n - r_d)E(r_d, n) - (n - r_{k-1})(n - r_d)E(r_{k-1}, n) \\ &= (n - r_{k-1})(n - r_d)\left[E(r_d, n) - E(r_{k-1}, n)\right] > 0 \end{aligned}$$

The inequality follows since $A_{r_d,n}$ stochastically dominates $A_{r_{k-1},n}$. ■

Refining a top category that contains several elements does not affect the rewards going to agents outside that category. The reward and effort of those agents in the (new) second highest category is lower than before since these agents lose their top status. But, this loss is more than offset by the effort increase coming from the highest ability types whose status is increased by the refinement since they perceive more inferior agents after the change.

[15] See Section 4 where this result need not hold if the designer is budget constrained and if agents must be monetarily compensated for low status.

Optimal Fine Partitions

Our next main result identifies a condition on the distribution of abilities that allows us to extend the above logic to all status categories, thus exhibiting an optimal partition that is the finest possible. We use a statistic result about stochastic monotonicity of normalized differences (also called spacings) of order statistics. We first need to remind the reader some well-known concepts: The failure rate (or hazard rate) of a distribution F is defined by:

$$\lambda(a) = \frac{f(a)}{1 - F(a)}$$

A distribution function F has an *increasing failure rate* (IFR) if $\lambda(a)$ is increasing or, equivalently, if $\log(1 - F(a))$ is concave. Analogously, F has an *decreasing failure rate* (DFR) if $\lambda(a)$ is decreasing, or, equivalently, if $\log(1 - F(a))$ is convex.[16]

Armed with these concepts, we can now state:

Lemma 2.1 *(Barlow and Proschan, 1966) Assume that a distribution F with $F(0) = 0$ satisfies IFR (DFR). Then, $(n-i+1)(A_{i,n} - A_{i-1,n})$ is stochastically decreasing (increasing) in i for a fixed n.*

In other words, up to a normalizing factor, the difference between the expected abilities of consecutively ranked contestants is higher at the bottom than at the top if the distribution is IFR, and the opposite holds for DFR distributions. An application of this result yields now:

Theorem 2.5 *Assume that F, the distribution of abilities, has an increasing failure rate. Then, the optimal partition is the finest possible one: each status category contains a unique element.*

Proof. See Appendix. ■

[16] Most well known distributions belong to these important and much studied categories. The relationships between IFR, DFR, convexity and concavity of F are as follows: Convexity implies IFR , while DFR implies concavity. The only distribution that is both concave and convex is the uniform, while the only distribution that is both IFR and DFR is the exponential.

The intuition behind the above result is analogous to one appearing in models of monopolistic quality/quantity discrimination: in "regular" settings, where marginal revenue is increasing in type (note that IFR is a sufficient condition for this to happen !), the optimal tariff allocates different qualities (here different status classes) to consumers with different types. In particular, lumping (or pooling) together several types cannot be optimal.

Splitting status class j in two sub-classes has two effects: there is a loss of expected effort stemming from the fact that several agents are now placed in the lower sub-class, and there is a gain from agents that are now placed in the higher sub-class (again, classes other than j are not affected by the split). The IFR condition ensures that the gain more than offsets the loss. For illustration purposes, assume that a category j has size two, and we refine it into two new categories, each with one element $r_j - r_d = r_d - r_{j-1} = 1$. This change is advantageous if the difference of expected efforts after and before the change is positive, i.e., if

$$\left\{ \begin{array}{c} (n - r_d + 1)\left[E\left(r_d, n\right) - E\left(r_{j-1}, n\right)\right] - (n - (r_d + 1) + 1)\left[E\left(r_d + 1, n\right) - E\left(r_d, n\right)\right] \\ + \left[E\left(r_d + 1, n\right) - E\left(r_d, n\right)\right] \geq 0 \end{array} \right\}$$

The first line is positive, because the normalized difference between the expected abilities of consecutively ranked contestants is higher at the bottom than at the top if the distribution is IFR, while the second line is positive because of usual stochastic dominance.[17]

Optimal Coarse Partitions

If the IFR condition (which represents, in fact, a convexity requirement with respect to the exponential distribution) is not satisfied, a coarse partition may be optimal. We now show that a very coarse partition with only two categories is optimal for sufficiently concave distributions. If there are only two categories, total effort is given by

$$E_{total}^{(2)} = n(n - r_1)E\left(r_1, n\right)$$

[17] The argument also indicates that the IFR condition is not necessary for class proliferation.

The intuition for the above expression is simple: this is a contest with $(n - r_1)$ equal prizes (for all those in the higher category), and each prize is worth here n (the difference in payoffs between the high and low categories). By Theorem 2.4, when looking for optimal partitions, we can restrict attention to those where the top category consists of a unique element. In this case $r_1 = n - 1$, and total effort is given by

$$E^{(2)}_{total} = nE\left(n - 1, n\right)$$

In order to prove the result, we need to show that any other partition yields less effort if the distribution of abilities is sufficiently concave. The proof uses the following Lemma:

Lemma 2.2 *(Barlow and Porschan, 1966) Consider two distributions F and G such that $F(0) = G(0) = 0$, and such that $G^{-1}F$ is convex on the support of F.*[18] *Then $E_F(i, n)/E_G(i, n)$ is decreasing in i.*

Proposition 2.1 *Assume that the optimal partition of status categories under distribution F consists only of two categories, and consider another distribution G such that $G^{-1}F$ is convex on the support of F. Then the optimal partition under G also consists of two categories.*

Proof. See Appendix. ■

If we can show that there exists a distribution function for which the optimal partition consists indeed of two categories, then the above result immediately implies that the same will hold for **all** more concave distributions. The existence of such a distribution is established in the Appendix.

The intuition for the optimality of very coarse partitions for sufficiently concave distributions of ability is simple: most of the mass is then concentrated at the bottom and high ability individuals are rare. Thus, many "mediocre" types are motivated by a high reward (a unique high status prize) since they have a reasonable chance to get it. Moreover, the

[18] This means that G is more concave than F.

rare high ability individual lacks sufficient competition, and is therefore best motivated by a large reward.

2.3.3 How Good Are Partitions with Two Categories ?

In the above subsection we have identified conditions under which a partition with two categories are optimal. Here we take a somewhat different perspective that is not based on optimality: we show that, for the large and important class of IFR distributions (for which the optimal partition is the finest possible one), the designer can nevertheless achieve a substantial share of the optimal performance with a simple partition in two categories.[19] Thus, if very fine partitions are for some reason costly, a designer may find it optimal to choose the simplest non-trivial coarse partition. This seems to us a powerful argument in favor of coarse partitions.

Proposition 2.2 *Assume that F, the distribution of abilities, has an increasing failure rate. Then, the optimal partition in the class of partitions with only two status categories yields at least half the performance obtained by the overall optimal partition.*

Proof. Recall that in the IFR case, the overall optimal partition is the finest possible one, and hence has n status categories. Thus, total effort in the overall optimal partition is given by :

$$E_{total}^{(n)} = 2\sum_{i=1}^{n-1}(n-i)E(i,n)$$

Total effort in the optimal partition with only two categories is given by

$$E_{total}^{(2)} = n(n-i^*)E(i^*,n)$$

[19] We were not able to find a direct technical relation between our result and McAfee's (2002) paper on complete information matching of two continuum of populations. In McAfee's model the "optimal partition" is always (i.e., irrespective of distribution) the finest possible — assortative matching, whereas we get the optimality of the finest partition only under IFR. His result requires IFR on both distributions of abilities and on their survival functions, whereas we require IFR only on the distribution itself. Finally, his result holds for the partition with two categories where the cutoff is at the mean of each population, whereas our result holds for the optimal partition in the class of partitions with two categories.

where $i^* \in \arg\max_i[n(n-i^*)E(i^*,n)]$. This immediately yields: $E^{(2)}_{total} > \frac{1}{2}E^{(n)}_{total}$. ∎

The above approximation is rough, and the coarse partition with only two classes yields for "well-behaved" distributions a much higher percentage of the optimal performance. For example, $E^{(2)}_{total} \geq \frac{3}{4}E^{(n)}_{total}$ for a uniform distribution of abilities.

Our final result in this section gives further information about the optimal partition with two categories. Its proof is also based on Lemma 2.2 above.

Proposition 2.3 *Let r^* be the division point defining the optimal partition in two status categories, i.e. the optimal number of contestants in the lower class. If the distribution of abilities F is convex (concave) then $r^* \leq (\geq)\ n/2$.*

Proof. Suppose that r^* is the optimal division point. Then, total effort in the optimal partition is higher than in any other partition. In particular, it is higher than total effort in the partition where $r = n - r^*$. This yields:

$$\begin{aligned} n\left(n-r^*\right)E\left(r^*,n\right) &\geq n\left[n-\left(n-r^*\right)\right]E\left(n-r^*,n\right) \Leftrightarrow \\ \left(n-r^*\right)E\left(r^*,n\right) &\geq r^*E\left(n-r^*,n\right) \Leftrightarrow \\ \frac{E\left(r^*,n\right)}{r^*} &\geq \frac{E\left(n-r^*,n\right)}{n-r^*} \end{aligned}$$

By taking one of the distributions to be uniform in Lemma 2.2, we obtain that, for a fixed n, $E(i,n)/i$ is decreasing (increasing) in i if the distribution of abilities is convex (concave). Then, for a convex F, the last inequality above can hold only if $r^* \leq (n-r^*)$, which is equivalent to $r^* \leq n/2$. Analogously, if F is concave, it must be the case that $r^* \geq (n-r^*)$, which yields $r^* \geq n/2$. ∎

A simple corollary is, of course, that exactly half of the agents should be in the low (high) category if abilities are uniformly distributed.

2.4 Status Derived from Monetary Prizes

So far, we focus on the pure effect of status in contests: there are no other real prizes to drive efforts. We now consider contests where status is being indirectly (and solely) induced by the rank of monetary prizes in the respective hierarchy. Higher effort leads to a (weakly) higher monetary prize, and, in addition, agents get positive utility proportional to the number of agents that have lower monetary prizes, and negative utility proportional to the number of agents that have higher monetary prizes.[20] In particular, we depart from the zero-sum world presented above.

