Common Belief Foundations of Global Games*

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Abstract

We provide a characterization of when an action is rationalizable in a binary action coordination game in terms of beliefs and higher order beliefs. The characterization sheds light on when a global game yields a unique outcome. In particular, we can separate those features of the noisy information approach to global games that are important for uniqueness from those that are merely incidental. We derive two sufficient conditions for uniqueness that do not make any reference to the relative precision of public and private signals.

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1 Introduction

Games often have many equilibria. Even when they have a single equilibrium, they often have many actions that are rationalizable, and are therefore consistent with common knowledge of rationality. Yet a pathbreaking paper by Carlsson and van Damme (1993) suggested a natural perturbation of complete information that gives rise to a unique rationalizable equilibrium for each player. They introduced the idea of "global games" - where any payoffs of the game are possible and each player observes the true payoffs of the game with a small amount of noise. They showed - for the case of two player two action games - that as the noise about payoffs become small, there is a unique equilibrium; the equilibrium strategies played also constitute the unique rationalizable strategies. This result has since been generalized in a number of directions and used in a number of applications.[1] When the global game approach can be applied to more general games, it can be used to derive unique predictions in settings where the underlying complete information game has multiple equilibria, making it possible to carry out comparative static and policy analysis.

However, a number of recent papers have raised questions both on the basic theoretical rationale for global games and the applicability of the framework for the analysis of real world problems. Three strands of the argument from the literature are particularly worthy of note.

1. In most economic environments where coordination is important, interactions endogenously generate informative public information that might be used as a coordination device. An especially important source

[1] Morris and Shin (1998) analyzed a global game with a continuum of players making binary choices, and this case has been studied in a number of later applications. Morris and Shin (2003) survey some theory and applications of global games.

of endogenous public information are market prices (see Atkeson (2001), Tarashev (2003), Hellwig, Mukherji and Tsyvinski (2006) and Angeletos and Werning (2006))[2]. When prices convey information, increased precision of private information will feed increased accuracy of (endogenous) public signals. Thus uniqueness conditions will fail if private signals are sufficiently accurate.

2. While asymmetric information may exist in a large variety of economic settings, it does not always conform to the global game notion of “noisy signals”. Global game results turn on the relative precision of private and public signals, but if we do not know what these noisy signals are in real life, debates about relative precisions have no conceptual basis (see, for example, Kurz (2006), Sims (2005a, 2005b), Svensson (2006), Woodford (2005)).

3. While common knowledge of payoffs is relaxed in global games, there is still assumed to be common knowledge of the information structure, which is surely a no more realistic assumption. A recent paper by Weinstein andYildiz (2007) shows that the exact form of the perturbation away from common knowledge of payoffs is crucial in determining the rationalizable outcome. The global game prediction is not the only possible perturbation that yields unique rationalizable outcomes. What claim does the global game approach have for being a “natural” or “reasonable” perturbation?

The objective of our paper is to evaluate these arguments and questions concerning the global game methodology, and to provide a framework that

[2] Angeletos, Hellwig and Pavan (2006a, 2006b) note (inter alia) how other sources of endogenous public information may lead to multiplicity in such coordination games.

can both deepen our understanding of the theoretical basis for global games and to provide guidance for applied researchers on the scope (and limitations) of the global game approach.

The canonical information structure associated with the global game approach is one where players observe the underlying fundamental variable with some noise. This is for the historical reason that the early papers (Carlsson and van Damme (1993), Morris and Shin (1998)) adopted this formalism. The noise is a convenient way to relax common knowledge of the fundamentals, but in subsequent applications of global games the noisy information structure has been taken more literally - as players failing (literally) to observe the true fundamentals perfectly. Many of the criticisms of the global game approach presumes such a literal interpretation of the global games approach.

However, there are pitfalls in taking the noisy information structure too literally, as the underlying logic of the argument becomes identified with a particular formalism, and the general scope of the approach becomes obscured by debates surrounding the merits or otherwise of the particular formalism. The logic underlying the global game approach turns out to be more robust, and is not tied to taking noisy signals literally.

In this paper, we revisit the belief foundations of global games. We know already that the failure of common knowledge of the fundamentals is a necessary condition for generating the global game outcome, but the more demanding task is to show precisely *how* beliefs depart from the complete information benchmark. We have two objectives in this paper.

First, we link the global game analysis with the earlier literature on common knowledge and interactive epistemology - to the framework popularized by Aumann (1976) and Monderer and Samet (1989). We provide a framework that can encompass global games (especially their countable state analogues) within a framework of interactive beliefs. We define an operator on the type space that has a strong resemblance to the p-belief operator of Monderer and Samet (1989), and show how rationalizability corresponds to common belief in this generalized belief operator. The perspective is that of an outside observer who observes only whether a player chooses one action or the other. The fact that a particular action has been chosen reveals much about the player's beliefs - both about the fundamentals of the environment, but also about the beliefs and higher order beliefs of *other* players. The belief operators that we identify correspond with to the *revealed strength of beliefs* that a player holds about the environment and the other players. In this sense, we take the viewpoint of an outside observer (such as an empirical economist) who attempts to piece together the beliefs from the action chosen. In this way, we can characterize the higher order beliefs that underpin play in global games, thereby answering the question of how the departure from common knowledge is achieved in global games.

Second, the revealed beliefs approach yields insights on the question of when there is a unique rationalizable outcome in the global game. By using the framework of the generalized belief operators, we identify two sets of sufficient conditions on the common beliefs of the types that ensure unique rationalizability. Essentially, the property that matters is the stationarity of beliefs with respect to the ordering of types. Global game arguments work because the beliefs that player types have over their neighboring types do not change abruptly as we consider types along the ordering. A special case of such insensitivity of beliefs along the type space is the case when each type believes he is "typical". We show that uniqueness in the noisy information approach to global games with public and private information uses precisely this strong version of insensitivity of beliefs to the order.

The rest of the paper is structured as follows. We begin in section 2 with a leading example that illustrates many of the features that will make an appearance in the general argument. We then characterize the higher order beliefs that are necessary and sufficient for rationalizability, and revisit some familiar examples of global games from the applied literature, and illustrate our result. Section 5 then builds on earlier results to shed light on uniqueness. We discuss two sufficient conditions for uniqueness that do not make reference to noisy signals, or relative precisions of private and public signals.

2 Example

There are I players who choose from $\{\text{invest, not invest}\}$. There is a cost of investing, $p \in (0,1)$. The payoff to investing depends on the fundamental state θ. There are dominance thresholds $\underline{\theta}$ and $\bar{\theta}$ with $\underline{\theta} < \bar{\theta}$ such that "not invest" is dominant when θ falls below the lower threshold $\underline{\theta}$ and "invest" is dominant when θ is above the upper threshold $\bar{\theta}$. When $\theta < \underline{\theta}$, the gross return to investing is zero irrespective of the actions of the other players, so that investing yields a sure payoff of $-p$. When $\theta > \bar{\theta}$, the gross return to investing is 1 irrespective of the actions of the other players so that investing yields a sure payoff of $1 - p$.

When $\underline{\theta} \leq \theta \leq \bar{\theta}$, the gross return to investing is 1 if and only if proportion q or more of the players (including oneself) invest, where $0 < q < 1$. The payoff matrix is

	at least q invest	less than q invest
invest	$1 - p$	$-p$
not invest	0	0

2.1 Reconstructing the Belief Hierarchy

For an outside observer (an empirical economist, say), the observable features of the problem are quite coarse. The outside observer sees only whether a player invests or not. But when combined with the knowledge of the payoffs and the players' rationality, the decision to invest reveals much about that player's beliefs - both about the fundamentals θ, but also about the beliefs of other players.

Suppose player i is seen to invest. Then, either i has a dominant action to invest, or he p-believes all of the following.

1. $\theta \geq \underline{\theta}$
2. proportion q or more either have a dominant action to invest or p-believe that $\theta \geq \underline{\theta}$
3. proportion q or more either have a dominant action to invest or p-believe that [proportion q or more either have a dominant action to invest or p-believe that $\theta \geq \underline{\theta}$]
4. and so on ...

p-belief of statement 1 is a necessary condition for investing, since otherwise the expected payoff to investing is negative irrespective of the actions of the other players. But then, other players will have reached a similar conclusion. So, player i must also p-believe statement 2, since otherwise there is probability less than p that proportion q or more players consider "invest" as being first-order undominated. Then, fewer than q will invest. In general, the failue to p-believe statement $n+1$ is a reason not to invest, because there is probability less than p that proportion q or more players consider "invest" as being n-th order undominated.

In this way, unless i finds it dominant to invest, p-belief of all the statements in the list is necessary for "invest" to be chosen. Conversely, when a player p-believes all of the statements in the list, this is also sufficient for "invest" to survive the iterated deletion of dominated strategies.

There is an exactly analogous hierarchy of beliefs that are revealed by a player who chooses *not* to invest. Player j who does not invest reveals that either he finds it dominant not to invest, of he $(1-p)$-believes of all of the following statements.

1. $\theta \leq \bar{\theta}$
2. proportion $1-q$ or more either have a dominant action not to invest or $(1-p)$-believe that $\theta \leq \bar{\theta}$
3. proportion $1-q$ or more either have a dominant action not to invest or $(1-p)$-believe that [proportion $1-q$ or more either have a dominant action not to invest or $(1-p)$-believe that $\theta \leq \bar{\theta}$]
4. and so on ...

These statements are individually necessary and jointly sufficient for "not invest" to survive the iterated deletion of interim dominated strategies.

2.2 Information Structure

To explore when one or other action may be supported as an iteratively undominated action, we introduce an information structure. Suppose θ takes realizations in the set of integers $\mathcal{Z} = \{\cdots, -2-1, 0, 1, 2, 3, \cdots\}$ and there is a prior density μ over $\mathcal{Z}$. There are $I = 2n+1$ players who play the investment game.

