Typical Types

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Abstract

Harsanyi (1967/68) introduced higher order belief types in order to capture all possible uncertainty of Bayesian players about payoffs, other players' beliefs about payoffs, and so on. It is natural to say that two such higher order belief types are close if they behave similarly in strategic situations and obtain similar outcomes. I argue that all the types in most of the type spaces studied in the economics and game theory literature are nongeneric with respect to this strategic topology on higher order belief types. For example, all the types in finite type spaces where players' beliefs are derived from a common support and all the types in continuum type spaces with uniformly bounded densities belong to a nongeneric set.

1. Introduction

Economists usually represent incomplete information by assuming a fixed set of types for each player. Each type has beliefs about states of the world and other players' types. Each mapping from a player's types into beliefs over states of the world and other players' types are informally assumed to be common knowledge.

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A natural question to ask is: what are reasonable assumptions to impose on players' beliefs in this setting?

One response is to assume that players' beliefs are derived from a common prior on the fixed type space and to assume the common prior is "generic".[1] This is a sensible view to take *if* one takes literally the view that type sets and common prior are publicly revealed to the players.

However, one standard justification for assuming common knowledge of the type sets and their beliefs over others' types is Harsanyi's (1967/68) argument that there is a canonical way of representing players' possible types that makes the common knowledge assumptions vacuous. Harsanyi proposed that a player's type should be a description of his beliefs about states of the world (payoff-relevant events), his beliefs about other players' beliefs about states of the world, and so on.

With this canonical representation of types, the fixed type spaces often studied in economics can be understood as subsets of the space of all possible higher order belief types. It is natural to ask if the standard type spaces are "representative" (or "generic", or "typical") of higher order belief types as a whole. In order to do this, one must ask what is a natural topology to impose on the higher order belief types. A mathematically natural topology to employ is the product topology: a sequence of higher order belief types converges if each level of higher order beliefs converges.[2] This implies that what happens in the tail of the sequence of higher order beliefs does not much matter. But if one is interested in type spaces to study strategic problems, it is natural to say that two types are close if they behave similarly and/or obtain similar outcomes in a variety of strategic problems. The one consistent lesson from the whole literature on strategic behavior and higher order beliefs is that tails do matter. In other words, there is a large difference between an event being common knowledge and that event being mutual knowledge to some large finite level.[3]

But there is nothing about the assumption of Bayesian rationality that limits the tail behavior of higher order beliefs. To illustrate this point, consider the simple case where there are two states of the world, 0 and 1, and two players, 1 and 2. We can ask what is player 1's expected value of the state; what is player 1's expectation of player 2's expectation of the state; what is player 1's expectation of player 2's expectation of player 1's expectation of the state; and so on. This will generate a sequence of numbers between zero and one. Any higher order belief type of player 1 can thus be mapped into such sequences. In section 2, we

[1]This approach is quite common in the literature. See, for example, Cremer and McLean (1988).

[2]This topology is employed in this context by Mertens and Zamir (1985) and Lipman (2001).

[3]As in, for example, Geanakoplos and Polemarchakis (1982) and Rubinstein (1989).

will describe a class of games where two types will behave similarly and obtain similar outcomes if and only if they are close in the uniform topology of higher order expectations, i.e., if the supremum of the differences between their higher order expectations is small. In section 3, we will show that any sequence of higher order expectations might emerge on common knowledge type spaces. However, most standard common knowledge type spaces studied in the economics literature have the property that such higher order expectations converge either to a point or to a cycle. This includes finite type spaces and continuum type spaces with uniformly bounded densities.

An interpretation of this result, a discussion of related literature and conclusions are postponed to a final section.