A set of k monetary prizes $V_k \geq V_{k-1} \geq ... \geq V_1$ and a family of division points $\{r_i\}_{i=0}^{k}$ where $r_0 = 0$ and $r_k = n$ determines a partition with k categories: a contestant ranked in the top category k (i.e., a contestant whose effort is among the top $r_k - r_{k-1}$) receives a monetary prize of V_k, a contestant in the second highest category receives a prize of $V_{k-1} \leq V_k$, and so on till the lowest $V_1 \leq V_2 \leq ... \leq V_k$.

Thus, a player who is awarded the i-th highest monetary prize V_i perceives in fact a total prize (money + status) of :

$$v_i = V_i + r_{i-1} - (n - r_i).$$

In order to make the problem non-trivial, we add here two realistic assumptions: 1) The contest designer is financially constrained: the total amount of monetary prizes cannot exceed a given amount P. Otherwise, it is obvious that large enough monetary prizes can always swamp any status effects. 2) We impose an individual rationality constraint: the expected payoff of each contestant should be non-negative. Otherwise, contestants will leave without competing (the outside option being normalized to zero).

By calculations similar to those performed for the case of pure status concerns, total

[20]Dubey and Geanakoplos (2005) consider a status model where monetary prizes are awarded on the basis of absolute performance.

effort in a symmetric equilibrium is given by

$$E_{total}^{(k)} = \sum_{i=1}^{k-1}(n-r_i)(r_{i+1}-r_{i-1})E(r_i,n) + \sum_{i=1}^{k-1}(n-r_i)E(r_i,n)(V_{i+1}-V_i)$$

Therefore, the designer's problem is as follows:

$$\begin{aligned}
\max_{k,\{r_i\}_{i=1}^{k},\{V_i\}_{i=1}^{k}} E_{total}^{(k)} &= \sum_{i=1}^{k-1}(n-r_i)(r_{i+1}-r_{i-1})E(r_i,n) + \sum_{i=1}^{k-1}(n-r_i)E(r_i,n)(V_{i+1}-V_i) \\
\text{subject to} &: \quad (1) \quad 1 \leq k \leq n \\
&: \quad (2) \quad \sum_{i=1}^{k}(r_i - r_{i-1})V_i = P \\
&: \quad (3) \quad V_1 \geq n - r_1 \\
&: \quad (4) \quad V_k \geq V_{k-1} \geq \ \ldots \ \geq V_1
\end{aligned}$$

Note that constraint (3) guarantees that the expected payoff of the lowest type, who does not make any effort, is non-negative. By a standard monotonicity argument, all other types will have positive expected payoffs.

Theorem 2.6 *If $P > n$, (i.e., if the available budget is as least as large as the number of contestants), the optimal solution to the designer's problem has the following structure: The designer induces a partition with two status categories such that the contestant with the highest effort receives a monetary prize $V_2 = P - (n-1)$,while all other contestants receive a monetary prize $V_1 = 1$. If $P \leq n$, it is optimal to restrict entry to the contest.*

Proof. See Appendix. ■

The intuition behind the optimality of the above described partition is as follows: Take a partition with two categories and a singleton in the top category, and refine it, for example, by dividing the low category in new "middle" and "low" categories. Then, the agents in the new low category perceive a decline in status, and this decline must be compensated by a higher monetary prize (in order to satisfy their individual rationality constraint). Since status is derived from monetary prizes, the agents in the new middle category must obtain a monetary prize that is at least as large as that of the agents in the new low category. Thus, via the budget constraint, the monetary prize of the agent in the top category must go down — this decline necessarily induces a decline in the effort of high ability types. Since the strongest effect of prizes is on high ability agents, the potential increase in effort of middle ability agents is not enough to compensate for the decline at the top. This insight is related to the optimality of a unique "first" prize in Moldovanu and Sela's (2001) contest model with linear cost functions and purely monetary prizes. That optimality naturally translates here into a partition in two status classes, with a singleton in the top category.

Our previous result suggests that an upward shift in the relative weight of the monetary part in determining status will lead to a larger gap between CEO compensation and the compensation of the other agents in a firm.[21] Frydman (2005) documents the relatively recent dramatic increase in this gap in the US, and offers an explanation based on a shift in the importance of general versus firm-specific skills.

2.5 Conclusion

We have studied a contest model where heterogeneous agents who care about relative standing are ranked according to output, and are then partitioned into status categories. Our main results describe the structure of the optimal partition into status classes from the point of view of a designer who maximizes total output. The model explains ubiquitous phenomena such as a top status class that contains a unique individual, and the proliferation of status classes in organizations where high-skilled individuals are not rare. We also studied the interplay between pure status and monetary prizes.

As already mentioned in the introduction, in most real-life situations status is only partly determined by measurable differences in monetary compensation. Social, cultural and other economic considerations that may be connected to a concern for relative position

[21]For example, the average ratio of highest to fifth highest compensation in US firms jumped from about 2.8 in the middle of 20th century to 6.1 at the beginning of the 21th century. The increase in the ratio of CEO compensation to average compensation in the firm is much more dramatic.

in a future interaction are also important determinants. Modeling a specific situation requires a simple combination of the two variants displayed here, and the corresponding results will be driven by the relative strengths of the monetary versus the less tangible parts.

Finally, note that, in principle, an analysis analogous to ours is possible for other agents' utility functions, or for other designer's goals. In particular, for given, fixed utility functions, the equilibrium analysis is not affected by the designer's goal which can be modified according to the desired application. Thus, our model offers a convenient framework for the study of the various implications of concerns for social status on organizational design.

2.6 Appendix

A few useful facts about order statistics:

It is well-known that:

$$\begin{aligned} F_{k,n}(s) &= \sum_{j=k}^{n} \binom{n}{j} F(s)^j [1-F(s)]^{n-j} \\ f_{k,n}(s) &= \frac{n!}{(k-1)!(n-k)!} F(s)^{k-1} [1-F(s)]^{n-k} f(s) \end{aligned}$$

Let $F_i^n(s)$, $i = 1, 2, ...n$ denote the probability that a player's type s ranks exactly i-th highest among n random variables distributed according to F. Then

$$F_i^n(s) = \frac{(n-1)!}{(i-1)!(n-i)!} [F(s)]^{i-1} [1-F(s)]^{n-i}$$

Defining $F_{n,n-1} \equiv 0$, and $F_{0,n-1} \equiv 1$, it is immediate that the relation between $F_{i,n}(s)$ and $F_i^n(s)$ is

$$F_i^n(s) = F_{i-1,n-1}(s) - F_{i,n-1}(s)$$

Finally, let $P_i(s)$ be the probability of a player with type s being ranked in category i, i.e., her type is between the r_i-th and r_{i-1}-th highest. Then:

$$P_i(s) = \sum_{j=1}^{r_i - r_{i-1}} F^n_{r_{i-1}+j}(s) = F_{r_{i-1},n-1}(s) - F_{r_i,n-1}(s)$$

Proof of Theorem 2.1:

Proof. Let a partition with k categories be given by $\{\ (0, r_1], (r_1, r_2], ...(r_{i-1}, r_i], ..., (r_{k-1}, n]\ \}$. Assuming a symmetric equilibrium in strictly increasing strategies,[22] the optimization problem of a player with ability a is

$$\max_s \left\{ \begin{array}{c} [1 - F_{r_1,n-1}(s)][-(n - r_1)] \\ +\sum_{i=2}^{k-1} \left[F_{r_{i-1},n-1}(s) - F_{r_i,n-1}(s)\right] [r_{i-1} - (n - r_i)] \\ +F_{r_{k-1},n-1}(s) r_{k-1} - \frac{\beta(s)}{a} \end{array} \right\}$$

where the first term is the utility of being in the lowest category, the second term is the utility of being in categories 2 till $(k-1)$, the third term is the utility of being in the highest category, and the last term is the disutility of exerting effort $\beta(s)$.

The solution of the resulting differential equation with boundary condition $\beta(0) = 0$ is

$$\beta(a) = \int_0^a x \left\{ f_{r_1,n-1}(x)(n - r_1) + \sum_{i=2}^{k-1} \left[f_{r_{i-1},n-1}(x) - f_{r_i,n-1}(x)\right] (r_{i-1} + r_i - n) + f_{r_{k-1},n-1}(x) r_{k-1} \right\} dx \tag{2.1}$$

Thus, total effort is given by:

$$E_{total} = n \int_0^1 \beta(a) f(a) da \tag{2.2}$$

The above integral can be calculated by inserting formula 2.1 in 2.2 and by integrating by parts the constituent terms, who all have the form $b \int_0^1 \left[\int_0^a x f_{r,n-1}(x)dx\right] f(a)da$ where b

[22] It can be shown that there is a unique symmetric equilibrium.

is a constant. Note that :

$$\begin{aligned}
& \int_0^1 \left[\int_0^a x f_{r,n-1}(x)dx\right] f(a)da \\
= & \left[F(a)\int_0^a x f_{r,n-1}(x)dx\right]_0^1 - \int_0^1 F(a)af_{r,n-1}(a)da \\
= & \int_0^1 a\left[1-F(a)\right] f_{r,n-1}(a)da \\
= & E(r,n-1) - \frac{r}{n}E(r+1,n) \\
= & \frac{n-r}{n}E(r,n)
\end{aligned}$$

The last equality follows by a well known identity among order statistics (see David and Nagaraja, 2003, page 44). Assembling all terms in equation 2.2, and recalling that $r_0 = 0$, and $r_k = n$ finally yields:

$$\begin{aligned}
E_{total}^{(k)} &= \left\{ \begin{array}{c} (n-r_1)^2 E\left(r_1,n\right) \\ +\sum_{i=2}^{k-1}(r_{i-1}+r_i-n)\left[(n-r_{i-1})E\left(r_{i-1},n\right)-(n-r_i)E(r_i,n)\right] \\ +r_{k-1}(n-r_{k-1})E(r_{k-1},n) \end{array} \right\} \\
&= \sum_{i=1}^{k-1}(r_{i+1}-r_{i-1})(n-r_i)E(r_i,n)
\end{aligned}$$

■

Proof of Theorem 2.3:

Proof. Consider a partition $\{r_i\}_{i=0}^k$ for a given number of contestants m. Total effort is given by

$$\begin{aligned}
E_{total} &= \sum_{i=1}^{k-1}(r_{i+1}-r_{i-1})(m-r_i)E(r_i,m) \\
&= r_2(m-r_1)E(r_1,m) + \sum_{i=2}^{k-1}(r_{i+1}-r_{i-1})(m-r_i)E(r_i,m)
\end{aligned}$$

Assume now that a designer faced with $m+1$ contestants expands by one the size of the lowest status category: thus, consider the new partition $\{r'_i\}_{i=0}^k$ where $r'_0 = 0, r'_1 = r_1 + 1, r'_2 = r_2 + 1, ..., r'_{k-1} = r_{k-1} + 1, r'_k = m + 1$.