The players receive noisy signals concerning θ. Let s_i be player i's signal realization. s_i takes values in Z. Conditional on θ, player i is equally likely to observe any signal between $\theta - n$ to $\theta + n$, but we depart from the familiar global game assumption that players' signals are independent conditional on θ. The purpose of this departure is to construct an information structure that is as close as possible in spirit to the continuum player global game, as we will elaborate below. Conditional on θ, each signal realization between $\theta - n$ and $\theta + n$ is observed by precisely one player. No two players observe the same signal, and each possible realization between $\theta - n$ and $\theta + n$ is observed by some player.

One way in which our information structure could be generated is through the following procedure. Conditional on θ, a player is selected randomly to receive the highest signal (namely $\theta + n$). Each player has equal chance of being selected. Next, the second highest signal realization, $\theta + n - 1$ is given to a player chosen from the remaining pool of players, where each player has equal chance of being selected, and so on. Once the ranking has been chosen (unknown to the players themselves), each player observes his signal, and makes inferences based on this signal. The information structure arrived at in this way has the following two features.

- Any two players can be strictly ranked according to their signal realizations.

- Conditional on θ, player i has equal chance of observing any signal realization between $\theta - n$ and $\theta + n$.

Conditional on observing signal realization s_i, player i's posterior density has support over the interval $[s_i - n, s_i + n]$, and

$$\frac{\mu(\theta \mid s_i)}{\mu(\theta' \mid s_i)} = \frac{\mu(\theta)}{\mu(\theta')}$$

for θ, θ' in the support. Among other things, this means that the posterior densities can be ranked by first-degree stochastic dominance.

We can trace a player's beliefs about his rank in the population, as measured by the realization of his signal relative to those of others. Player i with signal s_i has the highest signal realization when $\theta = s_i - n$. So, player i believes he has the highest signal with probability $\mu(s_i - n \mid s_i)$. In general, player i with signal s_i believes that he has the $k+1$-th highest signal in the population with probability $\mu(s_i - n + k \mid s_i)$. Let $\rho_k(s_i)$ be the probability that player i assigns to there being $k-1$ players with signals lower than himself, conditional on signal s_i. Then $\rho_k(s_i) = \mu(s_i + n - k + 1 \mid s_i)$. Let

$$\rho(s_i) \equiv (\rho_1(s_i), \rho_2(s_i), \cdots, \rho_I(s_i))$$

be the profile of i's beliefs over his rank order, conditional on s_i.

2.3 Evident Events

For the next step, see figure 1. Fix $\hat{\theta}$, and let $\hat{s}$ be the highest signal realization such that proportion q or more of players have signal realizations that are $\hat{s}$ or higher at $\hat{\theta}$. Denote by $\hat{p}$ the probability that $\theta \geq \hat{\theta}$ conditional on $\hat{s}$. We then have:

1. When $\theta \geq \hat{\theta}$, proportion q or more players receive signal $\hat{s}$ or higher. This follows from the first-degree stochastic dominance of signal realizations as θ increases.

2. When $s_i \geq \hat{s}$, player i $\hat{p}$-believes that $\theta \geq \hat{\theta}$. This follows from the first-degree stochastic dominance of the posterior density over θ as s_i increases.

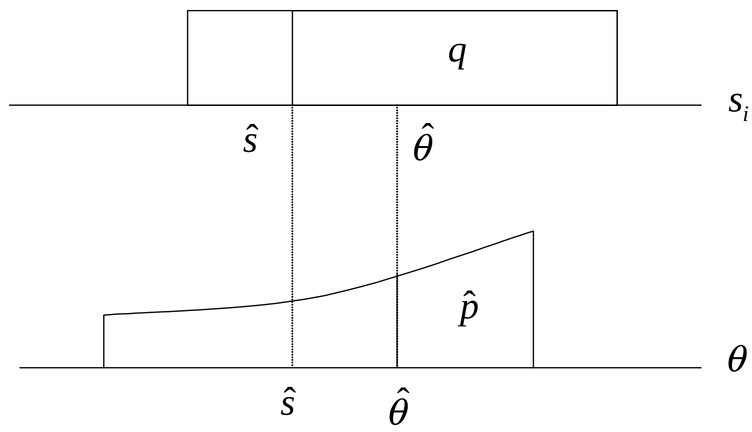

Figure 1: Evident events

So, when $\theta \geq \hat{\theta}$, proportion q or more of the players $\hat{p}$-believe that $\theta \geq \hat{\theta}$. We say that the event $\left\{\theta | \theta \geq \hat{\theta}\right\}$ is $(q, \hat{p})$-*evident* when this holds. Our definition generalizes Monderer and Samet's (1989) notion of p-evident events, where we keep track of the proportion q of players who p-believe an event.

Suppose now that $\hat{\theta} \geq \underline{\theta}$, and that player i $\hat{p}$-believes $\left\{\theta | \theta \geq \hat{\theta}\right\}$. Then he $\hat{p}$-believes all of the following:

1. $\theta \geq \underline{\theta}$

2. proportion q or more $\hat{p}$-believe that $\theta \geq \underline{\theta}$

3. proportion q or more $\hat{p}$-believe that [proportion q or more $\hat{p}$-believe that $\theta \geq \underline{\theta}$]

4. $\cdots$

Say that there is *common* (q, p)-*belief* that $\theta \geq \underline{\theta}$ when this list holds. From the list of statements in section 2.1, "invest" is rationalizable for i if $\hat{p} \geq p$. We also have the reverse implication. "Invest" is rationalizable *only*

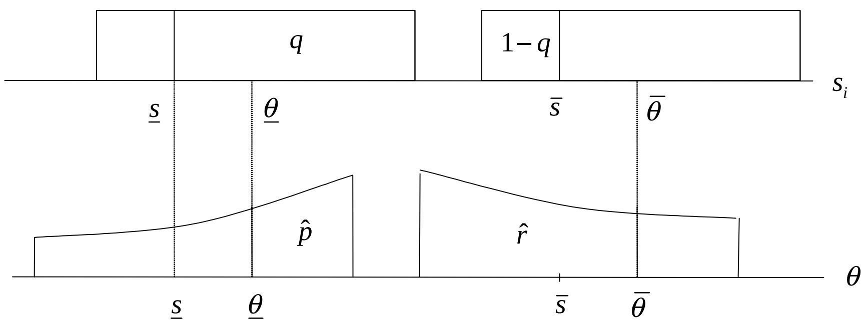

Figure 2: Case when $\hat{p} \geq p$ and $\hat{r} \geq 1 - p$

if $\hat{p} \geq p$. This is because when player i has a dominant action to invest, he $\hat{p}$-believes that $\theta \geq \hat{\theta}$. So, the "either-or" clause concerning dominant action types is subsumed under the $\hat{p}$-belief of $\left\{\theta | \theta \geq \hat{\theta}\right\}$. Since our space of signals and fundamentals is countable, common (q,p)-belief implies the existence of a (q,p)-evident event, since otherwise the countable intersection of events satisfying each clause yields the empty event. Following an exactly analogous line of reasoning for when "not invest" is rationalizable, we have:

Claim 1 *"Invest" is rationalizable for i if and only if i p-believes some (q,p)-evident subset of $\{\theta | \theta \geq \underline{\theta}\}$. "Not invest" is rationalizable for i if and only if i $(1-p)$-believes some $(1-q, 1-p)$-evident subset of $\left\{\theta | \theta \leq \bar{\theta}\right\}$.*

Figure 2 illustrates a case when both actions may be rationalizable. When $\hat{p} \geq p$ and $\hat{r} \geq 1 - p$, the event $\{\theta | \theta \geq \underline{\theta}\}$ is (q,p)-evident, and $\left\{\theta | \theta \leq \bar{\theta}\right\}$ is $(1-q, 1-p)$-evident. Unique rationalizability rests on ruling out such cases.

2.4 Uniqueness

Consider the rank profiles $\rho(s_i)$ and $\rho(s_i')$ at two different signal realizations s_i and s_i'. Write $\rho(s_i') \trianglerighteq \rho(s_i)$ when $\rho(s_i')$ weakly dominates $\rho(s_i)$ in the

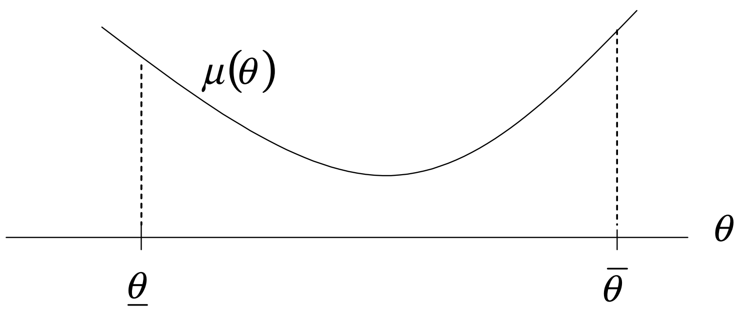

Figure 3: Decreasing rank beliefs

sense of first degree stochastic dominance. Say that rank beliefs are *weakly increasing* in signals when $s_i' \geq s_i$ implies $\rho(s_i') \trianglerighteq \rho(s_i)$.

Let $\underline{s}$ and $\bar{s}$ be signal realizations illustrated in figure 2. $\underline{s}$ is the highest signal such that at $\underline{\theta}$, proportion q or more have signal $\underline{s}$ or higher. $\bar{s}$ is the lowest signal such that at $\bar{\theta}$, $1-q$ or more have signal $\bar{s}$ or lower.

We then have the following sufficient condition for uniqueness.