2. The Uniform Topology on Higher Order Expectations

2.1. The Higher Order Belief Types

Consider a two player higher order belief type space construction with two payoff relevant states, $S = \{0, 1\}$. A player's *first order belief* is his belief about the states, S. His *second order belief* is his belief about the states S and the first order belief of the other player. His *nth order belief* (for any $n \geq 2$) is his belief about the states S and the $(n-1)$th belief of the other player. A type of a player specifies his nth order beliefs, for all $n \geq 1$, with the property that beliefs at different levels are *coherent*: for any n, beliefs at two different levels higher than n, projected onto the space of beliefs about states S and the nth level beliefs, are always the same. Formally, write $\Delta(X)$ for set of probability measures over the set X. A type $t = (\delta_1, \delta_2, ...) \in \times_{n=0}^{\infty} \Delta(X_n)$, where $X_0 = S$ and $X_n = S \times \Delta(X_{n-1})$. Write T for the set of coherent types.

2.2. Higher Order Expectations Types

Here we describe one very simple notion of closeness of types that is easy to visualize. For each type t, let $\widehat{\xi}_n(t)$ be the nth order expectation of $s \in S$. Thus $\widehat{\xi}_1(t)$ is type t's expected value of s; $\widehat{\xi}_2(t)$ is type t's expected value of the other player's expected value of s; $\widehat{\xi}_3(t)$ is type t's expected value of the other player's expected value of his expected value of s; and so on. Thus if $t = (\delta_1, \delta_2, ...)$,

$$\begin{aligned}
\widehat{\xi}_1(t) &= \widetilde{\xi}_1(\delta_1) = \delta_1[\{1\}] \\
\widehat{\xi}_2(t) &= \widetilde{\xi}_2(\delta_2) = \int_{(s,\delta_1') \in \{0,1\} \times \Delta(\{0,1\})} \widetilde{\xi}_1(\delta_1')\, d\delta_2 \\
&\cdots
\end{aligned}$$

$$\widehat{\xi}_n(t) = \widetilde{\xi}_n(\delta_n) = \int\limits_{(s,\delta'_{n-1})\in S\times\Delta(X_{n-2})} \widetilde{\xi}_{n-1}(\delta'_{n-1})\, d\delta_n$$

Thus

$$\widehat{\xi} : T \to [0,1]^{\infty}.$$

While universal types are nasty objects, we may sometimes be able to focus on such iterated expectations represented by a sequence of numbers between 0 and 1.

2.3. The Uniform Topology

Now one natural topology on types is generated by applying the uniform topology to the projection of T onto $[0,1]^{\infty}$. Thus we have pseudo-metric d with

$$d(t,t') = \sup_n \left|\widehat{\xi}_n(t) - \widehat{\xi}_n(t')\right|.$$

We write $t^k \to_{\xi} t$ if $d(t^k, t) \to 0$. We refer to the induced topology as the uniform topology on higher order expectations.

2.4. The Higher Order Expectations (HOE) Game

We will consider a particularly simple game where it is possible to characterize behavior as a function of higher order beliefs. The game parameterized by

$$\lambda \in \left\{\lambda = (\lambda_0, \lambda_1, ...) \in \mathbb{R}^{\infty}_{++} : \sum_{n=0}^{\infty} \lambda_n = 1\right\} \equiv \Lambda. \tag{2.1}$$

Each of two players picks an action $a_i \in [0,1]^{\infty}$. Player i's payoff is

$$u_i(a_i, a_j, s) = -\lambda_0 (a_{i1} - s)^2 - \sum_{n=1}^{\infty} \lambda_n (a_{i,n+1} - a_{jn})^2.$$

Now consider the incomplete information game, where payoffs are parameterized by $\lambda \in \Lambda$ and players' higher order beliefs about s are described by their universal types. A pure strategy in this game is a function $\sigma : T \to [0,1]^{\infty}$. This game has a unique equilibrium where (independent of $\lambda \in \Lambda$) each player sets his action equal to $\widehat{\xi}(t)$. In fact, this is the unique strategy profile surviving iterated deletion of strictly dominated strategies, since by induction on n, we have that if pure strategy σ survives n rounds of iterated deletion, then $\sigma_k(t) = \xi_k(t)$ for all $k \le n$.