Total effort for this new partition is given by

$$\begin{aligned} E'_{total} &= \sum_{i=1}^{k-1}(r'_{i+1} - r'_{i-1})(m + 1 - r'_i)E(r'_i, m + 1) \\ &= (r_2 + 1)(m - r_1)E(r_1 + 1, m + 1) + \sum_{i=2}^{k-1}(r_{i+1} - r_{i-1})(m - r_i)E(r_i + 1, m + 1) \end{aligned}$$

We obtain:

$$\begin{aligned} & E'_{total} - E_{total} \\ = \; & (m - r_1)E(r_1 + 1, m + 1) + \sum_{i=1}^{k-1}(r_{i+1} - r_{i-1})(m - r_i)[E(r_i + 1, m + 1) - E(r_i, m)] \geq 0 \end{aligned}$$

The last inequality holds since, for all i, m, $A_{i+1,m+1}$ stochastically dominates $A_{i,m}$.[23] The claim follows now by starting from an optimal partition for m contestants, and expanding the size of the lowest category as above. Further eventual optimization of the partition for $m + 1$ contestants must weakly increase the total effort even further, thus yielding the wished result. ■

Proof of Theorem 2.5:

Proof. Suppose that, in an optimal partition with k categories, the j-th $(1 \leq j \leq k)$ category contains more than one element. Divide the j-th category into two sub-categories and denote by r_d the dividing point, $r_{j-1} < r_d < r_j$. Letting $E(0, n) \equiv 0$, the difference in

[23] See Shaked and Shanthikumar (1994) for more details.

total effort between the new and the initial partition is given by:

$$E_{total}^{(k+1)} - E_{total}^{(k)} = \left\{ \begin{array}{c} (r_j - r_{j-1})(n - r_d)E(r_d, n) \\ -(r_j - r_d)(n - r_{j-1})E\left(r_{j-1}, n\right) \\ -(r_d - r_{j-1})(n - r_j)E(r_j, n) \end{array} \right\}$$

$$= \left\{ \begin{array}{l} (r_j - r_d)\left[(n - r_d)\, E\left(r_d, n\right) - (n - r_{j-1})\, E\left(r_{j-1}, n\right)\right] \\ -\left(r_d - r_{j-1}\right)\left[(n - r_j)\, E\left(r_j, n\right) - (n - r_d)\, E\left(r_d, n\right)\right] \end{array} \right\}$$

Let $t = r_j - r_{j-1}$, $r_d = r_{j-1} + 1$. Then,

$$E_{total}^{(k+1)} - E_{total}^{(k)}$$

$$= \left\{ \begin{array}{c} (t-1)\left[(n - r_d)\, E\left(r_d, n\right) - (n - (r_d - 1))\, E\left(r_{j-1}, n\right)\right] \\ -\left[(n - (r_d + t - 1))\, E\left(r_d + t - 1, n\right) - (n - r_d)\, E\left(r_d, n\right)\right] \end{array} \right\}$$

$$= \left\{ \begin{array}{c} (t-1)\left[(n - r_d)\, E\left(r_d, n\right) - (n - (r_d - 1))\, E\left(r_d - 1, n\right)\right] \\ -\left[(n - (r_d + t - 1))\, E\left(r_d + t - 1, n\right) - (n - (r_d + t - 2))\, E\left(r_d + t - 2, n\right)\right] \\ -\left[(n - (r_d + t - 2))\, E\left(r_d + t - 2, n\right) - (n - (r_d + t - 3))\, E\left(r_d + t - 3, n\right)\right] \\ -.... \\ -\left[(n - (r_d + 1))\, E\left(r_d + 1, n\right) - (n - r_d)\, E\left(r_d, n\right)\right] \end{array} \right\}$$

Note that

$$\begin{aligned} & (n - r)\, E\left(r, n\right) - (n - (r - 1))\, E\left(r - 1, n\right) \\ = \; & (n - r + 1)\left[E\left(r, n\right) - E\left(r - 1, n\right)\right] - E\left(r, n\right) \end{aligned}$$

By Barlow and Proschan's Lemma about IFR distributions, and by the fact that $-E\left(r, n\right)$ is decreasing in r, it immediately follows that $\left[(n - r)\, E\left(r, n\right) - (n - (r - 1))\, E\left(r - 1, n\right)\right]$ is decreasing in r. Therefore $E_{total}^{(k+1)} - E_{total}^{(k)} > 0$. This contradicts the assumption that the initial partition was optimal. Therefore, each category in the optimal partition must contain a unique element. ■

Proof of Proposition 2.1:

Proof. By Theorem 2.4 we can restrict the argument to partitions for which the top status class contains a unique element. By Theorem 2.1, the total effort in a partition with k status categories is given by

$$\begin{aligned} E^{(k)}_{total} &= \sum_{i=1}^{k-1}(r_{i+1}-r_{i-1})(n-r_i)E(r_i,n) \\ &= \sum_{i=1}^{k-2}(r_{i+1}-r_{i-1})(n-r_i)E(r_i,n)+(n-r_{k-2})E(n-1,n) \end{aligned}$$

The optimal partition contains only two status classes iff $E^{(2)}_{total} \geq E^{(k)}_{total}$ for all $2 \leq k \leq n$. That is, the following claim must hold for all $2 \leq k \leq n$ and all admissible partition sequences $\{r_i\}_{i=1}^{k}$:

$$\begin{aligned} nE(n-1,n) &\geq \sum_{i=1}^{k-2}(r_{i+1}-r_{i-1})(n-r_i)E(r_i,n)+(n-r_{k-2})E(n-1,n) \\ &\Rightarrow r_{k-2}E(n-1,n) \geq \sum_{i=1}^{k-2}(r_{i+1}-r_{i-1})(n-r_i)E(r_i,n) \\ &\Rightarrow r_{k-2} \geq \sum_{i=1}^{k-2}(r_{i+1}-r_{i-1})(n-r_i)\frac{E(r_i,n)}{E(n-1,n)} \end{aligned} \tag{2.3}$$

By Lemma 2.2 above, we know that $E_F(i,n)/E_G(i,n)$ is decreasing in i. This yields

$$\frac{E_F(r_i,n)}{E_G(r_i,n)} \geq \frac{E_F(n-1,n)}{E_G(n-1,n)}$$

which in turn implies

$$\frac{E_F(r_i,n)}{E_F(n-1,n)} \geq \frac{E_G(r_i,n)}{E_G(n-1,n)} \tag{2.4}$$

Thus, if inequality (2.3) holds under F, it must also hold under G, and the desired result follows. ■

Existence of a distribution for which a partition with two categories is optimal:

Proof. By the proof of Proposition 2.1, it is sufficient to show that there exists a distribution function for which condition (2.3) is satisfied. Consider $F(x) = x^{\frac{1}{w}}$ $, w > 1$. Then

$$E(r,n) = \frac{n!(w+r-1)!}{(r-1)!(n+w)!}$$

and

$$\frac{E(r,n)}{E(n-1,n)} = \frac{(n-2)!}{(r-1)!}\frac{(w+r-1)!}{(w+n-2)!}$$

It can be easily verified that $\lim_{w\to\infty} \frac{(w+r-1)!}{(w+n-2)!} = 0$. Therefore, for a sufficiently large w, condition (2.3) is satisfied, and the result follows. ■

Proof of Theorem 2.6

Proof. The designer's problem is:

$$\begin{aligned}
\max_{k,\{r_i\}_{i=1}^k,\{V_i\}_{i=1}^k} E_{total}^{(k)} &= \sum_{i=1}^{k-1}(n-r_i)(r_{i+1}-r_{i-1})E(r_i,n) + \sum_{i=1}^{k-1}(n-r_i)E(r_i,n)(V_{i+1}-V_i) \\
\text{subject to} &: \quad 1)\ \ 1 \leq k \leq n \\
&: \quad 2)\ \ \sum_{i=1}^{k}(r_i - r_{i-1})V_i = P \\
&: \quad 3)\ \ V_1 \geq n - r_1 \\
&: \quad 4)\ \ V_k \geq V_{k-1} \geq .... \geq V_1
\end{aligned}$$

Assume first that a given partition with k status categories is fixed. We derive the optimal allocation of money prizes consistent with such a partition. Subsequently, we find the optimal partition.

Note that $\frac{dE_{total}^{(k)}}{dV_1} < 0$, and therefore $V_1 = n - r_1$. The maximization problem reduces to:

$$\begin{aligned}
&\max_{\{V_i\}_{i=1}^k} \sum_{i=1}^{k-1}(n-r_i)(r_{i+1}-r_{i-1})E(r_i,n) + \sum_{i=1}^{k-1}(n-r_i)E(r_i,n)(V_{i+1}-V_i) \\
&\text{subject to: } \sum_{i=1}^{k}(r_i - r_{i-1})V_i = P \\
:\quad &V_k \geq V_{k-1} \geq ....V_1 = n - r_1
\end{aligned}$$

Assuming that all the constraints $V_k \geq .... \geq V_1 = n - r_1$ are binding, the Lagrangian is

$$\begin{aligned} L &= \sum_{i=1}^{k-1}(n-r_i)(r_{i+1}-r_{i-1})E(r_i,n) + \sum_{i=1}^{k-1}(n-r_i)E(r_i,n)(V_{i+1}-V_i) - \\ & \alpha_0(\sum_{i=1}^{k}(r_i-r_{i-1})V_i - P) + \sum_{i=1}^{k}\alpha_i(V_i-(n-r_1)) \end{aligned}$$

The first order conditions are

$$\frac{dL}{dV_i} = [(n-r_{i-1})E\left(r_{i-1},n\right) - (n-r_i)E\left(r_i,n\right)] - \alpha_0(r_i-r_{i-1}) - \alpha_i = 0, \quad i = 1, ..., k$$

The solution of this problem is:

$$\begin{aligned} V_{k-1} &= ... = V_1 = (n-r_1); \\ V_k &= \frac{P - r_{k-1}(n-r_1)}{n-r_{k-1}} \\ \alpha_0 &= E(r_{k-1},n); \\ \alpha_i &= [(n-r_{i-1})E\left(r_{i-1},n\right) - (n-r_i)E\left(r_i,n\right)] - \alpha_0(r_i-r_{i-1}), \; i = 1, .., k \end{aligned}$$