Claim 2 *If rank beliefs are weakly decreasing in signals in* $\{s_i \mid \underline{s} \leq s_i \leq \bar{s}\}$*, then there is a unique rationalizable outcome in the investment game, except possibly at one value of* θ*.*

When rank beliefs are weakly decreasing in signals, a player believes that his rank is low when he receives a high signal. Suppose that a student finds out that his test score is high, and what matters is just his relative ranking in the class. Is the high score good news or bad news? When rank beliefs are decreasing in signals, this is bad news. The fact that he has received a high score indicates that the test must have been very easy, so that others have received even higher scores. Such beliefs correspond to priors that are locally U-shaped, such as that illustrated in figure 3.

Let $\hat{p}(\theta)$ be the largest probability h with which $\{\theta' | \theta' \geq \theta\}$ is common (q, h)-belief. When rank beliefs are weakly decreasing, $\hat{p}(\theta)$ is increasing in θ. If $\hat{p}(\theta) < p$ for all θ in the undominated region, then there is no (q, p)-evident subevent of $\{\theta | \theta \geq \underline{\theta}\}$. Thus, suppose $\hat{p}(\theta) \geq p$ above some threshold θ^*. Then, $\{\theta | \theta \geq \theta^*\}$ is common (q, p)-belief at all $\theta \geq \theta^*$, but $\{\theta | \theta \leq \theta^*\}$ is not $(1-q, 1-p)$-belief at all $\theta < \theta^*$. Below the threshold θ^*, $\{\theta | \theta \geq \theta^*\}$ is not common (q, p)-belief , but $\{\theta | \theta \leq \theta^*\}$ is $(1-q, 1-p)$-belief. At θ^* itself, both actions may be rationalizable, but this is due to the probability atom on θ^* arising from the fact that θ is drawn from a discrete space. Otherwise, there is a unique rationalizable outcome.

We note the following corollaries, bearing in mind that the results hold except possibly at one value of θ.

Corollary 3 *If $\rho(\cdot)$ is a constant function over $\{s_i \mid \underline{s} \leq s_i \leq \bar{s}\}$, then there is a unique rationalizable outcome in the investment game.*

For instance, $\rho(\cdot)$ would be constant over $\{s_i \mid \underline{s} \leq s_i \leq \bar{s}\}$ if the prior μ is a geometric density over the relevant interval, so that $\mu(\theta)/\mu(\theta+1) = \mu(\theta+j)/\mu(\theta+j+1)$.

Also, although we have conducted the discussion with a common prior μ, our argument could easily be extended for the case where players hold different priors over θ. Izmalkov and Yildiz (2006) examine an information structure where some players are systematically more optimistic than others. Our framework could accommodate such information structures.

An even more restrictive special case is when ρ is not only constant over signals, but its cross-section is uniform over the possible rank orders, in the sense that

$$\rho(s_i) = \left(\tfrac{1}{I}, \tfrac{1}{I}, \cdots, \tfrac{1}{I}\right) \tag{1}$$

If (1) holds, player i believes he has equal probability of being ranked anywhere in the population. Player i believes that he is "typical" in quite a strong sense. The uniqueness result for continuum action global games with Gaussian fundamentals and signals rests of approaching the analogue of (1). When $\rho(s_i)$ is uniform, we can characterize the unique rationalizable outcome crisply.

Corollary 4 *Suppose* $\rho(s_i) = \left(\frac{1}{I}, \frac{1}{I}, \cdots, \frac{1}{I}\right)$ *whenever* $\underline{s} \leq s_i \leq \bar{s}$. *Then, "invest" is the unique rationalizable action in the first-order undominated region when* $p + q < 1$. *"Not invest" is the unique rationalizable action in the first-order undominated region when* $p + q > 1$.

The corollary follows from the fact that when ρ uniform, $\hat{p} = 1 - q$. "Invest" is rationalizable when $\hat{p} > p$. That is, when $p + q < 1$. "Not Invest" is rationalizable when $1 - \hat{p} > 1 - p$. That is, when $p + q > 1$.

2.5 Comparison to Gaussian Information Structures

Given the importance of rank order beliefs, let us retrace what the analogous rank order beliefs are in the familiar Gaussian information structure that is commonly used in continuum player global games. Player i's private signal is given by

$$x_i = \theta + \varepsilon_i$$

where θ is a Gaussian random variable with mean y and variance $1/\alpha$, and ε_i is Gaussian with mean zero and variance $1/\beta$. The random variables $\{\varepsilon_i\}$ are mutually independent, and independent of θ.

Denote by $\lambda(x)$ the proportion of players whose signal is x or less. The "λ" stands for "lower". Then, $\lambda(x)$ is a random variable with realizations in the unit interval, and which is a function of the random variables $\{\theta, \varepsilon_i\}_{i\in[0,1]}$

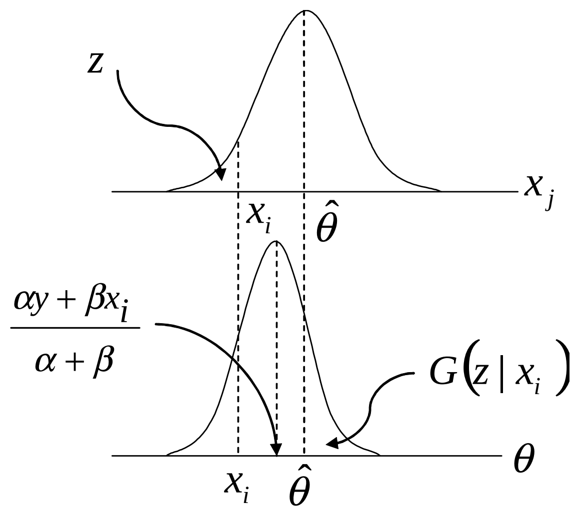

Figure 4: Deriving $G(z|x_i)$

and the threshold x. We derive the density function of $\lambda(x_i)$ conditional on x_i. Let

$$G(z|x_i) \tag{2}$$

be the cumulative distribution function of $\lambda(x_i)$ conditional on x_i, evaluated at z. In other words,

$$G(z|x_i) = \Pr(\lambda(x_i) \leq z|x_i) \tag{3}$$

so that, $G(z|x_i)$ is the probability that the proportion of players with signal lower than x_i is z or less, conditional on x_i. Figure 4 illustrates the derivation of $G(z|x_i)$.

Given θ, the proportion of players who have signal below x_i is

$$\Phi\left(\sqrt{\beta}(x_i - \theta)\right) \tag{4}$$

where $\Phi(\cdot)$ is the cumulative distribution function for the standard normal. Let $\hat{\theta}$ be the realization of θ at which this proportion is exactly z. In other words,

$$\hat{\theta} = x_i - \frac{\Phi^{-1}(z)}{\sqrt{\beta}} \tag{5}$$

The top panel of figure 4 illustrates $\hat{\theta}$. When $\theta \geq \hat{\theta}$, the proportion of players that have signal below x_i is z or less. In other words, $\lambda(x_i) \leq z$ whenever $\theta \geq \hat{\theta}$. Hence, $G(z|x_i)$ is the probability of $\left\{\theta|\theta \geq \hat{\theta}\right\}$ conditional on x_i. The bottom panel of figure 4 illustrates the argument. Conditional on x_i, the density over θ is normal with mean

$$\frac{\alpha y + \beta x_i}{\alpha + \beta} \tag{6}$$

and precision $\alpha + \beta$. The probability that $\theta \geq \hat{\theta}$ is the area under this density to the right of $\hat{\theta}$, namely

$$1 - \Phi\left(\sqrt{\alpha+\beta}\left(\hat{\theta} - \tfrac{\alpha y + \beta x_i}{\alpha+\beta}\right)\right) \tag{7}$$

This expression gives $G(z|x)$. Substituting out $\hat{\theta}$ by using (5) and re-arranging, we can re-write (7) to give:

$$G(z|x_i) = \Phi\left(\tfrac{\alpha}{\sqrt{\alpha+\beta}}(y - x_i) + \sqrt{\tfrac{\alpha+\beta}{\beta}}\Phi^{-1}(z)\right) \tag{8}$$

In the special case when $\beta \to \infty$, the private signal becomes infinitely precise. In this limit,

$$G(z|x) \to \Phi\left(\Phi^{-1}(z)\right) = z$$

so that G is the identity function. In other words, the c.d.f. of $\lambda(x_i)$ is the 45 degree line, and hence the density over $\lambda(x_i)$ is uniform. Thus, in this limit, player i believes that he is "typical" in quite a strong sense, in that he puts equal weight on every realization of $\lambda(x_i)$. In this sense, the uniform density over λ is exactly analogous to the rank belief profile $\rho(.)$ being uniform.

3 Common Belief in Global Games

We now generalize our argument of the previous section. In so doing, we characterize the hierarchy of beliefs that underpin actions in global games. We will also apply these insights in considering the hierarchy of beliefs that ensure a unique rationalizable outcome in the global game.