2.5. The Strategic Topology for the HOE Game

Define the strategic distance between two types t, t' to be the maximum loss of expected utility if type t behaves as if he is type t' (or vice-versa). Formally, writing $\widehat{u}(a,t,\lambda)$ for the expected utility of a type t player who chooses action a in the λ-game, when he expects his opponent to follow his (unique) optimal strategy,

$$\widehat{u}(a,t,\lambda) = -\sum_{s\in S}\delta_1(s)\lambda_0(a_1-s)^2 - \sum_{n=1}^{\infty}\int_{(\delta_n'',s)}\lambda_n\left(a_{n+1}-\widetilde{\xi}_n(\delta_n'')\right)^2 d\delta_{n+1}(t),$$

and $d^*(\lambda,t,t')$ for the difference in expected payoff if t behaves as if he is type t', instead of following his optimal strategy,

$$d^*(\lambda,t,t') = \widehat{u}_i(\xi(t),t,\lambda) - \widehat{u}_i(\xi(t'),t,\lambda),$$

the strategy distance between a pair of types is

$$d^{**}(t,t') = \sup_{\lambda\in\Lambda}\left(\max\left\{d^*(\lambda,t,t'),d^*(\lambda,t',t)\right\}\right).$$

2.6. Result

Now we show that the strategic topology for the HOE game is the iterated expectations topology. This is an easy implication of the following exact characterization of $d^{**}(t,t')$:

Lemma 2.1. $d^{**}(t,t') = \sup_{n=1,2,\ldots}(\xi_n(t)-\xi_n(t'))^2$.

PROOF. If a player is of type t, his expected payoff to following his optimal action is

$$\widehat{u}(\xi(t),t,\lambda) = -\sum_{s\in S}\delta_1(t)\lambda_0\left(\widehat{\xi}_1(t)-s\right)^2 - \sum_{n=1}^{\infty}\int_{(\delta_n'',s)}\lambda_n\left(\widehat{\xi}_{n+1}(t)-\widetilde{\xi}_n(\delta_n'')\right)^2 d\delta_{n+1}(t).$$

His expected payoff to behaving as if he were type t' is

$$\widehat{u}(\xi(t'),t,\lambda) = -\sum_{s\in S}\delta_1(s)\lambda_0\left(\widehat{\xi}_1(t')-s\right)^2 - \sum_{n=1}^{\infty}\int_{(\delta_n'',s)}\lambda_n\left(\widehat{\xi}_{n+1}(t')-\widetilde{\xi}_n(\delta_n'')\right)^2 d\delta_{n+1}(t).$$

Recall that $\widehat{\xi}_1(t)$ is the expected value of s under $\delta_1(t)$ and each $\widehat{\xi}_{n+1}(t)$ (for $n \geq 1$) is the expected value of $\widetilde{\xi}_n(\delta_n'')$ under $\delta_{n+1}(t)$, when δ_n'' is the belief of the opponent; and recall that for any random variable $\widetilde{x}$ with expectation $\overline{x}$,

$$E(\widetilde{x}-c)^2 = (\overline{x}-c)^2 + E(\widetilde{x}-\overline{x})^2.$$

Thus

$$\begin{aligned}
\widehat{u}(\xi(t'),t,\lambda) &= \left\{\begin{array}{l}
-\sum_{s\in S}\delta_1(s)\lambda_0\left(\widehat{\xi}_1(t')-s\right)^2 \\
-\lambda_0\left(\widehat{\xi}_1(t')-\widehat{\xi}_1(t)\right)^2 \\
-\sum_{n=1}^{\infty}\int_{(\delta_n'',s)}\lambda_n\left(\widehat{\xi}_{n+1}(t')-\widetilde{\xi}_n(\delta_n'')\right)^2 d\delta_{n+1}(t) \\
-\sum_{n=1}^{\infty}\lambda_n\left(\widehat{\xi}_{n+1}(t')-\widehat{\xi}_{n+1}(t)\right)^2.
\end{array}\right\} \\
&= \widehat{u}(\xi(t),t,\lambda) - \lambda_0\left(\widehat{\xi}_1(t')-\widehat{\xi}_1(t)\right)^2 - \sum_{n=1}^{\infty}\lambda_n\left(\widehat{\xi}_{n+1}(t')-\widehat{\xi}_{n+1}(t)\right)^2.
\end{aligned}$$