Note that :

$$\begin{aligned} \alpha_i &= [(n-r_{i-1})E\left(r_{i-1},n\right) - (n-r_i)E\left(r_i,n\right)] - \alpha_0(r_i-r_{i-1}) \\ &< (r_i-r_{i-1})(E(r_i,n) - E(r_{k-1},n)) \leq 0 \end{aligned}$$

That is, our assumption that all the constraints $V_{k-1} \geq ... \geq V_1 = n - r_1$ are binding ($V_k \geq n - r_1$ is not binding) was correct. Now, at the optimal solution, total effort is given by

$$E_{total}^{(k)} = \sum_{i=1}^{k-1}(n-r_i)(r_{i+1}-r_{i-1})E(r_i,n) + E(r_{k-1},n)(P - n(n-r_1))$$

For a partition with $k = 2$ with division point r_1', the above formula yields:

$$E_{total}^{(2)} = PE(r_1', n)$$

which is maximized for $r'_1 = n-1$. Noting that $\sum_{i=1}^{k-1}(n-r_i)(r_{i+1}-r_{i-1}) = n(n-r_1)$, and that for any k, $r_{k-1} \leq n-1$, we obtain that

$$\begin{aligned}
& E_{total}^{(2)} - E_{total}^{(k)} \\
= \quad & PE(n-1,n) - \left(\sum_{i=1}^{k-1}(n-r_i)(r_{i+1}-r_{i-1})E(r_i,n) + E(r_{k-1},n)(P-n(n-r_1))\right) \\
= \quad & P[E(n-1,n) - E(r_{k-1},n)] - \sum_{i=1}^{k-1}(n-r_i)(r_{i+1}-r_{i-1})[E(r_i,n) - E(r_{k-1},n)] \geq 0
\end{aligned}$$

Thus, a partition with two status categories where the top category contains a unique element is optimal. ■

2.7 References

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Chapter 3

Split-Award Auctions with Entry

3.1 Introduction

Split-award auctions (also known as second sourcing or multiple sourcing) refer to the practice in which the seller divides one object (contract) into several units (subcontracts) and each bidder can win at most one unit. Split-award auctions are popular in the sale of public assets, defence procurement and industrial practice. In US FCC spectrum auctions, the national market is divided into blocks and two licences are issued for each blocks. In European 3G auctions, several licences to the market are sold to different firms. In weapon procurement, two or more suppliers are awarded contracts if their bids are in "competitive" range (e.g. 10%). Woodside and Vyas (1987) study the purchasing strategies of six industrial firms and find that multiple sources are used for eight out of eighteen industrial products. All the purchasing agents they interviewed preferred to purchase from more than one source if possible, because "having more than one source is the cheapest way of buying insurance. If one source fails, you can fall back on the other." (p. 27).

Three arguments are commonly proposed in favor of split-award auctions. First, split-award auctions can help cultivate healthy competition in the long run. The government can set the number of winners to improve market structure of the particular industry. In the procurement setting, split-award enables the buyer to introduce yardstick competition to discipline the suppliers and smooth unanticipated surges in input demand. Second, split-award auction attract more bidders and enhance competition in bidding. Finally, it can be used as an instrument to favor disadvantaged bidders and level the playing field.

Split-award auctions have two distinct features. On one hand, the split-award auction is ex-post inefficient because the object is divided into several units and only one unit goes to the bidder who value them most. On the other hand, it can attract entry. In the split-award auction, the strong/advantageous bidder can win at most one unit, so several units are set aside for disadvantaged bidders. Therefore, it favors disadvantaged bidders and they will enter the auction more often. The combination of these two features is quite attractive when the seller (or buyer in procurement auctions) wants to favor disadvantaged bidders. For example, in FCC auctions, the Congress prefers to increasing the chance of winning of minorities. In government procurement auctions, the government favors domestic firms who might have higher costs than their foreign opponents. If all bidders are treated symmetrically as in the standard auctions, preferred disadvantaged bidders have low chance to win. However, if there are only one foreign firm and a bunch of domestic firms, the government can divide the original single contract into two or more smaller contracts, which will ensure that domestic firms win at least one contracts.[1]

These two features, however, have quite different implications for the seller's revenue. Allocation inefficiency hurts the seller's revenue while the entry effect improves it. As mentioned above, split-award auctions provide insurance and promote long-run competition. But an interesting question is: can it increase revenue in the short run? In other words, when will the entry effect dominates the allocation inefficiency? We show that split-award auctions could raise revenue or reduce procurement cost in the presence of both bidder asymmetry and endogenous entry.

[1] Various other instruments, such as bidding credits and set asides, are used in auction design to handicap strong bidders and help disadvantaged parties. Both bidding credits and set asides are used in FCC auctions to help minority bidders to win auctions (see Cramton (1997) for details). But these instruments have two defects. First, bidders are treated aymmetrically, which may violate domestic law or international treaty (e.g., WTO). Second, detailed information about bidder asymmetry is required for them to perform well, which is unlikely in many situations. In contrast, split-award auctions treat bidders symmetrically and information required is mild.

The goal of this paper is to justify split-award auctions in terms of revenue and highlight the importance of the interaction between asymmetry and entry in auction design. We want to compare the two auction formats used to sell two unit of homogenous goods: (1) standard second price auction: the highest bidder gets both units and pays the second highest bid; and (2) split-award auction (a multi-unit uniform second price auction): the two highest bidders each gets one unit and pays the highest losing bid. We show that the split-award is always inferior to the standard auction either when bidders are asymmetric or when entry is endogenous (but not both). When there is both bidder asymmetry and endogenous entry, we show that the split-award *could* generate higher revenue than standard second price auctions.

Although participation constraints are one of the two type of fundamental constraints that the designer faces in the mechanism literature, it is largely missing when specific auction formats, such as sealed-bid or open bid auctions, are discussed. Results on auctions with entry are quite limited (McAfee and McMillan, 1987a, 1987b, Levin and Smith, 1994, Ye, 2004 among others). More importantly, bidders are usually asymmetric in applications. Incumbent firms or suppliers have information advantage or cost advantage over new entrants or suppliers. A few recent studies on asymmetric auctions (Maskin and Riley 2000, and Krishna 2003) show that general insights gained from symmetric auctions (for example, the celebrated revenue equivalence theorem) cannot carry through under asymmetric setting. The interaction between bidder asymmetry and endogenous entry may have important implications, which are the focus of the current paper.

The literature on split-award auctions starts from the share auction (Wilson, 1979) and menu auction (Bernheim and Whinston,1987). In a study of the split-award auctions with two bidders, Anton and Yao (1989, 1992) show the split-award auctions are desirable when the technology is decreasing return to scale. Seshadri, Chatterjee and Lilien (1991) model the procurement as a first price auction and examine the tradeoff of multiple sourcing between low revenue and supply flexibility. The buyer chooses the number of winners to minimize total costs. However, they don't consider the entry issue and assume bidders are symmetric.

Perry and Sakovics (2003) study the split-award auction with entry. In their model, an order is spit into two contracts which are auctioned off in two sequential auctions. They assume entry is sequential and bidders are symmetric. The buyer chooses the relative size of the two contracts to minimize purchasing costs. In contrast, this paper assumes entry is simultaneous, bidders are asymmetric, and there is only one *multi-unit* auction. Athey, Levin and Seira (2004) compare open-price versus first price auctions theoretically and empirically with bidder asymmetry and endogenous entry. Gilbert and Klemperer (2000) model rationing with an interpretation of second sourcing. Both papers emphasize the interaction of bidder asymmetry and entry. Gilbert and Klemperer (2000) is closer to our paper in spirit. We will discuss the difference between their paper and our paper later.

A few papers study how other instruments (such as bidding credits and a bid cap) can help improve the seller's revenue. These papers share the same theme with the current paper that an *ex post* inefficient mechanism may turn out to be better for the seller *ex ante*. Based on the insights from optimal auction design by Myerson (1981), McAfee and McMillan (1989) analyze how bidding credit can help government reduce cost in procurement. They show that the government can minimize its purchasing cost by providing bidding credit to the weak bidders. Che and Gale (1998), Gavious, Moldovanu and Sela (2002) study how the bid cap can be used as an instrument to handicap strong bidders to increase the total lobby expenditure and total effort input respectively.[2]

The rest of the paper is arranged as follows. Section 3.2 sets up the model framework. Section 3.3 studies two benchmarks: auctions without entry and symmetric auctions with entry. In both benchmarks, the standard auction is always better than the split-award. The main results of the paper is contained in section 3.4, where we study asymmetric auctions with entry. We first define the type-symmetric equilibrium and show that it is unique under mild restrictions. Our numerical results show that the split-award auction

[2]The difference is that, Che and Gale study a setting with complete information and asymmetric bidders, while Gavious et al study a setting with incomplete information and symmetric bidders.

outperforms the standard auction for a wide range of parameters. Section 3.5 discusses the results and concludes the paper.

3.2 Model

A seller wants to sell a divisible object to $N+1$ potential bidders. The seller can either divide the object into two units and restrict each bidder to win at most one unit, or sell it as a single object. In principle, the seller could divide the object in an arbitrarily way. For simplicity, we focus on 50-50 split: if the seller choose to split the object, the resulted two units will be identical. There is no complementarity or substitutability between the two units,[3] so bidders' valuations for the units are additive.

There are two type of bidders: a group of weak bidders ($i = 1, .., N$) and a strong bidder ($S = N+1$). The weak bidders' valuations are independently drawn from a common distribution with support $[0,1]$, that is, $v_i \sim F(\cdot)$. The strong bidder's valuation is drawn from distribution $F(\cdot)^{\lambda}$ ($\lambda > 1$) with support $[0,1]$ .Therefore, the valuation distribution of the strong bidder stochastically dominates the valuation distribution of weak bidders.[4] For all $i = 1, .., N$, v_i and v_S are independent.

We consider the following two auction formats. The first format is the standard second price auction: the highest bidder gets the two units and pays the second highest bid. The other one is a split-award auction: a multi-unit uniform second price auction, In the split-award auction, the two highest bidders each gets one unit and pays the highest losing bid. We want to compare the revenue performance of the two auction formats with and without the interaction of bidder asymmetry and endogenous entry.