3.1 Setting

There are I players, $\mathcal{I} = \{1, 2, ..., I\}$ and a countable set of payoff states, Θ. A type space is a collection $\mathcal{T} = (T_i, \pi_i)_{i=1}^{I}$, where T_i is the set of types of player i and $\pi_i : T_i \to \Delta(T_{-i} \times \Theta)$. We consider binary action games, where each player i will choose $a_i \in \{0, 1\}$. We write $\lambda_i(Z, \theta)$ for the payoff gain to player i of choosing action 1 over choosing action 0 if $Z \subseteq \mathcal{I} \setminus \{i\}$ is the set of his opponents who choose action 1 and the payoff state is θ. In other words, if $u_i(a, \theta)$ were player i's payoff if action profile a is chosen and state is θ, the function λ_i is defined as

$$\lambda_i(\{j \neq i | a_j = 1\}, \theta) = u_i(1, a_{-i}, \theta) - u_i(0, a_{-i}, \theta).$$

Thus a game is parameterized by payoffs $\boldsymbol{\lambda} = (\lambda_1, .., \lambda_I)$. Throughout the paper, we will consider supermodular games which in this context means:

Assumption. (Supermodularity) $\lambda_i(Z, \theta)$ is increasing in Z, i.e., $Z \subseteq Z' \Rightarrow \lambda_i(Z, \theta) \leq \lambda_i(Z', \theta)$

3.2 Product Events

The relevant state space for our problem is $\Omega = T_1 \times T_2 \times ... \times T_I \times \Theta$ and an event would ordinarily be defined as a subset of Ω. However, we will be interested in a special class of *product events* corresponding to each player

i's type t_i belonging to a subset $F_i \subseteq T_i$. Thus a product event is a vector $F = (F_1, ..., F_I) \in \overset{I}{\underset{i=1}{\times}} 2^{T_i}$. We will be highlighting two interpretations of product events.

First, a product event F uniquely defines an *equivalent ordinary event* $X_F \subseteq \Omega$ with

$$X_F = \{(t_1, ..., t_I, \theta) \in \Omega \, | t_i \in F_i \text{ for each } i = 1, ..., I\} .$$

Where no confusion arises, we will identify a product event F with its equivalent ordinary event X_F. In keeping with this interpretation, we will write $t \in F$ if $t_i \in F_i$ for each $i = 1, ..., I$ and we will define a natural partial order on product events by set inclusion, so $F \subseteq E$ if $F_i \subseteq E_i$ for each $i = 1, .., I$.

Second, because we are focussing on binary action games, the set of product events is isomorphic to the set of strategy profiles. Thus we can identify the product event F with the strategy profile where player i chooses action 1 if and only if $t_i \in F_i$.

Denote by $\mathcal{S}$ the class of product events. Now $\mathcal{S}$ is a complete lattice under the partial order $\subseteq$ and the *join* $E \vee F$ and *meet* $E \wedge F$ of a pair of events E and F are defined as

$$\begin{aligned} E \vee F &\equiv (E_i \cup F_i)_{i=1}^{I} \\ E \wedge F &\equiv (E_i \cap F_i)_{i=1}^{I} \end{aligned}$$

We write

$$\varnothing = \left(\overbrace{\varnothing, .., \varnothing}^{I \text{ times}} \right) \text{ and } T = (T_1, ..., T_I)$$

for the smallest and largest elements of $\mathcal{S}$, respectively.

Notice that the meet operation corresponds to intersection of the equivalent ordinary events, i.e.,

$$X_{E \wedge F} = X_E \cap X_F$$

and that the (set inclusion) ordering on product events generates the same ordering as set inclusion on their equivalent ordinary events, i.e.,

$$F \subseteq E \text{ if and only if } X_F \subseteq X_E.$$

There is also a natural definition of the negation of an event, $\neg F$, with

$$\neg F = \neg (F_i)_{i=1}^{I} \equiv (\backsim F_i)_{i=1}^{I}.$$

Now the class of product events is closed under $\{\vee, \wedge, \neg\}$. The definitions of join can be extended to any countable collection of simple events in the natural way, and we will appeal to these definitions later. Also, we note the following properties of these operations.

$$\begin{aligned} \neg\neg F = F, \quad \neg\emptyset &= T, \quad \neg T = \emptyset \\ \neg (E \vee F) &= \neg E \cap \neg F \end{aligned}$$

3.3 Generalized Belief Operators

We will define player i's λ_i-*belief function* $B_i^{\lambda_i} : \mathcal{S} \to 2^{T_i}$ as follows. Let $Z_{F,i}(t)$ be the set of players other than i such that $t_j \in F_j$; thus $Z_{F,i} : T \to 2^{\mathcal{I}}$ is defined as

$$Z_{F,i}(t_1, \cdots, t_I) = \{j \in \mathcal{I} \mid j \neq i \text{ and } t_j \in F_j\}.$$

For any random variable $f : T \times \Theta \to \mathbb{R}$, write $E_{t_i}(f)$ for type t_i's expectation of f, so

$$E_{t_i}(f) = \sum_{t_{-i},\theta} \pi_i [t_i] (t_{-i}, \theta) f((t_i, t_{-i}), \theta).$$

Now

$$B_i^{\lambda_i}(F) = \{t_i \in F_i \,| E_{t_i}(\lambda_i (Z_{F,i}, \theta)) \geq 0\},$$

Thus $B_i^{\lambda_i}(F)$ is player i's best response to the strategy profile F, since $t_i \in B_i^{\lambda_i}(F)$ exactly if action 1 is a best response for player i if he thinks each opponent j chooses action 1 only if $t_j \in T_j$. We dub $B_i^{\lambda_i}$ a "belief function" because $t_i \in B_i^{\lambda_i}(F)$ reveals that type t_i puts sufficiently high probability on some or all of his opponents having types $t_j \in T_j$. The more likely is F, the greater is player i's incentive to play action 1 himself. Hence, his taking action 1 reveals that he places high weight on F.

Define $B^{\boldsymbol{\lambda}}(F)$ as the product set:

$$B^{\boldsymbol{\lambda}}(F) = \left(B_i^{\lambda_i}(F)\right)_{i=1}^{I};$$

$B^{\boldsymbol{\lambda}}(F)$ identifies the set of type profiles for whom playing 1 is a best reply when other players play 1 on event F; equivalently, it is the set of types with high beliefs that F is true.

The *generalized belief operator* $B^{\boldsymbol{\lambda}} : \mathcal{S} \to \mathcal{S}$ satisfies the following properties:

B1. $F \subseteq F' \Rightarrow B^{\boldsymbol{\lambda}}(F) \subseteq B^{\boldsymbol{\lambda}}(F')$

B2. $B^{\boldsymbol{\lambda}}(F) \subseteq F$ for all F

B3. If F^n is a decreasing sequence, then $B^{\boldsymbol{\lambda}}(\wedge_n F^n) = \wedge_n B^{\boldsymbol{\lambda}}(F^n)$.

B4. $\left(B^{\boldsymbol{\lambda}}\right)^n(F)$ is a decreasing sequence

B1 states that $B^{\boldsymbol{\lambda}}$ is an increasing operator on the lattice $\mathcal{S}$; it is an implication of supermodularity, and shows that our interpretation of "revealed beliefs" is consistent with the deductive closure of beliefs. That is, if F implies F', then belief in F implies belief in F'. B2 follows from the definition. B3 is a continuity axiom: it is implied by B1 if the type space is finite. In B4, $\left(B^{\boldsymbol{\lambda}}\right)^k$ denotes the k-fold application of the $B^{\boldsymbol{\lambda}}$ operator. B4 follows from B1 and B2.

Definition 5 *Event F is $\boldsymbol{\lambda}$-evident if it is a fixed point of $B^{\boldsymbol{\lambda}}$, i.e.,*

$$F = B^{\boldsymbol{\lambda}}(F)$$

By B2, this is equivalent to the requirement that $F \subseteq B^{\boldsymbol{\lambda}}(F)$. Note that event F is $\boldsymbol{\lambda}$-*evident* if and only if the strategy profile F is an equilibrium of the incomplete information game (where indifferent types choose action 1).

Definition 6 *Event $C^{\boldsymbol{\lambda}}(F)$ is the largest $\boldsymbol{\lambda}$-evident contained in F, so (by B1)*

$$C^{\boldsymbol{\lambda}}(F) \equiv \underset{k \geq 1}{\wedge} \left(B^{\boldsymbol{\lambda}}\right)^k (F).$$

If $t \in C^{\boldsymbol{\lambda}}(F)$, we say that there is *common* $\boldsymbol{\lambda}$-*belief* at t. At t, everyone $\boldsymbol{\lambda}$-believes F, everyone $\boldsymbol{\lambda}$-believes that everyone $\boldsymbol{\lambda}$-believes F, and so on. These definitions parallel definitions in the formal economics literature on common beliefs, and we can use them to report a fixed point characterization of *common* $\boldsymbol{\lambda}$-*belief* in the manner of Aumann (1976) and Monderer and Samet (1989):

Proposition 7 *Event F is common $\boldsymbol{\lambda}$-belief at t if and only if there exists a $\boldsymbol{\lambda}$-evident event F' such that $t \in F' \subseteq F$;*

Proof. For the 'if' direction, note that since F' is $\boldsymbol{\lambda}$-evident, we have $F' \subseteq B^{\boldsymbol{\lambda}}(F') \subseteq B^{\boldsymbol{\lambda}}\left(B^{\boldsymbol{\lambda}}(F')\right) \subseteq \cdots$. From property B1, we then have $F' \subseteq F \subseteq B^{\boldsymbol{\lambda}}(F) \subseteq B^{\boldsymbol{\lambda}}\left(B^{\boldsymbol{\lambda}}(F)\right) \subseteq \cdots$. Hence, F is $\boldsymbol{\lambda}$-evident at t. For the 'only if' direction, if F is common $\boldsymbol{\lambda}$-belief at t, then $C^{\boldsymbol{\lambda}}(F) = B^{\boldsymbol{\lambda}}\left(C^{\boldsymbol{\lambda}}(F)\right)$, so that $C^{\boldsymbol{\lambda}}(F)$ is $\boldsymbol{\lambda}$-evident. ■

Lemma 8 *$C^{\boldsymbol{\lambda}}(T)$ is the largest $\boldsymbol{\lambda}$-evident event, i.e., if F' is $\boldsymbol{\lambda}$-evident then $F' \subseteq T$.*

This lemma shows that $C^{\boldsymbol{\lambda}}(T)$ is the equilibrium of the incomplete information game where action 1 is played the most. It is therefore a very special case of the observation of Vives (1990) that the largest equilibrium of a supermodular game can be found looking at the limit of best response dynamics starting at the largest strategy profile.