This in turn implies that

$$\begin{aligned}
d^*(\lambda,t,t') &= \widehat{u}_i(\xi(t),t,\lambda) - \widehat{u}_i(\xi(t'),t,\lambda) \\
&= \lambda_0\left(\widehat{\xi}_1(t')-\widehat{\xi}_1(t)\right)^2 + \sum_{n=1}^{\infty}\lambda_n\left(\widehat{\xi}_{n+1}(t')-\widehat{\xi}_{n+1}(t)\right)^2 \\
&= d^*(\lambda,t',t)
\end{aligned}$$

So

$$\begin{aligned}
d^{**}(t,t') &= \sup_{\lambda\in\Lambda}\left(\max\{d^*(\lambda,t,t'),d^*(\lambda,t',t)\}\right) \\
&= \sup_{\lambda\in\Lambda}\left(\lambda_0\left(\widehat{\xi}_1(t')-\widehat{\xi}_1(t)\right)^2 + \sum_{n=1}^{\infty}\lambda_n\left(\widehat{\xi}_{n+1}(t')-\widehat{\xi}_{n+1}(t)\right)^2\right) \\
&= \sup_{n=1,2,\ldots}\left(\xi_n(t')-\xi_n(t)\right)^2.
\end{aligned}$$

Now we immediately have

Proposition 2.2. *The uniform topology on higher order expectations and the strategic topology are the same.*

This says that two types' expected utility is almost the same in the higher order beliefs game if and only if they are close in the iterated expectations topology. Of course, it is also clear that their optimal actions are also close (in the uniform closeness sense) if and only they are close in the iterated expectations topology.

2.7. Some Properties of "Typical Types"

Let

$$T_N^* = \left\{ t \in T : \exists\, x \in [0,1]^N \text{ such that } \widehat{\xi}_{nN+m}(t) \to x_m \text{ as } n \to \infty, \text{ for each } m = 1, ..., N \right\}$$

Thus

$$T_1^* = \left\{ t \in T : \exists x \in [0,1] \text{ such that } \widehat{\xi}_n(t) \to x \text{ as } n \to \infty \right\}$$

Thus T_N^* is the set of types whose higher order expectations converge to N-cycles. Samet (1998) has shown that all types derived from finite type spaces where the common prior holds belong to T_1^*. This will also be true for types drawn from continuum type spaces with common prior characterized by a uniformly bounded, continuous density (Morris (2002)).

Without the common prior, all types derived from finite type spaces with common *support* priors will belong to T_2^*. This will also be true for types drawn from continuum type spaces where each player's prior is given by a uniformly bounded continuous density (Morris (2002)).

Clearly, the set T_2^* will be non-generic in the space T under the uniform topology on higher order expectations, under any formal topological or probabilitistic notion of genericity. We will check that this is true for one simple topological notion of genericity. We will show that T_2^* is a closed set while the complement of T_2^* is dense in T.

Proposition 2.3. *(1)* $\overline{T_2^*} = T_2^*$; *(2)* $\overline{T/T_2^*} = T$.

PROOF. (1) Let $t^k \in T_2^*$ and $t^k \to_\xi t$. The former implies that exists $x^k > 0$ such that $\widehat{\xi}_{2n}(t^k) \to x^k$ as $n \to \infty$. Without loss of generality, we can assume $x^k \to x^*$ as $k \to \infty$. The latter implies that $\left| \widehat{\xi}_{2n}(t^k) - \widehat{\xi}_{2n}(t) \right| \le \varepsilon^k$ for all n (where $\varepsilon^k \to 0$). Now for all k,

$$\left| \widehat{\xi}_{2n}(t) - x^* \right| \le \left| \widehat{\xi}_{2n}(t) - \widehat{\xi}_{2n}(t^k) \right| + \left| \widehat{\xi}_{2n}(t^k) - x^k \right| + \left| x^k - x^* \right| \qquad (2.2)$$

Now for any $\delta > 0$, choose k such that $\varepsilon^k < \frac{1}{3}\delta$ and $\left| x^k - x^* \right| < \frac{1}{3}\delta$ and then choose N such that $\left| \widehat{\xi}_{2n}(t^k) - x^k \right| < \frac{1}{3}\delta$ for all $n \ge N$. By (2.2),

$$\left| \widehat{\xi}_{2n}(t) - x^* \right| < \delta$$

for all $n \ge N$. Thus $\widehat{\xi}_{2n}(t) \to x^*$ as $n \to \infty$. A similar argument shows that $\widehat{\xi}_{2n+1}(t)$ also converges as $n \to \infty$.