The timing of the game is following. First, the seller announces the auction format and commits to it. Next, potential bidders decide whether to incur a entry cost c to gather value information and enter the auction. By paying c, bidder i learns his private value for

[3]The no-complementarity assumption between the two objects is crucial for our model, but the no-substutability assumption can be dropped without affecting our results generated below

[4]This specific form of stochastic dominance is assumed to simplify the calculation in asymmetric auctions with entry.

the object, v_i or v_S, and submit a bid in the auction. If bidder i chooses not to incur cost c, then he will stay out of the auction. Finally, bids are revealed and the winners are chosen.

Therefore, the bidders' strategy can be characterized by two decision variables: entry decision and bidding decision. Let D_i $(i = S, W)$ denote the bidders' entry decision and let $b_i\,(i = S, W)$ denote bidders' bidding strategy once they enter. We will focus on the type-symmetric equilibrium defined as follows.

Definition 3.1 *A vector* (D_S, D_W, b_S, b_W) *is a type-symmetric equilibrium if and only if*

(1) the strong bidder uses strategy (D_S, b_S)*, and weak bidders use strategy* (D_W, b_W)*;*

(2) each bidder's bidding strategy (b_S *or* b_W) *maximizes their profits conditional on entry;*

(3) A bidder enters if and only if his expected profit from entry is higher than the cost c.

For the rest of the paper, we will argue that bidder asymmetry and entry are two important considerations for mechanism design. Their interaction may reverse the revenue ranking of the two auction formats we consider because the presence of strong bidders will make the entry effect more prominent. In order to isolate the two effects (allocation and entry) and highlight the importance of interaction of bidder asymmetry and endogenous entry, we first consider two benchmarks: auctions without entry and symmetric auctions with entry.

3.3 Benchmarks

In this section, we will study two benchmarks: one with asymmetric bidders and one with endogenous entry. The goal of this section is to show the neither bidder asymmetry nor endogenous entry alone can make split-award auctions superior to the standard second price auctions. But as we show in the next section, split-award auctions may perform better indeed when there is an interaction between bidder asymmetry and endogenous entry.

3.3.1 Auctions without Entry

The first benchmark considers the case where the number of bidders is fixed. So there is allocation inefficiency but no entry effect in the split-award auction.

Let $X^{(i:N+1)}$ denote the i-th order statistic of the $N+1$ random variables, $v_1, v_2, ..., v_N, v_{N+1}$. Note that $X^{(N+1:N+1)}$ represents the highest order statistic. The following proposition shows that it is never optimal for the seller to split the object.

Proposition 3.1 *In auctions without entry, the standard second auction generates higher revenue than the split-award auction.*

Proof. In the standard second auction, all bidders bid their true valuations for the two units. The highest bidder wins both units and pays the second highest bid. Therefore, the seller's revenue is[5]

$$\Pi^B = \mathbb{E}\left[2X^{(N:N+1)}\right] = 2\mathbb{E}\left[X^{(N:N+1)}\right].$$

In the split-award auction, all bidders bid their true valuations for one unit, and the top two bidders each wins one unit and pays the third highest bid. Thus, the seller's revenue is

$$\Pi^S = 2\mathbb{E}\left[X^{(N-1:N+1)}\right].$$

Since

$$\mathbb{E}\left[X^{(N:N+1)}\right] \geq \mathbb{E}\left[X^{(N-1:N+1)}\right],$$

we have $\Pi^B \geq \Pi^S$. ■

Because the number of bidders is fixed, the seller gains nothing in the short run by using the split-award auction. The allocation inefficiency hurts the seller's revenue, so the standard auction always outperforms split-award auction in (symmetric or asymmetric) auctions without entry. Thus, if the seller wants to use split-award auction to favor disadvantaged bidders or to gain long term benefits, she must sacrifice her revenue.

[5]The superscript "B" and "S" represent "bundle" and "split", respectively.

3.3.2 Symmetric Auctions with Entry

The second benchmark allows endogenous entry restricts bidders to be symmetric. Suppose there are N potential ex ante symmetric bidders and their valuations $\{v_1, ..., v_n\}$ are i.i.d. draws from the same distribution F. The seller first announces the auction format $m \in \{B, S\}$ that will be used to sell the object. And then bidders decide whether to incur an entry cost c and bid in the auction. If they enter then they will submit bids in the auction. Let n denote the number of bidders who enter the auction and bid. Since bidders are ex ante symmetric, we focus on symmetric mixed strategy equilibrium where all potential bidders choose to incur the cost c and enter the auction with some probability p.

Denote by $\mathbb{E}[\pi|n, m]$ each potential bidder's ex ante expected gain from entering, paying c, learning n, and bidding according to the symmetric undominated strategy implied by m and n. We assume that the entry cost c is moderate in the sense that it is profitable for some but not all potential bidders to enter the auction, that is $0 < n^* < N$.As showed in Levin and Smith (1994), $\mathbb{E}[\pi|n, m]$ is decreasing in n, so there are exists a unique integer, n^*, such that $\mathbb{E}[\pi|n^*, m] \geq 0 > \mathbb{E}[\pi|n^* + 1, m]$. Furthermore, there is a unique symmetric equilibrium where all potential bidders enter the auction with the same probability. First, we introduce a lemma proved in Levin and Smith (1994).[6]

Lemma 3.1 (Levin and Smith, 1994) *The second price auction with zero entry fee and zero reserve price generates optimal entry for the society and the seller.*

Let $X^{(n:n)}$ denote the highest order statistic among the valuations of n active bidders. When potential bidders enter the auction with probability p, the expected social surplus is

$$\sum_{n=1}^{N} \binom{N}{n} p^n (1-p)^{N-n} [2X^{(n:n)}] - pNc.$$

Denote p_B the probability of entry in the standard second price auction, and p_S the prob-

[6]See proposition 6 and 7 in Levin and Smith (1994).

ability of entry in the split-award auction. Then above lemma implies that

$$p_B \in \arg\max_p \left[\sum_{n=1}^{N} \binom{N}{n} p^n (1-p)^{N-n} [2X^{(n:n)}] - pNc \right].$$

Applying above results, we can show that standard auction is better when entry is endogenous but bidders are symmetric.

Proposition 3.2 *In symmetric second price auction with endogenous entry, the standard second price auction dominates the split-award auction in terms of seller's revenue.*

Proof. Again, let $X^{(n:n)}, X^{(n-1:n)}$ and $X^{(n-2:n)}$ denote the first, second and third highest order statistics among n valuations v_i, $i = 1, ..., n$. In equilibrium, bidders are indifferent between entering and not entering. Therefore, the expected gain from attending auction is zero for all bidders. As a result, the seller's revenue coincides with the social surplus. According to Lemma 3.1,

$$p_B \in \arg\max_p \left[\sum_{n=1}^{N} \binom{N}{n} p^n (1-p)^{N-n} [2X^{(n:n)}] - pNc \right]$$

Thus,

$$\begin{aligned} \Pi^B &= \sum_{n=1}^{N} \binom{N}{n} p_B^n (1-p_B)^{N-n} [2X^{(n:n)}] - p_B Nc \\ &\geq \sum_{n=1}^{N} \binom{N}{n} p_S^n (1-p_S)^{N-n} [2X^{(n:n)}] - p_S Nc \\ &\geq \sum_{n=1}^{N} \binom{N}{n} p_S^n (1-p_S)^{N-n} [X^{(n:n)} + X^{(n-1:n)}] - p_S Nc \\ &= \Pi^S. \end{aligned}$$

■

Hence, if bidders are symmetric, the standard auction is still better than the split-award in terms of revenue. In addition, the social surplus is higher in the standard auction than in the split-award. Thus, with endogenous entry alone, the revenue of the split-award auction is always lower than the standard second price auction.

3.3.3 Summary

To summarize, we have showed that the standard second price auction generates higher revenue than the split-award auction when the interaction between bidder asymmetry and endogenous entry is absent.

Proposition 3.1 shows that when entry is not an issue the standard second auction is always better in generating seller's revenue. Therefore, if the seller only cares about revenue, then she should bundle the two units and sell them together. If entry is an important concern but bidders are symmetric, then the standard auction is still better in terms of revenue, as showed in proposition 3.2. The intuition is the following. In the symmetric auctions with entry, the seller's revenue is equivalent to social surplus. The split-award auction indeed attracts more entry, but it is socially excessive because too many bidders pay entry cost. Under private value setting, standard second price auction generates the optimal entry, thereby the highest revenue. Thus, neither bidder asymmetry or endogenous entry alone can justify the use of the split-award auctions in terms of seller's revenue. It is optimal for the seller to sell the object as a single piece.

The effects of splitting the object on the seller's revenue can be summarized as follows:

Effects of Split-Award	Allocation Effect	Entry Effect	Gross Effect
Without Entry	Negative	/	Negative
Symmetry & Entry	Negative	Positive	Negative
Asymmetry & Entry	Negative	Positive	?

Table 3.1 Revenue effect of split-award

As we can see later, when there is an interaction between bidder asymmetry and endogenous entry, the gross effect of split-award on seller's revenue can be positive.

3.4 Asymmetric Auctions with Entry

In the previous discussion, we already show that standard auctions outperform split-award auctions in both benchmarks: auctions without entry and symmetric auctions with entry. This section will show that split-award auctions might perform better in asymmetric auctions with entry with a range of parameters.

For simplicity, we assume there is only one strong bidder and N potential weak bidders. The unit valuation of weak bidders $v_i(i = 1, ...N)$ are independently drawn from the distribution $F(\cdot)$ with support $[0, 1]$. The strong bidder's unit valuation v_S is drawn from $F^{\lambda}(\cdot)$ with support $[0, 1]$ and $\lambda > 1$.

We first characterize the unique type-symmetric equilibrium under mild restrictions. Then we use a numerical example to show that the split-award auction performs better than the standard second price auctions for a wide range of parameters when there are both bidder asymmetry and endogenous participation.

3.4.1 Existence and Uniqueness of the Type-Symmetric Equilibrium

We focus on type-symmetric equilibrium in which all weak bidders adopt the same mixed strategy when they make their entry decision. Let q denote the probability of entry of the strong bidder, and p denote the probability of entry of weak bidders. Denote by n the number of weak bidders who choose to enter. Finally, let $\pi_W(n, 1)$ denote a weak bidders' expected payoff from attending the auction when the stronger bidder and n weak bidders enter the auction, $\pi_W(n, 0)$ denote a weaker bidder's payoff when the strong bidder stays out, and $\pi_S(n)$ denote the strong bidder's expected payoff when n weak bidders enter.