3.4 Characterizing Rationalizability

We now characterize rationalizable strategy profiles in terms of our generalized belief operators, in the analogous way that we characterized rationalizable strategies in our leading example of the investment game. We first define rationalizable actions as follows.

Definition 9 *Action a_i is rationalizable for type t_i if $a_i \in R_i^*(\boldsymbol{\lambda}, t_i)$, where*

$$\begin{aligned}
R_i^0(\boldsymbol{\lambda}, t_i) &= \{0,1\} \\
R_i^{k+1}(\boldsymbol{\lambda}, t_i) &= \left\{ a_i \in R_i^k(\boldsymbol{\lambda}, t_i) \;\middle|\; \begin{array}{l} \text{there exists } \mu_i \in \Delta(T_{-i} \times \Theta \times \{0,1\}) \text{ such that} \\ \text{(1) } \mu_i(t_{-i}, \theta, a_{-i}) > 0 \Rightarrow a_j \in R_j^k(\boldsymbol{\lambda}, t_j) \text{ for all } j \neq i \\ \text{(2) } \sum\limits_{a_{-i}} \mu_i(t_{-i}, \theta, a_{-i}) = \pi_i(t_{-i}, \theta \,|t_i) \\ \text{(3) } a_i \in \arg\max\limits_{a_i'} \sum\limits_{t_{-i}, \theta, a_{-i}} \mu_i(t_{-i}, \theta, a_{-i}) \, u_i((a_i', a_{-i}), \theta) \end{array} \right\} \\
R_i^*(\boldsymbol{\lambda}, t_i) &= \bigcap_{k \geq 1} R_i^k(\boldsymbol{\lambda}, t_i)
\end{aligned}$$

This corresponds to the definition of "interim correlated rationalizability" in Dekel, Fudenberg and Morris [DFM] (2007), who gave a formal epistemic argument that the interim correlated rationalizable actions are exactly those that are consistent with common knowledge of rationality and a type's higher order beliefs about Θ. They also show that there is the standard equivalence between (correlated) rationalizability and iterated dominance. An action is interim correlated rationalizable if and only if it survives iterated deletion of strictly interim dominated strategies (claim 1). The "correlation" in the definition arises because a player's type is allowed to have any - perhaps correlated - beliefs over others' actions, types and payoff states θ as long as he puts probability 1 on others' actions being rationalizable for their types (part (1) of the definition) and his beliefs are consistent with that type's beliefs about others types and payoff state. The alternative "interim independent rationalizability" solution concept discussed in DFM puts conditional independence restrictions on those beliefs. However, there will not be a difference between the ex and interim solution concepts in this environment because supermodularity will ensure that the critical conditional beliefs over opponents' actions will be point beliefs.

Now we have our characterization of rationalizable actions.

Proposition 10 *Action* 1 *is rationalizable for type* t_i *if and only if* $t_i \in C_i^{\boldsymbol{\lambda}}(T)$.

Recall that a product event F can be understood as a strategy profile, where F_i is the set of types of player i. The operator B is then the best response map on strategy profiles. Now T corresponds to the largest strategy profile and $C^{\boldsymbol{\lambda}}(T)$ is the strategy profile that arises in the limit when we iteratively apply the best response function. Thus the above proposition reflects the well known fact that best response dynamics starting with the largest strategy profile converges to the largest equilibrium in an incomplete information game with supermodular payoffs (Vives (1990)) and the largest equilibrium also correspondence to the largest rationalizable strategy profile (Milgrom and Roberts (1991)). As noted above, the difference between ex ante and interim rationalizability will not matter in this setting. For completeness, we will report a direct argument for the proposition which highlights the "infection argument" logic from the higher order beliefs literature and introduces some techniques we will appear to later.

Proof. In proving this result, it is insightful to introduce a dual operator to the $B^{\boldsymbol{\lambda}}$ operator. For any product event F, define $S^{\boldsymbol{\lambda}}(F)$ as

$$S^{\boldsymbol{\lambda}}(F) \equiv \neg B^{\boldsymbol{\lambda}}(\neg F) \tag{9}$$

To interpret $S^{\boldsymbol{\lambda}}(F)$, note that

$$S^{\boldsymbol{\lambda}}(F) = \left(\backsim B_i^{\boldsymbol{\lambda}_i}\left((\backsim F_i)_{i=1}^I\right)\right)_{i=1}^I$$

$B_i^{\boldsymbol{\lambda}_i}\left((\backsim F_i)_{i=1}^I\right)$ is the set of player i's types for whom action 1 is a best reply when, for all $j \neq i$, player j plays action 0 on F_j. Then $\backsim B_i^{\lambda_i}\left(\times_{i=1}^I \backsim F_i\right)$ is the set of player i's types who *strictly* prefer to play action 0 when player j plays action 0 on F_j, for all $j \neq i$. Thus, $S^{\boldsymbol{\lambda}}(F)$ is the set of type profiles who strictly prefer to play action 0 when action zero is played on F. Note that $S^{\boldsymbol{\lambda}}(F)$ is a simple event, when F is a simple event.

In particular, when $F = \emptyset$, the event $S^{\boldsymbol{\lambda}}(\emptyset)$ consists of the type profiles for whom playing action 0 is strictly dominant. This is so, since the these types strictly prefer to play action 0 even if no other types play action 0. The event

$$S^{\boldsymbol{\lambda}}\left(S^{\boldsymbol{\lambda}}(\emptyset)\right)$$

consists of type profiles who strictly prefer to play action 0 when all type profiles in $S^{\boldsymbol{\lambda}}(\emptyset)$ play action 0. In other words, $S^{\boldsymbol{\lambda}}\left(S^{\boldsymbol{\lambda}}(\emptyset)\right)$ is the set of type profiles who strictly prefer action 0 when faced with types who do not use first-order dominated actions. Iterating the $S^{\boldsymbol{\lambda}}$ operator, the event

$$\left(S^{\boldsymbol{\lambda}}\right)^{k+1}(\emptyset)$$

is the set of type profiles who strictly prefer action 0 when faced with types who do not use kth order dominated actions. Then, the join defined as

$$\bigvee_{k\geq 1}\left(S^{\boldsymbol{\lambda}}\right)^{k}(\emptyset) \tag{10}$$

is the simple event consisting of type profiles who strictly prefer to play action 0 after the iterated deletion of strictly dominated strategies. Thus, action 1 is rationalizable for player i if only if action 1 is a best reply when other types play action 1 in the negation of (10). That is, action 1 is rationalizable for type t_i player i if and only if

$$\begin{aligned} t_i &\in B_i^{\lambda_i}\left(\neg\bigvee_{k\geq 1}\left(S^{\boldsymbol{\lambda}}\right)^{k}(\emptyset)\right) \\ &= B_i^{\lambda_i}\left(\bigcap_{k\geq 1}\neg\left(S^{\boldsymbol{\lambda}}\right)^{k}(\emptyset)\right) \\ &= B_i^{\lambda_i}\left(\bigcap_{k\geq 1}\left(B^{\boldsymbol{\lambda}}\right)^{k}(T)\right) \\ &= B_i^{\lambda_i}\left(C^{\boldsymbol{\lambda}}(T)\right) \end{aligned}$$

This proves the proposition. ■

Naturally, we can carry out an exactly analogous analysis for action 0. Define $\widetilde{\lambda}_i$ be

$$\widetilde{\lambda}_i(Z,\theta) = -\lambda_i\left(\mathcal{I}\setminus(Z\cup\{i\})\right).$$

Then we have

Proposition 11 *Action* 0 *is rationalizable for type* t_i *if and only if* $t_i \in C_i^{\widetilde{\boldsymbol{\lambda}}}(T)$.

Say that *dominance solvability* holds if $R_i^*(\boldsymbol{\lambda}, t_i) = \{0\}$ or $\{1\}$ for all i and t_i.

Corollary 12 *There is a unique rationalizable action for each type if and only if* $C^{\boldsymbol{\lambda}}(T) = \neg C^{\widetilde{\boldsymbol{\lambda}}}(T)$.

4 Characterizing Belief Hierarchies

We are now in a position to utilize our result on rationalizability to characterize the belief hierarchies of players in a global game. We take the point of view of an outside observer. We have just observed a player taking action 1. What can we infer from the action about the beliefs of the player? We illustrate the scope of the generalized belief operator by listing a number of examples of global games, some of which have received attention in the applied literature in financial economics and macroeconomics.

We start with our leading example, discussed in an earlier section.

Investment Game Revisited

When $\underline{\theta} \leq \theta \leq \bar{\theta}$, then successful coordination is possible only if proportion q or more invest. The cost of investing is $p \in (0,1)$, and the gross return to investing is 1. The payoff to not investing is 0. In this case, we have

$$\lambda_i (Z, \theta) = \begin{cases} 1 - p \text{ if } \theta > \bar{\theta} \\ 1 - p, \text{ if } \frac{\#Z+1}{I} \geq q \text{ and } \underline{\theta} \leq \theta \leq \bar{\theta} \\ -p, \text{ otherwise} \end{cases}$$

From our proposition on rationalizability, "invest" is rationalizable for a player if and only if the player p-believes all of the following

1. $\theta \geq \underline{\theta}$
2. $\theta \geq \overline{\theta}$ or proportion at least q p-believe that $\theta \geq 0$
3. $\theta \geq \overline{\theta}$ or proportion at least q p-believe that [$\theta \geq \overline{\theta}$ or proportion at least q p-believes that $\theta \geq 0$]
4. and so on...