(2) Let $t \in T_2^*$, so that $\widehat{\xi}_{2n}(t) \to x_1^*$ and $\widehat{\xi}_{2n+1}(t) \to x_2^*$ as $n \to \infty$. Let $\varepsilon^k \to 0$ and consider a type t^k with

$$\begin{aligned}
\widehat{\xi}_{2n}\left(t^k\right) &= \begin{cases} \max\left(x_1^* + \varepsilon^k, 1\right), \text{ if } n \text{ is even} \\ \min\left(x_1^* - \varepsilon^k, 0\right), \text{ if } n \text{ is odd} \end{cases} \\
\text{and } \widehat{\xi}_{2n+1}\left(t^k\right) &= \begin{cases} \max\left(x_2^* + \varepsilon^k, 1\right), \text{ if } n \text{ is even} \\ \min\left(x_2^* - \varepsilon^k, 0\right), \text{ if } n \text{ is odd} \end{cases}
\end{aligned}$$

By construction, $t^k \to_\xi t$ as $k \to \infty$, but each $t^k \notin T_2^*$.

Thus we conclude that all of the types in most of the type spaces studied by economists are not generic according to any notion of genericity based on strategic closeness.

We should finally note that finite spaces even without the common support assumption generate types in T_N^*, where N is twice the number of types of the player (this can be shown using the arguments in Morris (2002)). Since one could presumably also show that the set of types $\underset{N}{\cup} T_N^*$ is nongeneric, we have that all finite types belong to a nongeneric set.

3. Example

For any $x \in [0,1]^\infty$, we exhibit a common knowledge discrete type space containing a type t with $\xi(t) = x$. Let $T_1 = T_2 = \{1, 2, ...\}$; $S = \{0, 1\}$. Let player 1's prior be given by

$$\begin{aligned}
P_1\left((t_1, t_2, s)\right) &= \begin{cases} \alpha_1^k \pi_1^k, \text{ if } t_1 = t_2 = k \text{ and } s = 1 \\ \alpha_1^k\left(1 - \pi_1^k\right), \text{ if } t_1 = t_2 = k \text{ and } s = 0 \\ 0, \text{ otherwise} \end{cases} \\
P_2\left((t_1, t_2, s)\right) &= \begin{cases} \alpha_2^k \pi_2^k, \text{ if } t_2 = k,\ t_1 = k+1 \text{ and } s = 1 \\ \alpha_2^k\left(1 - \pi_2^k\right), \text{ if } t_2 = k,\ t_1 = k+1 \text{ and } s = 0 \\ 0, \text{ otherwise} \end{cases}
\end{aligned}$$

where each $\pi_i^k \in [0,1]$, each $\alpha_i^k \in (0,1)$ and

$$\sum_{k=1}^{\infty} \alpha_i^k = 1.$$

Let

$$X^*\left(k, k', s\right) = \begin{cases} 1, \text{ if } s = 1 \\ 0, \text{ if } s = 0 \end{cases}.$$

Writing $E_i(X)$ for i's expectation of random variable x, we have

$$\begin{aligned} E_1(X^*)[(k,k',s)] &= \pi_1^k \\ E_1E_2(X^*)[(k,k',s)] &= \pi_2^k \\ E_1E_2E_1(X^*)[(k,k',s)] &= \pi_1^{k+1} \\ E_1E_2E_1E_2(X^*)[(k,k',s)] &= \pi_2^{k+1} \\ & \ldots.. \end{aligned}$$

and, by induction,

$$\begin{aligned} (E_1E_2)^n E_1(X^*)[(k,k',s)] &= \pi_1^{k+n} \\ (E_1E_2)^{n+1}(X^*)[(k,k',s)] &= \pi_2^{k+n} \end{aligned}$$

for all $n \geq 0$. So

$$\widehat{\xi}(t_1) = \left(\pi_1^{t_1}, \pi_2^{t_2}, \pi_1^{t_1+1}, \pi_2^{t_2+1}, ...\right)$$

Since π_1 and π_2 can be chosen arbitrarily, we have shown that any sequence of higher order expectations can arise on a common knowledge type space.