Proposition 3.3 *Consider an asymmetric auction with one strong bidder and N potential weak bidders. Denote by p, q the entry probability of the weak bidders and the strong bidder respectively. If $0 < p \leq q \leq 1$, then there is a unique type-symmetric equilibrium with $q = 1$.*

Proof. (1) Existence. Fix $q = 1$, and a probability p such that weak bidders are indifferent between entering and staying out.[7] All bidders bid their true value after entry. Then (p, q) constitutes a type-symmetric equilibrium.

(2) Uniqueness. First notice that given $q = 1$ there is a unique p such that $\mathbb{E}_n [\pi_W(n, 1)] - c = 0$, because $\mathbb{E}_n [\pi_W(n, 1)]$ is monotonically decreasing in p.

Second, if $0 < p \leq q \leq 1$, then in the type-symmetric equilibrium, it must be $q = 1$. Suppose not, that is, $q < 1$. Then in equilibrium, both the strong bidder and the weak bidders earn expected payoff zero. That is,

$$\mathbb{E}_n [q\pi_W(n, 1) + (1 - q)\pi_W(n, 0)] - c = 0, \text{ and } \mathbb{E}_n [\pi_S(n)] - c = 0.$$

We want to show that these two equations cannot hold simultaneously, which means $q < 1$ cannot be true in equilibrium.

Define

$$\varphi(q) = \mathbb{E}_n [\pi_S (n)] - \mathbb{E}_n [q\pi_W(n, 1) + (1 - q)\pi_W(n, 0)]$$

Then

$$\begin{aligned}
\varphi(q) &= \sum_{n=0}^{N} \binom{N}{n} p^n (1-p)^{N-n} \pi_S(n, 1) - \sum_{n=1}^{N} \binom{N-1}{n-1} p^{n-1} (1-p)^{N-n} [q\pi_W(n, 1) + (1 - q)\pi_W(n, 0)] \\
&= \sum_{n=1}^{N} \binom{N}{n} p^n (1-p)^{N-n} \left\{ \pi_S(n, 1) - \frac{n}{Np} [q\pi_W(n, 1) + (1 - q)\pi_W(n, 0)] \right\} + (1-p)^N \pi_S(0, 1).
\end{aligned}$$

Note that

$$\frac{\partial \varphi(q)}{\partial q} = \sum_{n=1}^{N} \binom{N-1}{n-1} p^{n-1} (1-p)^{N-n} [\pi_W(n, 0) - \pi_W(n, 1)] > 0.$$

[7] Because as one can show, a weaker bidder's expected payoff is decreasing in p. Therefore, there is a unique p to make weak bidders to be indifferent between participation and staying out.

In addition,

$$
\begin{aligned}
& \varphi(p) \\
= & \sum_{n=1}^{N}\binom{N}{n}p^{n}(1-p)^{N-n}\left\{\pi_S(n,1)-\frac{n}{Np}\left[p\pi_W(n,1)+(1-p)\pi_W(n,0)\right]\right\}+(1-p)^N\pi_S(0,1) \\
> & \sum_{n=1}^{N}\binom{N}{n}p^{n}(1-p)^{N-n}\left\{\frac{N-n}{N}\pi_S(n,1)-\frac{n\,(1-p)}{Np}\pi_W(n,0)\right\}+(1-p)^N\pi_S(0,1) \\
= & \sum_{n=1}^{N-1}\binom{N-1}{n}p^{n}(1-p)^{N-n}\pi_S(n,1)-\sum_{n=1}^{N}\binom{N-1}{n-1}p^{n-1}(1-p)^{N-n+1}\pi_W(n,0)+(1-p)^N\pi_S(0,1) \\
= & \sum_{n=1}^{N-1}\binom{N-1}{n}p^{n}(1-p)^{N-n}\pi_S(n,1)-\sum_{t=0}^{N-1}\binom{N-1}{t}p^{t}(1-p)^{N-t}\pi_W(t+1,0)+(1-p)^N\pi_S(0,1) \\
= & \sum_{n=1}^{N-1}\binom{N-1}{n}p^{n}(1-p)^{N-n}\left[\pi_S(n,1)-\pi_W(n+1,0)\right]+(1-p)^N\left[\pi_S(0,1)-\pi_W(1,0)\right] \\
> & 0
\end{aligned}
$$

The last inequality holds because $\pi_S(n,1)\geq\pi_W(n+1,0)$ and $\pi_S(0,1)\geq\pi_W(1,0)$ by stochastic dominance of bidders' valuation distributions. Since $0<p\leq q\leq 1$ and $\varphi'(\cdot)>0$, $\varphi(q)\geq\varphi(p)>0$. Therefore, the two equations cannot hold simultaneously in equilibrium. Hence, $q=1$. ■

The intuition of the proof is the following. If weak bidders are indifferent between entry and staying out, the strong bidder must prefer to entering for sure, because his valuation stochastically dominates the valuations of weak bidders. However, the restriction $0<p\leq q\leq 1$ is important for the uniqueness. If we allow weak bidders to enter more frequently than the strong bidders, then alternative equilibrium is possible. For example, suppose there one weak bidder and one strong bidder, and assume that it is profitable for one bidder to enter only when the other stays out. Then the following entry strategy could be an equilibrium: the weak bidder chooses to enter for sure, while the strong bidder choose to stay out for sure.

3.4.2 Revenue Comparison: Numerical Results

Now we want to compare the performance of the two auction formats: the standard second price auction and the split-award auction. Ideally, one would like to provide analytically a range of parameters under which the split-award auction performs better. Unfortunately, we are unable to do that. Instead, we provide some numerical results to illustrate that the split-award auction does perform better for a wide range of parameters.

In order to calculate the seller's revenue, we need to derive the distribution of order statistics for bidders' valuations. Pool the strong bidder's valuation V_S and n weak bidders' valuations $(V_1, ..., V_n)$ together, and denote the order statistic as $X^{(1:n+1)}, X^{(2:n+1)}, ..., X^{(n+1:n+1)}$. In addition, we denote by $Z^{(1:n)}, Z^{(2:n)}, ..., Z^{(n:n)}$ the order statistic for the weak bidders' valuations $V_1, ..., V_n$.Let $G^{(n:n)}(x), G^{(n-1:n)}$ and $G^{(n-2:n)}$ are the cumulative distribution of $Z^{(n:n)}$, $Z^{(n-1:n)}$, $Z^{(n-2:n)}$ respectively, and $H^{(n+1:n+1)}, H^{(n:n+1)}$ and $H^{(n-1:n+1)}$ are the distribution of $X^{(n+1:n+1)}$, $X^{(n:n+1)}$ and $X^{(n-1:n+1)}$ respectively. The G's and H's are derived in the Appendix. Those distributions are extensively used in proving the following two lemmas.

Lemma 3.2 *In standard second price auction with one strong bidder and n weak bidders, the seller's expected revenue is*

$$\Pi^B(n) = 2\left[1 - \int_0^1 \{F(x)^n + n[1 - F(x)]F(x)^{\lambda+n-1}\}dx\right]$$

The strong bidder's expected profit is

$$\pi_S^B(n) = 2\int_0^1 F(x)^n[1 - F(x)^\lambda]dx$$

Each weak bidder's expected profit is

$$\pi_W^B(n) = 2\int_0^1 F(x)^{n+\lambda-1}[1 - F(x)]dx$$

Proof. See Appendix. ■

Similarly, we can calculate the payoffs to the three parties for the split-award auction with one strong bidder and n weak bidders.

Lemma 3.3 *In split-award auction with one strong bidder and* n *weak bidders, the seller's expected revenue is*

$$\Pi^S(n) = 2 - \int_0^1 \left\{ 2F(x)^n + 2nF(x)^{n-1}[1 - F(x)] + n(n-1)F(x)^{\lambda+n-2}[1 - F(x)]^2 \right\} dx$$

The strong bidder's expected profit is

$$\pi_S^S(n) = \int_0^1 [n - (n-1)F(x)][1 - F(x)^\lambda]F(x)^{n-1}dx$$

Each weak bidder's expected profit is

$$\pi_W^S(n) = \int_0^1 \left\{ 1 + (n-1)F(x)^{\lambda-1}[1 - F(x)] \right\} [1 - F(x)]F(x)^{n-1}dx$$

Proof. See Appendix. ■

Denote by p_B the equilibrium probability of entry in the standard auction, and p_S the equilibrium probability of entry in the split-award auction. Then the indifference conditions of both auctions pin down p_B and p_S, respectively:

$$\text{Standard Auction} \quad : \quad \sum_{n=1}^{N} \binom{N-1}{n-1} p_B^{n-1}(1-p_B)^{N-n}\pi_W^B(n) - c = 0$$

$$\text{Split-Award Auction} \quad : \quad \sum_{n=1}^{N} \binom{N-1}{n-1} p_S^{n-1}(1-p_S)^{N-n}\pi_W^S(n) - c = 0$$

From the equilibrium condition, we can solve the equilibrium entry probability p_B and p_S.

With p_B and p_S, we can calculate the seller's revenue in two auction formats:

$$E\Pi^B = \sum_{n=1}^{N} \binom{N}{n} p_B^n (1-p_B)^{N-n} \pi^B(n)$$
$$E\Pi^S = \sum_{n=2}^{N} \binom{N}{n} p_S^n (1-p_S)^{N-n} \pi^S(n)$$

Then we can compare seller's revenue under different parameter profile (λ, c, N).

We cannot solve the equilibrium entry probability analytically, but we can use numerical example to illustrate when the split-award auction may outperform the standard second price auction. To simplify the numerical computation, we assume uniform valuation distribution for the weaker bidders' valuation, that is, $F(x) = x, F(x)^\lambda = x^\lambda$.

First, we investigate how the entry probability and the seller's revenue vary with different level of entry cost c, holding the asymmetry level λ, the number of potential bidders N.