"Regime Change" Game

There is a cost of investing of $p \in (0,1)$. The gross return to investing is 1 if proportion investing is at least $f(\theta)$, and it is 0 otherwise. The payoff to not investing is 0. These are the payoffs in Morris and Shin's (1998) paper on currency attacks. The λ_i function that corresponding to these payoffs is given by

$$\lambda_i(Z,\theta) = \begin{cases} 1-p, \text{ if } \frac{\#Z+1}{I} \geq f(\theta) \\ -p, \text{ otherwise} \end{cases}$$

Coordination is successful only if the proportion investing is least $f(\theta)$, where f is a non-increasing function of the fundamentals θ.

Assume that $f(\theta) > 1$ if and only if $\theta < 0$. In this case, Invest is a rationalizable action for a player if and only if he p-believes all of the following.

1. $\theta \geq 0$
2. the proportion of players who p-believe that $\theta \geq 0$ is at least $f(\theta)$
3. the proportion of players who p-believe that [the proportion of players who p-believe that $\theta \geq 0$ at least $f(\theta)$] is at least $f(\theta)$
4. and so on....

Linear "Regime Change" Game

This is the special case of the regime change game where

$$f(\theta) = 1 - \theta.$$

Thus, the gross return to investing is 1 if proportion investing is at least $1-\theta$, and it is 0 otherwise. The payoff to not investing is 0. The λ_i function corresponding to these payoffs is

$$\lambda_i(Z,\theta) = \begin{cases} 1-p, \text{ if } \frac{\#Z+1}{I} \geq 1-\theta \\ -p, \text{ otherwise} \end{cases}$$

These payoffs have become the canonical global game payoff structure in recent papers, such as Dasgupta (2001), Metz (2002), Angeletos, Hellwig and Pavan (2006, 2007), and others.[3]

For the linear regime change game, invest is a rationalizable action for a player if and only if he p-believes all of the following.

1. $\theta \geq 0$

2. the proportion of players who p-believe that $\theta \geq 0$ is at least $1-\theta$

3. the proportion of players who p-believe [the proportion of players who p-believe that $\theta \geq 0$ at least $1-\theta$] is at least $1-\theta$

4. and so on....

Linear Payoff Game

Payoff to invest is $\theta - l$, where l is the proportion of opponents not investing. Payoff to not invest is 0.

$$\lambda_i(Z,\theta) = \theta - 1 + \frac{\#Z}{I-1}$$

These payoffs were examined by Morris and Shin (2001, 2003), and has figured in applied papers such as Plantin, Sapra and Shin (2005). Invest is rationalizable for player 1 only if all of the following hold.

1. player 1's expectation of θ is at least 0, i.e., $E_1(\theta) \geq 0$

[3] Morris and Shin introduced these payoffs in their 1999 invited lecture at the Econometric Society European meetings in Santiago de Compostella, eventually published as Morris and Shin (2004).

2. player 1's expectation of θ is at least one minus player 1's expectation of the proportion of others with expectation of θ at least 0, i.e., $E_1(\theta) \geq 1 - \Pr_1(E_2(\theta) \geq 0)$

3. player 1's expecation of θ is at least one minus player 1's expectations of the proportion of others with expectation of θ at least one minus others' expectation of the proportion of others with expectation of θ at least 0

4. and so on ...

The two person version of this game has an especially simple structure. The payoff function is

	Invest	Not Invest
Invest	θ, θ	$\theta - 1, 0$
Not Invest	$0, \theta - 1$	$0, 0$

Then invest is rationalizable for player 1 if and only if all of the following hold.

1. player 1's expectation of θ is at least 0, i.e., $E_1(\theta) \geq 0$

2. player 1's expectation of θ is at least one minus player 1's probability that player 2's expectation of θ is at least 0, i.e., $E_1(\theta) \geq 1 - \Pr_1(E_2(\theta) \geq 0)$

3. player 1's expecation of θ is at least one minus player 1's probability that player 2's probability that player 1's expectation of θ is at least 0, i.e., $E_1(\theta) \geq 1 - \Pr_1(E_2(\theta) \geq 1 - \Pr_2(E_1(\theta) \geq 0))$

4. and so on ...

Contribution Game

The public good contribution game is a "private values" version of a global game. Let $\Theta \subseteq \mathbb{R}^I$. The cost of investing is $-\theta_i$. The return to investing is 0 if proportion at least κ invest, -1 otherwise.

$$\lambda_i(Z,\theta) = \begin{cases} \theta_i, \text{ if } \frac{\#Z}{I-1} \geq \kappa \\ \theta_i - 1, \text{ otherwise} \end{cases}$$

In this context, invest is rationalizable for player 1 only if all of the following hold.

1. player 1's expectation of θ_1 is at least 0, e.g., $E_1(\theta_1) \geq 0$
2. player 1's expectation of θ_1 is at least one minus player 1's probability that the proportion of others with expectation of θ_i at least 0 is at least κ.
3. and so on ...

5 Uniqueness

We now turn our attention to sufficient conditions for dominance solvability. The perspective of common belief gives us new insights into the properties belief hierarchies that yield uniqueness. We report on two sufficient conditions for uniqueness. We begin with the common certainty of rank beliefs.

5.1 Common Certainty of Rank Beliefs

Common certainty of rank beliefs relies on a large degree of symmetry in the game, and has considerable affinity with many uses of global games seen in the applied literature. The argument for uniqueness is a generalization of the argument we gave for the example of the investment game given in section 2.

Payoffs $\boldsymbol{\lambda}$ are *separable-symmetric* if there exist a non-decreasing function $g : \{0, 1, .., I - 1\} \to \mathbb{R}$ and a function $h : \Theta \to \mathbb{R}$ such that

$$\lambda_i (Z, \theta) = g (\#Z) + h (\theta)$$

for all $i = 1, .., I$, $Z \subseteq \mathcal{I}/\{i\}$ and $\theta \in \Theta$. We will maintain this assumption throughout this section. With separable-symmetric payoffs, a type $t_i \in T_i$ has a *strictly dominant strategy to choose action* 1 if

$$g (0) + \sum_{t_{-i},\theta} \pi_i (t_i) [t_{-i}, \theta] h (\theta) > 0;$$

and a type $t_i \in T_i$ has a *strictly dominant strategy to choose action* 0 if

$$g (I - 1) + \sum_{t_{-i},\theta} \pi_i (\underline{t}_i) [t_{-i}, \theta] h (\theta) < 0.$$

Limit dominance is satisfied if there exists at least one type of one player with a strictly dominant strategy to choose action 1 and at least one type of one player with a strictly dominant strategy to choose action 0. A type is said to be *strategic* if neither action is strictly dominant for that type.

We introduce the following complete order on the union of all types, $T_\cup = \cup_{i=1}^I T_i$:

$$t_i \succeq t_j \text{ if } \sum_{t_{-i},\theta} \pi_i (t_i) [t_{-i}, \theta] h (\theta) \geq \sum_{t_{-j},\theta} \pi_j (t_j) [t_{-j}, \theta] h (\theta) .$$

In other words, each type is ordered by his beliefs on the fundamentals θ. High types are those that have high expectations of fundamentals. Now let $\rho_i : T_i \to \Delta (\{0, ..., I - 1\})$ be a player's belief about his rank, so

$$\rho_i (t_i) [k] = \sum_{t_{-i},\theta} \pi_i (t_i) [\{(t_{-i}, \theta) | \# \{j \neq i | t_j \prec t_i\} \} = k - 1] .$$

Now $\rho_i(t_i)[k]$ is the probability that player i attaches to there being exactly $k-1$ players' having a lower expectation of θ. Define

$$\rho_i(t_i) \equiv (\rho_i(t_i)[1], \rho_i(t_i)[2], \cdots, \rho_i(t_i)[I])$$

as the mapping that associates with each type the density over possible ranks for that player. *Constant common rank beliefs of strategic types* is satisfied if there exists $r^* \in \Delta(\{0, ..., I-1\})$ such that for each player i and each strategic type $t_i \in T_i$, $\rho_i(t_i) = r^*$.

Finally, we will use three "technical" assumptions. We label them technical assumptions because they satisfied for free in the standard continuous signal global game environment with smooth densities. One merit of our discrete formulation is that it forces us to make explicit assumptions that are implicit in the standard formulation.

There is *uniform separation* if there exists $\varepsilon^* > 0$ such that for any i and $t_i, t_i' \in T_i$,

$$\begin{aligned} \sum_{t_{-i},\theta} \pi_i(t_i)[t_{-i},\theta] h(\theta) &\neq \sum_{t_{-i},\theta} \pi_i(t_i')[t_{-i},\theta] h(\theta) \\ &\Rightarrow \left| \sum_{t_{-i},\theta} \pi_i(t_i)[t_{-i},\theta] h(\theta) - \sum_{t_{-i},\theta} \pi_i(t_i')[t_{-i},\theta] h(\theta) \right| \geq \varepsilon^* \end{aligned}$$

In other words, if one type of a player has a higher expectation of θ than another, the difference exceeds some uniform amount ε^*. There are *no rank ties* if $t_i \succ t_j$ or $t_j \succ t_i$ for all $i \neq j$. There are *no common rank payoff ties* if

$$\sum_{n=0}^{I-1} r^*(n+1) g(n) + \sum_{t_{-i},\theta} \pi_i(t_i)[t_{-i},\theta] h(\theta) \neq 0$$

for all i and $t_i \in T_i$.