4. Discussion and Conclusion

4.1. An Interpretation of the Result

Under any natural strategic topology on higher order belief types, the types usually studied in the economics and game theory literature are atypical. What are the practical implications of this claim?

First, arguments that "typical types" can be identified by fixing a type space and picking a "generic" prior on the space make no sense if we think type spaces are trying to capture incomplete information.

Second, consider Harsanyi's (1967/68) hope that incomplete information could incorporated without loss of generality by looking at the space of higher order belief types. It is occasionally argued that Mertens and Zamir's (1985) observation that finite types are dense in the set of higher order belief types, under the product topology, justifies the assumption of finite type spaces without loss of generality. This makes no sense: by using the product topology as a notion of closeness, one is merely *assuming* that the tails of higher order beliefs do not matter. The general point is that the type spaces actually studied - including finite type spaces - are not at all representative of all higher order belief types. If one wants to restrict attention to such well-behaved type spaces, one should be clear that apparent "technical" or "tractibility" assumptions are devices to build in the substantive content of the common knowledge assumption. Finite types and continuum types with bounded densities, may be interesting models to study. But it would be preferable to upfront and explicit about the common knowledge assumptions in our type spaces.

Third, although it will often be impossible to work with the space of all possible higher order beliefs, it is sometimes possible to find type spaces that are tractible and yet capture typical properties of higher order belief types. Two examples illustrate this point. First, coordination problems have interesting features when players are well informed about payoffs but do not have approximate common knowledge of payoffs (Rubinstein (1989), Carlsson and van Damme (1993), Morris and Shin (1998, 2000)). The information structures giving rise to a large divergence between first order belief and common belief look extreme and contrived to some, viewed from the small type / asymmetric information perspective. However, they have tails that matter and thus seem more natural higher order belief type perspective. Second, a crucial property in mechanism design is the *belief extraction property*: if one knows a player's beliefs about other players' types, one can deduce that player's beliefs over payoff relevant events (see, e.g., Cremer and McLean (1988)). This property holds for "generic" priors over a fixed finite type space, but nonetheless fails on the universal type space (Neeman (1999) and Bergemann and Morris (2001)). Neeman (1999) proposed a tractable, finite type space failing the belief extraction property for use in mechanism design.

4.2. Implications for the Universal Type Space Construction?

Mertens and Zamir (1989) and extensions show that if some topological structure is imposed on the construction of higher order belief types, the construction "closes" in the following sense. There exists a homeomorphism $f : T \to \Delta(S \times T)$, such that for any type $t = (\delta_1, \delta_2, ...)$, the belief $f(t) \in \Delta(S \times T)$ correctly reproduces all the finite level beliefs. For example, Brandenburger and Dekel (1993) show how this is a corollary of Kolmogorov's Existence Theorem if the belief spaces are Polish spaces (complete separable metric spaces).

This construction uses the product topology on higher order belief types. The product topology is not natural if one is interested in using the construction to study strategic problems. The arguments in this note certainly have implications for the interpretation of the universal type space of Mertens and Zamir (1985). For example, the continuity of f is with respect to the strategically irrelevant product topology. We would like to pursue these implications in later work. However, the argument in this note do not depend on whether the higher order beliefs construction closes. This note restricts attention to countable hierarchies of beliefs and games where countable hierarchies are certainty sufficient to characterize rational behavior.

4.3. A General Strategic Topology

The strategic topology constructed in this note was illustrative. It sufficed to make the genericity arguments we wanted to make. However, it would obviously be interesting to provide a general construction. Here, we briefly discuss some issues that arise.

The two players and two payoff-relevant states were notationally convenient, but presumably not too important. The main issue is what happens if we define strategic closeness using other games or classes of games. This is discussed in the next section. Then we discuss what topologies might result.