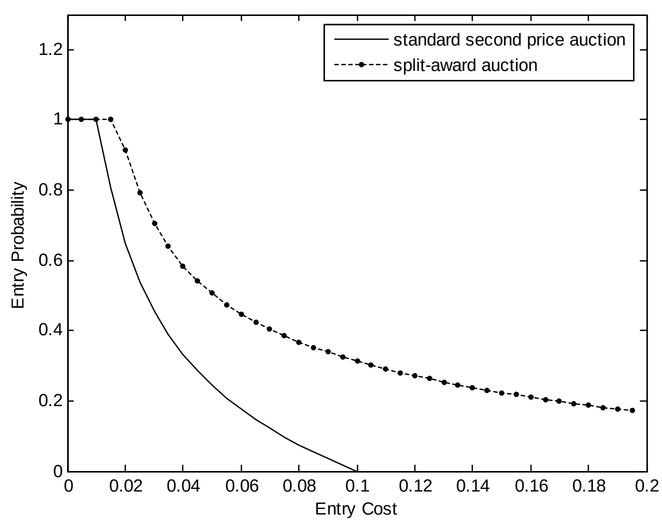

Figure 3.1: Entry probability and entry cost $(N = 10, \lambda = 3)$

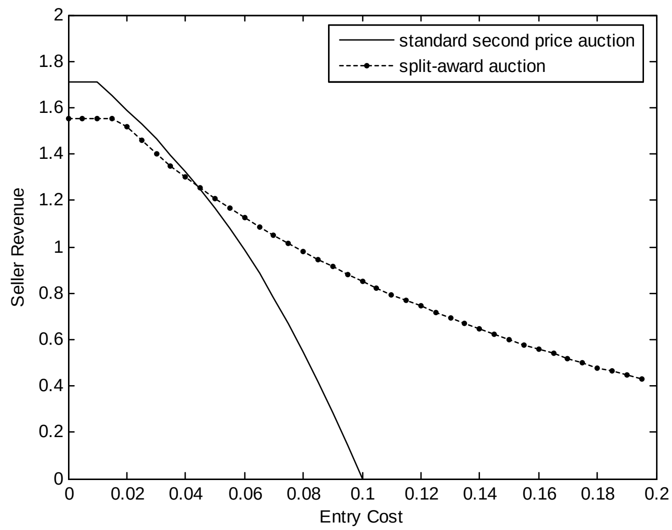

Figure 3.2: Seller's revenue and entry cost $(N = 10, \lambda = 3)$

It is clear from above graphs that the split-award auction is better when entry cost is higher. Moderately high entry cost means that bidders' entry decision is not trivial. So that the presence of a strong bidder has strong effect on the weak bidder's incentive to enter. As we can see from the right graph, the entry probability of the split-award auction is systematically higher than the standard second price auction. From the left graph, for the low entry cost, the entry induced by the split-award is excessive so the seller's revenue is lower. But when the entry cost is high, split-award did a pretty good job in attracting weak bidder to increase competition in the auction. The pattern is robust to the number of potential bidders N.

Next, we investigate how entry probability and seller's revenue change with different level of asymmetry λ, holding entry cost and the number of potential bidders fixed.

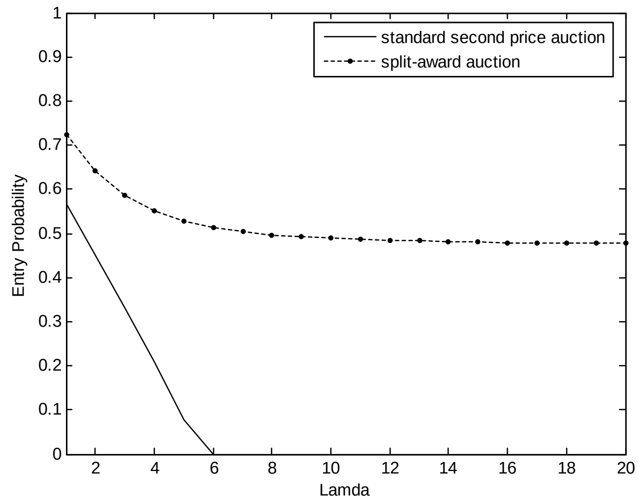

Figure 3.3: Entry probability and bidder asymmetry $(N = 10, c = 0.04)$

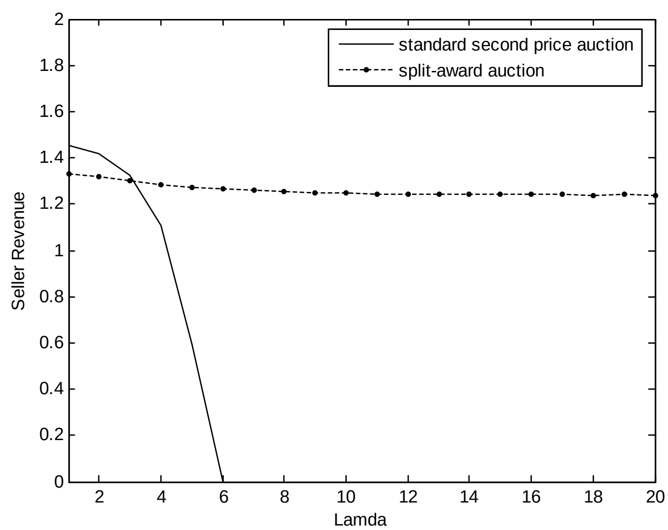

Figure 3.4: Seller's revenue and bidder asymmetry $(N = 10, c = 0.04)$

As we can see, the split-award auction is better when bidder asymmetry is strong. This is intuitive given our results on symmetric auctions with entry. When bidders are almost symmetric, the standard second price auction is close to optimal. So bundling the two unit together is optimal. On the other hand, if bidders are strongly asymmetric, then the participation of the weak bidders will be a problem. In order to encourage them to enter, the seller can use split-award to set aside one unit for them and thus induce them to enter with higher probability.

To summarize, we show that neither bidder asymmetry or endogenous entry alone can justify the use of split-award auction in terms of seller's revenue. But the coexistence of bidder asymmetry and endogenous participation amplifies the positive entry effect on seller's revenue, which makes the split-award auction attractive for a broad range of parameters.

3.5 Conclusion and Discussion

Most of the existing literature impose two assumptions: bidder symmetry and a fixed number of bidders. But in reality, both assumptions are not realistic. This paper shows that the interaction of asymmetry and entry has important implications in auction design. We show without the interaction between asymmetry and entry, the standard second price auction always generates higher revenue than the split-award auction. But when the interaction is important, the reverse could be true.

Our paper is closely related to Gilbert and Klemperer (2000). In their paper, a monopolist commits a pricing policy to sell one object to two potential buyers. Each buyer independently chooses whether to incur a seller-specific sunk investment c, success with some probability. They publicly report whether the investment is successful. If successful, the strong(weak) bidder has valuation v_S (v_W) and chooses whether to buy. They find that committing a single price across all demand states and rationing when necessary can be more profitable than the market-clearing price schedule. In their setting, rationing arises when in three circumstances: first, the inefficiency caused by rationing is small, i.e., the level of asymmetry is low; second, entry cost is high; and third, the weak bidder gains more from the rationing in high demand than a low price in low demand.

There are several important differences between their paper and ours. In our model, the price is formed through the bidding process which depends on the bidders' entry decision. In contrast, in their model, the seller determines the price and because consumers' valuation is binary the seller's pricing decision is quite simple. Moreover, when bidders are symmetric, rationing is ex-post efficient in Gilbert and Klemperer (2000), but split-award is not efficient in our setting. In addition, our model shows that rationing should not appear in the absence of bidder asymmetry, but in their model, rationing arises with symmetric bidders. Finally, in our model, higher level of asymmetry facilitates rationing while in their model, higher level of asymmetry hinders rationing.

This paper, however, shows the desirability of the split-award auction only through numerical examples. One important line of future research is to investigate whether the optimal auction involves split-award, when both entry and bidder asymmetry are important.

3.6 Appendix

Here we will review a few standard results for order statistics, and then provide proofs for Lemma 3.2 and 3.3 in the text.

Distributions for Order Statistics

Recall that we use $X^{(1:n+1)}, X^{(2:n+1)}, ..., X^{(n+1:n+1)}$ are the order statistics of pooled valuations, $(v_S, v_1, ..., v_n)$ and $Z^{(1:n)}, Z^{(2:n)}, ..., Z^{(n:n)}$ are order statistics for the weak bidders' valuations $(v_1, ..., v_n)$.

Let $G^{(n:n)}(x)$ denote the cumulative distribution of the first highest order statistic $Z^{(n:n)}$,then $G^{(n:n)}(x) = F(x)^n$. For the second highest order statistic $Z^{(n-1:n)}$, we can derive its distribution $G^{(n-1:n)}(x)$ as follows:

$$\begin{aligned} G^{(n-1:n)}(x) &= \Pr(Z^{(n-1:n)} \leq x) \\ &= \Pr(\text{at least } n-1 \text{ of the } v_i \text{ are less than or equal to } x) \\ &= \sum_{j=n-1}^{n} \binom{n}{j} F(x)^j [1-F(x)]^{n-j} \\ &= nF(x)^{n-1}[1-F(x)] + F(x)^n \\ &= nF(x)^{n-1} - (n-1)F(x)^n \end{aligned}$$

Let $H^{(n+1:n+1)}(x)$ denote the distribution for the order statistic $X^{(n+1:n+1)}$, then

$$H^{(n+1:n+1)}(x) = \Pr(X^{(n+1:n+1)} \leq x) = F(x)^{\lambda+n}$$

The distribution of the second highest order statistic $X^{(n:n+1)}$ is quite involved. We know, the event that $X^{(n:n+1)}$ is less than or equal to x is the union of: (1) all v_S and v_i's are less than or equal to x; (2) v_S is greater than x and all v_i's are less than or equal to x; (3) v_S is less than x, $(n-1)$ of the v_i's are less than or equal to x, and one of v_i's is greater than x.

$$\begin{aligned} H^{(n:n+1)}(x) &= \Pr(X^{(n:n+1)} \leq x) \\ &= F(x)^{\lambda+n} + [1-F(x)^{\lambda}]F(x)^n + n[1-F(x)]F(x)^{\lambda+n-1} \\ &= F(x)^n + n[1-F(x)]F(x)^{\lambda+n-1} \end{aligned}$$