Proposition 13 *If separable-symmetric payoffs, limit dominance, constant common rank beliefs of strategic types, uniform separation, no rank ties and no common rank payoff ties satisfied, then dominance solvability holds. If* r^* *is the common rank belief held by all strategic types, action* 1 *is the unique rationalizable action for type* t_i *of player* i *if*

$$\sum_{n=0}^{I-1} r^* (n+1)\, g(n) + \sum_{t_{-i},\theta} \pi_i (t_i) [t_{-i}, \theta]\, h(\theta) > 0;$$

and action 0 *is the unique rationalizable action of type* t_i *of player* i *if*

$$\sum_{n=0}^{I-1} r^* (n+1)\, g(n) + \sum_{t_{-i},\theta} \pi_i (t_i) [t_{-i}, \theta]\, h(\theta) < 0.$$

We can paraphrase our result as: *Common certainty of common rank beliefs for strategic types implies dominance solvability*, where "common certainty" denotes "common 1-belief," which is often described as common knowledge in the economics literature.

Proof. Limit dominance implies that there exists a player j and type $\bar{t}_j$ such that

$$c = g(0) + \sum_{t_{-j},\theta} \pi_j \left(\bar{t}_j\right) [t_{-j}, \theta]\, h(\theta) > 0. \tag{11}$$

Now for each i,

$$\begin{aligned} \left\{t_i \in T_i \left| t_i \succeq \bar{t}_j \right.\right\} &= \left\{ t_i \in T_i \left| g(0) + \sum_{t_{-i},\theta} \pi_i (t_i) [t_{-i}, \theta]\, h(\theta) \geq c \right.\right\} \\ &\subseteq S_i^{\lambda_i} (\varnothing). \end{aligned} \tag{12}$$

Now we establish the following claim by induction on k: for each i and

$k = 0, 1, ...$

$$\left\{ t_i \in T_i \left| \sum_{n=0}^{I-1} \left\{ \begin{array}{c} r^*(n+1)\, g(n) \\ + \sum_{t_{-i},\theta} \pi_i(t_i)[t_{-i},\theta]\, h(\theta) \end{array} \right\} \begin{array}{l} \geq c + \sum_{n=0}^{I-1} r^*(n+1)\, g(n) - g(0) - \varepsilon^* k \\ > 0 \end{array} \right. \right\} \tag{13}$$

$$\subseteq \quad S_i^{\lambda_i}\left[S^{\boldsymbol{\lambda}}\right]^k(\varnothing),$$

where ε^* is defined by the uniform separation assumption. Recall that $S_i^{\lambda_i}\left[S^{\boldsymbol{\lambda}}\right]^k(\varnothing)$ is the set of types of player i such that his unique kth level rationalizable action is to play 1. For $k = 0$, the claim follows from (11) and (12). Suppose that the claim holds for $k-1$ and that $t_i \in T_i$ satisfies

$$\sum_{n=0}^{I-1} r^*(n+1)\, g(n) + \sum_{t_{-i},\theta} \pi_i(t_i)[t_{-i},\theta]\, h(\theta) = c + \sum_{n=0}^{I-1} r^*(n+1)\, g(n) - g(0) - \varepsilon^* k > 0. \tag{14}$$

If t_i has a dominant strategy to play action 1, then $t_i \in S_i^{\lambda_i}(\varnothing) \subseteq S_i^{\lambda_i}\left[S^{\boldsymbol{\lambda}}\right]^k(\varnothing)$; (14) implies that t_i does not have a dominant strategy action to play action 0. If t_i does not have a dominant strategy, then common rank beliefs implies $\rho_i(t_i) = r^*$. Type t_i is certain (by uniform separation and the induction hypothesis) that all higher ranked players have types $t_j \in S_j^{\lambda_j}\left[S^{\boldsymbol{\lambda}}\right]^{k-1}(\varnothing)$ and therefore have a unique $(k-1)\,th$ rationalizable action to play action 1. So the expected payoff to playing action 1 is at least

$$\sum_{n=0}^{I-1} r^*(n+1)\, g(n) + \sum_{t_{-i},\theta} \pi_i(t_i)[t_{-i},\theta]\, h(\theta)$$

and $t_i \in S_i^{\lambda_i}\left[S^{\boldsymbol{\lambda}}\right]^k(\varnothing)$. This establishes the inductive step.

Now (13) implies

$$\begin{aligned} \left\{ t_i \in T_i \left| \sum_{n=0}^{I-1} r^*(n+1)\, g(n) + \sum_{t_{-i},\theta} \pi_i(t_i)[t_{-i},\theta]\, h(\theta) > 0 \right. \right\} &\subseteq \cup_{k\geq 1} S_i^{\lambda_i}\left[S^{\boldsymbol{\lambda}}\right]^k(\varnothing) \\ &= \left\{ t_i \in T_i \left| R_i(\boldsymbol{\lambda}, t_i) = \{1\} \right. \right\}. \end{aligned}$$

A symmetric argument implies

$$\left\{ t_i \in T_i \left| \sum_{n=0}^{I-1} r^* (n+1) g(n) + \sum_{t_{-i},\theta} \pi_i (t_i) [t_{-i}, \theta] h(\theta) < 0 \right. \right\} \begin{array}{l} \subseteq \cup_{k \geq 1} S_i^{\widetilde{\lambda}_i} \left[S^{\widetilde{\boldsymbol{\lambda}}} \right]^k (\varnothing) \\ = \{ t_i \in T_i \, | R_i (\boldsymbol{\lambda}, t_i) = \{0\} \} . \end{array}$$

No payoff ties implies

$$\left\{ t_i \in T_i \left| \sum_{n=0}^{I-1} r^* (n+1) g(n) + \sum_{t_{-i},\theta} \pi_i (t_i) [t_{-i}, \theta] h(\theta) < 0 \right. \right\} = \varnothing .$$

■

We briefly report two simple weakenings of the common rank beliefs under which the result will continue to hold.

First, consider the first order stochastic dominance order on rank beliefs, so that $r \trianglerighteq r'$ if, for each $n = 1, ..., I$,

$$\sum_{i=1}^{n} r(i) \leq \sum_{i=1}^{n} r'(i)$$

Say that there is *decreasing common rank beliefs* if, for any $t_i \in T_i$ and $t_j \in T_j$,

$$t_i \succeq t_j \Rightarrow \rho_i (t_i) \trianglelefteq \rho_j (t_j)$$

Now if we replaced the assumption of common and constant rank beliefs of strategic types with common and decreasing rank beliefs of strategic types, we would again have dominance solvability. In particular, action 1 (0) would be the unique rationalizable action if

$$\sum_{n=0}^{I-1} \rho_i (t_i) [n+1] + \sum_{t_{-i},\theta} \pi_i (t_i) [t_{-i}, \theta] h(\theta) > (<) 0.$$

Second, suppose that rank beliefs were not constant but that they did not change too fast relative to the expectations of fundamentals. Let

$$\xi = g(I-1) - g(0)$$

measure the strategic sensitivity of the game. Write $\Delta(r, r')$ for the distance between the rank beliefs r and r':

$$\Delta(r, r') = \max_{n=1,..,I} \left| \sum_{i=1}^{n} r(i) - \sum_{i=1}^{n} r'(i) \right|.$$

Say that there is *near constant common rank beliefs* if, for any $t_i \in T_i$ and $t_j \in T_j$,

$$\Delta\left(\rho_i(t_i), \rho_j(t_i)\right).\xi \leq \left| \sum_{t_{-i},\theta} \pi_i(t_i)[t_{-i},\theta] h(\theta) - \sum_{t_{-j},\theta} \pi_j\left(t'_j\right)[t_{-j},\theta] h(\theta) \right|.$$

Now if we replaced the assumption of common and constant rank beliefs of strategic types with near constant common rank beliefs of strategic types, we would again have dominance solvability. Morris and Shin (2005) describe a uniqueness result using this idea (where the near constant rank beliefs is delivered by "bounded marginals on differences" property. Mason and Valentinyi (2006) also used a related idea.

5.2 Common Certainty of Beliefs in Differences

We now present a second set of sufficient conditions for uniqueness that allows for asymmetry across players. Payoffs $\boldsymbol{\lambda}$ are *separable* if there exist increasing functions $\lambda_i^1 : 2^{\mathcal{I}/\{i\}} \to \mathbb{R}$ and $\lambda_i^2 : \Theta \to \mathbb{R}$ such that

$$\lambda_i(Z, \theta) = \lambda_i^1(Z) + \lambda_i^2(\theta)$$

The type space of each player is two-dimensional. A type has two components. The first component is completely ordered and we identify it with the set of integers $\mathcal{Z}$. The second component is any finite set Ψ_i. Thus, for each i, we have a bijection

$$g_i : T_i \to \mathcal{Z} \times \Psi_i.$$

The first component of a type can be interpreted as a signal received about the fundamentals θ, so that higher first components are associated with higher beliefs about θ. The second component is some other dimension along which players vary. However, note that the ordering applies only to the types of a single player, whereas the condition of common certainty of rank beliefs applied the ordering to the union of all types, and so we were ranking across players, also.

We now introduce our assumptions. Denote by $g_{i1}(t_i)$ the first component of $g_i(t_i)$.

Assumption (Uniform Monotonicity): There exists $\varepsilon > 0$ such that

$$\begin{aligned} g_{i1}(t_i) &> g_{i1}(t_i') \\ &\Rightarrow \sum_{t_{-i},\theta} \pi_i(t_i)[t_{-i},\theta]\,\lambda_i^2(\theta) > \sum_{t_{-i},\theta} \pi_i(t_i)[t_{-i},\theta]\,\lambda_i^2(\theta) + \varepsilon \end{aligned}$$

for all i, t_i, t_i'.