4.3.1. Alternative Games

The particular game chosen to generate the strategic topology is complicated (the action space is $[0,1]^{\infty}$) and is more than a little contrived to deliver the right results. One would like to show similar strategic topologies would be generated (1) if simpler, more intuitive, games were substituted for the HOB game; and (2) if natural *classes* of games were studied instead. We discuss these two questions in turn.

A Continuum Action Coordination Game A game is parameterized by $\lambda \in [0,1)$. Each of two players picks an action $a_i \in [0,1]$. Player i's payoff is

$$u_i(a_i, a_j, s) = -(1-\lambda)(a_i - s)^2 - \lambda(a_i - a_j)^2 .$$

Morris and Shin (2001) show that the optimal action is this game for type t is to set his action equal to

$$\sum_{n=1}^{\infty} (1-\lambda)\lambda^{n-1}\widehat{\xi}_n(t) .$$

Employing the same notation as above, the expected utility of a type t player who chooses action a in the λ-game, when he expects his opponent to follow his (unique) optimal strategy,

$$\widehat{u}(a,t,\lambda) = -(1-\lambda)\sum_{s\in S}\delta_1(s)(a-s)^2 - \lambda \int\limits_{(t'',s)} \left(a - \sum_{n=1}^{\infty}(1-\lambda)\lambda^{n-1}\widehat{\xi}_n(t'')\right)^2 df(t),$$

where $f(t)$ is the belief of type t over the (s,t''), where t'' is the type of the other player. Now

$$d^*(\lambda,t,t') = \sum_{n=1}^{\infty}(1-\lambda)\lambda^{n-1}\left(\widehat{\xi}_n(t) - \widehat{\xi}_n(t')\right)^2,$$

so the strategy distance between a pair of types is

$$d^{**}(t,t') = \sup_{\lambda \in [0,1)} \left(\sum_{n=1}^{\infty} (1-\lambda)\lambda^{n-1} \left(\widehat{\xi}_n(t) - \widehat{\xi}_n(t') \right) \right)^2 .$$

This is a weaker topology than the iterated expectations topology, but still is highly sensitive to what happens in the tail of the iterated expectations.

Both this game and the higher order expectations game have the feature that for any fixed λ, we can choose N such that beliefs above level N do not effect strategic outcomes too much. In other words, the order of quantifiers becomes crucial. One can also show that in a binary action coordination game, in the spirit of Rubinstein (1989), the dependence of arbitrarily high beliefs will hold in a fixed game. However, in order to analyze that game, it is necessary to consider whether the countable additive hierarchies of beliefs exhaust players' uncertainty (i.e., the question addressed by Mertens and Zamir (1985)) and we have been trying to bypass that issue in this note.

A Class of Games The right way to define strategic closeness would be to fix a class of simple games (say, with finite actions and bounded utility), and say that two types are close if they behave similarly and obtain similar outcomes in almost all games. In such a definition, one would need a solution concept from rational behavior. Of course, this was sidestepped in the examples above by focussing on games where there was a unique rationalizable outcome, always. Presumably, the most natural thing to do would be to look at the set of rationalizable behavior for each type and each game and use a set-based notion of similar behavior.

4.3.2. Alternative Topologies

The special feature of games described in this note was that, because of linearity, only higher order expectations, and not the whole of higher order beliefs, were important in explaining players' behavior. In general, the whole structure of higher order beliefs will matter. Monderer and Samet (1996) and Kajii and Morris (1997) describe strategic topologies for fixed state spaces (the former fixes beliefs and varies partitions, while the latter fixes partitions and varies beliefs). In the latter case, the strategic topology is stronger than the weak topology on priors but is weaker than the topology of uniform convergence of conditional probabilities. By analogy, the strategy topology on higher order belief types for a general class of games is probably weaker than requiring uniform closeness of beliefs at all levels. However, it will be stronger than the uniform topology of higher order expectations in this note. Of course, those types that are nongeneric in the uniform topology of higher order expectations would continue to be nongeneric in the stronger topology.

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