Similarly, we can calculate the distribution of the third highest order statistic $X^{(n-1:n+1)}$. The event that $X^{(n-1:n+1)}$ is less than or equal to x is the union of: (1) all v_S and v_i's are less than or equal to x; (2) v_S is less than x and all v_i's except one are less than or equal to x; (3) v_S is less than x, all v_i's except two are less than or equal to x;(4) v_S is greater than x, and all v_i's are less than x; (5) v_s is greater than x, all v_i's except one are less than or equal to x.That is,

$$\begin{aligned} H^{(n-1:n+1)}(x) &= \Pr\{X^{(n-1:n+1)} \leq x\} \\ &= F(x)^{\lambda+n} + nF(x)^{\lambda+n-1}[1-F(x)] + \frac{n(n-1)}{2}F(x)^{\lambda+n-2}[1-F(x)]^2 \\ &\quad +F(x)^n[1-F(x)^\lambda] + nF(x)^{n-1}[1-F(x)][1-F(x)^\lambda] \\ &= F(x)^n + nF(x)^{n-1}[1-F(x)] + \frac{n(n-1)}{2}F(x)^{\lambda+n-2}[1-F(x)]^2 \end{aligned}$$

Proof of Lemma 3.2:

The seller's expected revenue is

$$\begin{aligned} \Pi^B(n) &= 2\mathbb{E}[Y^{(n:n+1)}] = 2\int_0^1 x dH^{(n:n+1)}(x) \\ &= 2\int_0^1 x[nF(x)^{n-1} + n(\lambda+n-1)F(x)^{\lambda+n-2} - n(\lambda+n)F(x)^{\lambda+n-1}]f(x)dx \\ &= 2\left[1 - \int_0^1 \{F(x)^n + n[1-F(x)]F(x)^{\lambda+n-1}\}dx\right] \end{aligned}$$

For the strong bidder, he pays the highest bid $2Z^{(n:n)}$ among the n weak bidders if he wins. The distribution of $Z^{(n:n)}$ conditional on the strong bidder's winning is:

$$F(x|Z^{(n:n)} < v_S) = \frac{F(x)^n}{F(v_S)^n}$$

Therefore, expected payment for strong bidder with valuation v_S is,

$$\mathbb{E}[Z^{(n:n)}|Z^{(n:n)} \leq v_S] = \frac{1}{F(v_S)^n}\int_0^{v_S} xnF(x)^{n-1}f(x)dx = v_S - \frac{\int_0^{v_S} F(x)^n dx}{F(v_S)^n}$$

And the expected profit of the strong bidder is,

$$\begin{aligned} \pi_S(n) &= 2\int_0^1 F(v_S)^n\{v_S - \mathbb{E}[Z^{(n:n)}|Z^{(n:n)} \leq v_S]\}dF(v_S)^\lambda \\ &= 2\int_0^1 F(x)^n[1-F(x)^\lambda]dx \end{aligned}$$

For weak bidder i, its expected payment is $\frac{1}{n}$ share of the difference between seller's revenue and the strong bidder's expected payment. The ex-ante expected payment for the strong bidder is

$$\begin{aligned}
\mathbb{E}[t_S|n] &= 2\int_0^1 F(v_S)^n \mathbb{E}(Z^{(n:n)}|Z^{(n:n)} \leq v_S) dF(v_S)^\lambda \\
&= 2\int_0^1 F(v_S)^n v_S dF(v_S)^\lambda - \pi_S(n) \\
&= 2\left\{\frac{\lambda}{n+\lambda} - \int_0^1 F(x)^n [1 - \frac{n}{n+\lambda} F(x)^\lambda] dx\right\}
\end{aligned}$$

Then the expected payment for bidder i is,

$$\begin{aligned}
\mathbb{E}[t_W|n] &= \frac{1}{n}(\Pi^B(n) - \mathbb{E}[t_S|n]) \\
&= \frac{2}{n}\left\{1 - \int_0^1 [F(x)^n + n[1-F(x)]F(x)^{\lambda+n-1}]dx - \frac{\lambda}{n+\lambda} + \int_0^1 F(x)^n[1 - \frac{n}{n+\lambda}F(x)^\lambda]dx\right\} \\
&= 2\left\{\frac{1}{n+\lambda} - \frac{1}{n}\int_0^1 F(x)^{n+\lambda-1}[n(1-F(x)) + \frac{n}{n+\lambda}F(x)]dx\right\}
\end{aligned}$$

His profit is equal to the expected value of the object conditional on winning minus his expected payment. That is,

$$\begin{aligned}
\pi_W(n) &= \frac{1}{n}\int_0^1 xF(x)^\lambda dF(x)^n - \mathbb{E}[t_W|n] = \frac{1}{n}\int_0^1 nxF(x)^{\lambda+n-1}dF(x) - \mathbb{E}[t_W|n] \\
&= \frac{1}{n}\{\left[x\frac{nF(x)^{\lambda+n}}{\lambda+n}\right]_0^1 - \int_0^1 \frac{n}{\lambda+n}F(x)^{\lambda+n}dx\} - \mathbb{E}[t_W|n] \\
&= \int_0^1 F(x)^{n+\lambda-1}[1-F(x)]dx
\end{aligned}$$

This completes the proof. ■

Proof of Lemma 3.3:

In the split-award auction, each bidder is restricted to buy only one unit. The two highest bidders win the objects and pay the price of the third highest bid. The transaction price is the third highest order statistic $Y^{(n-1:n+1)}$,which has distribution

$$H^{(n-1:n+1)}(x) = F(x)^n + nF(x)^{n-1}[1-F(x)] + \frac{n(n-1)}{2}F(x)^{\lambda+n-2}[1-F(x)]^2$$

Hence, the seller's expected revenue is

$$\begin{aligned}
\Pi^S(n) &= 2E[Y^{(n-1:n+1)}] = 2\int_0^1 xdH^{(n-1:n+1)}(x) = 2[1-\int_0^1 H^{(n-1:n+1)}(x)dx] \\
&= 2-\int_0^1 \left\{2F(x)^n + 2nF(x)^{n-1}[1-F(x)] + n(n-1)F(x)^{\lambda+n-2}[1-F(x)]^2\right\} dx
\end{aligned}$$

Now consider the expected profits of the strong bidder. If the strong bidder wins, he pays the second highest bid among the n weak bidders, that is $Z^{(n-1:n)}$. The distribution of $Z^{(n-1:n)}$ conditional on the strong bidder's winning is :

$$F(x|Z^{(n-1:n)} < v_S) = \frac{G^{(n-1:n)}(x)}{G^{(n-1:n)}(v_S)}$$

Then, the expected payment of the strong bidder with valuation v_S is,

$$\mathbb{E}[Z^{(n-1:n)}|Z^{(n-1:n)} \leq v_S] = \frac{1}{G^{(n-1:n)}(v_S)}\int_0^{v_S} xdG^{(n-1:n)}(x) = v_S - \frac{\int_0^{v_S} G^{(n-1:n)}(x)dx}{G^{(n-1:n)}(v_S)}$$

And the expected profit of the strong bidder is,

$$\begin{aligned}
\pi_S(n) &= \int_0^1 G^{(n-1:n)}(v_S)\{v_S - \mathbb{E}[Z^{(n-1:n)}|Z^{(n-1:n)} \leq v_S]\}dF(v_S)^\lambda \\
&= \int_0^1 G^{(n-1:n)}(v_S)\frac{\int_0^{v_S} G^{(n-1:n)}(x)dx}{G^{(n-1:n)}(v_S)}dF(v_S)^\lambda \\
&= \left[F(v_S)^\lambda \int_0^{v_S} G^{(n-1:n)}(x)dx\right]_0^1 - \int_0^1 F(v_S)^\lambda G^{(n-1:n)}(v_S)dv_S \\
&= \int_0^1 G^{(n-1:n)}(x)[1-F(x)^\lambda]dx \\
&= \int_0^1 [nF(x)^{n-1} - (n-1)F(x)^n][1-F(x)^\lambda]dx
\end{aligned}$$

Next, consider weak bidders. The total expected payment from bidder weakers is equal to the difference between seller's revenue and strong bidder's expected payment. The ex-ante expected payment for the strong bidder is

$$\begin{aligned}\mathbb{E}[t_S|n] &= \int_0^1 G^{(n-1:n)}(v_S)\mathbb{E}(Z^{(n-1:n)}|Z^{(n-1:n)} \leq v_S)dF(v_S)^\lambda \\ &= \int_0^1 G^{(n-1:n)}(v_S)v_S dF(v_S)^\lambda - \pi_S(n) \\ &= \int_0^1 \lambda x G^{(n-1:n)}(x)F(x)^{\lambda-1}dF(x) - \int_0^1 G^{(n-1:n)}(x)[1-F(x)^\lambda]dx\end{aligned}$$

Then the expected payment from weak bidders is,

$$\begin{aligned}& n\mathbb{E}[t_W|n] \\ =\ & \Pi^S(n) - \mathbb{E}[t_S|n] \\ =\ & 2[1-\int_0^1 H^{(n-1:n+1)}(x)dx] - \left\{\int_0^1 \lambda x G^{(n-1:n)}(x)F(x)^{\lambda-1}dF(x) - \int_0^1 G^{(n-1:n)}(x)[1-F(x)^\lambda]dx\right\}\end{aligned}$$

Its profit is equal to the expected value of the object conditional on winning minus the expected payment. That is,

$$\begin{aligned}n\pi_W(n) &= \int_0^1 x dG^{(n:n)}(x) + \int_0^1 xF(x)^\lambda dG^{(n-1:n)}(x) - nE[t_W|n] \\ &= 1 - \int_0^1 F(x)^n dx + 1 - \int_0^1 F(x)^\lambda G^{(n-1:n)}(x)dx - \int_0^1 \lambda x F(x)^{\lambda-1}G^{(n-1:n)}(x)dF(x) \\ &\quad -2[1-\int_0^1 H^{(n-1:n+1)}(x)dx] + \int_0^1 \lambda x G^{(n-1:n)}(x)F(x)^{\lambda-1}dF(x) \\ &\quad -\int_0^1 G^{(n-1:n)}(x)[1-F(x)^\lambda]dx \\ &= 2\int_0^1 H^{(n-1:n+1)}(x)dx - \int_0^1 F(x)^n dx - \int_0^1 G^{(n-1:n)}(x)dx \\ &= n\int_0^1 \left\{F(x)^{n-1}[1-F(x)] + (n-1)F(x)^{\lambda+n-2}[1-F(x)]^2\right\}dx \\ &= n\int_0^1 \left\{1+(n-1)F(x)^{\lambda-1}[1-F(x)]\right\}[1-F(x)]F(x)^{n-1}dx.\end{aligned}$$

Thus, the proof is complete. ■

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