Assumption (Limit Dominance): For each i, there exist $\underline{t}_i$ and $\bar{t}_i$ such that

$$\begin{aligned} \lambda_i^1(\mathcal{I}/\{i\}) + \sum_{t_{-i},\theta} \pi_i(\underline{t}_i)[t_{-i},\theta]\,\lambda_i^2(\theta) &< 0 \\ \text{and } \lambda_i^1(\varnothing) + \sum_{t_{-i},\theta} \pi_i(\bar{t}_i)[t_{-i},\theta]\,\lambda_i^2(\theta) &> 0. \end{aligned}$$

Assumption (η-Diffuse Beliefs): There exists $\eta > 0$ such that, for each i and, for each $j \neq i$, $h_j : \Psi_j \rightarrow \mathcal{Z}$,

$$\sum_{\{t_{-i}: g_{j1}(t_j) = h_j(g_{j2}(t_j)) \text{ for some } j\},\theta} \pi_i(t_i)[t_{-i},\theta] < \eta$$

The last assumption and the uniformity requirement in the first assumption can be thought of as technical assumptions: they are required only because we are allowing for discrete type spaces and are not required (or are implicit) in the standard continuous signals global games framework.

Finally, we come to our key definition. Define player i's *beliefs about differences* $\xi_i : T_i \to \Delta\left(\left(\mathcal{Z} \times \ominus_j\right)_{j \neq i}\right)$ as follows:

$$\xi_i\left(t_i\right)\left[\left(\left(\delta_{j,}\psi_j\right)_{j\neq i},\theta\right)\right] = \pi_i\left(t_i\right)\left[\left\{\left(g_j^{-1}\left(g_{i1}\left(t_i\right)+\delta_j,\psi_j\right)\right)_{j\neq i}\right\}\times\Theta\right]. \quad (15)$$

To grasp the expression on the right hand side, note that $g_j^{-1}\left(g_{i1}\left(t_i\right)+\delta_j,\psi_j\right)$ is the type of player j whose first component is $g_{i1}\left(t_i\right)+\delta_j$, and whose second component is ψ_j. Thus, type t_i's beliefs about differences are t_i's beliefs over other players' types where player j's type is distance δ_j away along the first component.

Our sufficient condition for uniqueness rests on the beliefs about differences being insensitive to the ranking of a particular player's type. In other words, the function ξ_i defined in (15) is a constant function with respect to the first component of a player's type.

Proposition 14 *Assume uniform monotonicity and limit dominance. Then there exists $\overline{\eta} > 0$ such that, if $\eta \leq \overline{\eta}$ and there are η-diffuse beliefs, then common certainty of beliefs in differences implies dominance solvability.*

Proof. Figure 5 illustrates the argument. For each $k = 0, 1, \ldots$, there exists $\underline{h}_i^k : \Psi_i \to \mathcal{Z}$ and non-increasing $\overline{h}_i^k : \Psi_i \to \mathcal{Z}$ such that $\underline{h}_i^k\left(\psi_i\right)$ is non-decreasing in k for each ψ_i, $\overline{h}_i^k\left(\psi_i\right)$ is non-increasing in k for each ψ_i,

$$\begin{aligned} 1 &\in R_i^k\left(\boldsymbol{\lambda},t_i\right) \text{ if and only if } g_{i1}\left(t_i\right) \geq \underline{h}_i^k\left(g_{i2}\left(t_i\right)\right) \\ \text{and } 0 &\in R_i^k\left(\boldsymbol{\lambda},t_i\right) \text{ if and only if } g_{i1}\left(t_i\right) \leq \overline{h}_i^k\left(g_{i2}\left(t_i\right)\right) \end{aligned}$$

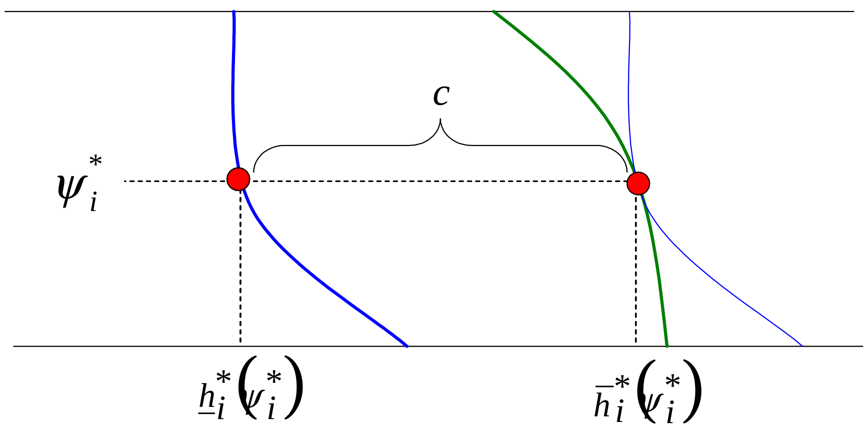

Figure 5: Translation

This can be proved by standard monotone methods, see, e.g., van Zandt and Vives (2006). Thus there exist $\underline{h}_i^*, \overline{h}_i^* : \Psi_i \rightarrow \mathcal{Z}$ such that $\underline{h}_i^*(\psi_i) \leq \overline{h}_i^*(\psi_i)$ for all $\psi_i \in \Psi_i$ and

$$\begin{aligned} 1 &\in R_i^*(\boldsymbol{\lambda}, t_i) \text{ if and only if } g_{i1}(t_i) \geq \underline{h}_i^*(g_{i2}(t_i)) \\ \text{and } 0 &\in R_i^*(\boldsymbol{\lambda}, t_i) \text{ if and only if } g_{i1}(t_i) \leq \overline{h}_i^*(g_{i2}(t_i)) \end{aligned} \tag{16}$$

Now we prove uniqueness by first supposing that $\underline{h}_i^* \neq \overline{h}_i^*$ (and then proving a contradiction). Let c be the smallest integer such that $\overline{h}_i^*(\psi_i) \leq \underline{h}_i^*(\psi_i) + c$ for all i and $\psi_i \in \Psi_i$. Observe that $c > 0$ and that there exists i and $\psi_i^* \in \Psi_i$ satisfying $\overline{h}_i^*(\psi_i^*) = \underline{h}_i^*(\psi_i^*) + c$. Now observe that by (16), we know that

$$1 \in R_i^*\left(\boldsymbol{\lambda}, g_i^{-i}\left(\underline{h}_i^*(\psi_i^*), \psi_i^*\right)\right)$$

and thus

$$\left\{ \begin{array}{l} \sum_{t_{-i},\theta} \pi_i\left(g_i^{-i}\left(\underline{h}_i^*(\psi_i^*), \psi_i^*\right)\right)[t_{-i},\theta]\, \lambda_i^1\left(\left\{j : g_{j1}(t_j) \geq \underline{h}_j^*(g_{j2}(t_j))\right\}\right) \\ + \sum_{t_{-i},\theta} \pi_i\left(g_i^{-i}\left(\underline{h}_i^*(\psi_i^*), \psi_i^*\right)\right)[t_{-i},\theta]\, \lambda_i^2(\theta) \end{array} \right\} \geq 0$$

Now suppose that player i is type $g_i^{-1}\left(\underline{h}_i^*\left(\psi_i^*\right)+c,\psi_i^*\right)$ and believes this his opponents are choosing action 1 if and only if $g_{j1}\left(t_j\right)\geq\underline{h}_j^*\left(g_{j2}\left(t_j\right)\right)+c$. Then by common knowledge of beliefs in differences

$$\begin{aligned}&\sum_{t_{-i},\theta}\pi_i\left(g_i^{-i}\left(\underline{h}_i^*\left(\psi_i^*\right)+c,\psi_i^*\right)\right)\left[t_{-i},\theta\right]\lambda_i^1\left(\left\{j:g_{j1}\left(t_j\right)\geq\underline{h}_j^*\left(g_{j2}\left(t_j\right)\right)+c\right\}\right)\\=&\sum_{t_{-i},\theta}\pi_i\left(g_i^{-i}\left(\underline{h}_i^*\left(\psi_i^*\right),\psi_i^*\right)\right)\left[t_{-i},\theta\right]\lambda_i^1\left(\left\{j:g_{j1}\left(t_j\right)\geq\underline{h}_j^*\left(g_{j2}\left(t_j\right)\right)\right\}\right)\end{aligned}$$

These are the payoffs from the strategic part of the payoff function that depends on the actions of others. On the other hand, the part of the payoff function that depends on the fundamentals θ can be ordered by the assumption of monotonicity

$$\begin{aligned}&\sum_{t_{-i},\theta}\pi_i\left(g_i^{-i}\left(\underline{h}_i^*\left(\psi_i^*\right)+c,\psi_i^*\right)\right)\left[t_{-i},\theta\right]\lambda_i^2\left(\theta\right)\\\geq&\sum_{t_{-i},\theta}\pi_i\left(g_i^{-i}\left(\underline{h}_i^*\left(\psi_i^*\right),\psi_i^*\right)\right)\left[t_{-i},\theta\right]\lambda_i^2\left(\theta\right)+c\varepsilon\end{aligned}$$

so, adding the two parts of the payoff function together, we have

$$\left\{\begin{array}{l}\sum_{t_{-i},\theta}\pi_i\left(g_i^{-i}\left(\underline{h}_i^*\left(\psi_i^*\right)+c,\psi_i^*\right)\right)\left[t_{-i},\theta\right]\lambda_i^1\left(\left\{j:g_{j1}\left(t_j\right)\geq\underline{h}_j^*\left(g_{j2}\left(t_j\right)+c\right)\right\}\right)\\+\sum_{t_{-i},\theta}\pi_i\left(g_i^{-i}\left(\underline{h}_i^*\left(\psi_i^*\right)+c,\psi_i^*\right)\right)\left[t_{-i},\theta\right]\lambda_i^2\left(\theta\right)\end{array}\right\}\geq c\varepsilon$$

Now observe that $\left\{j:g_{j1}\left(t_j\right)\geq\underline{h}_j^*\left(g_{j2}\left(t_j\right)+c\right)\right\}\subseteq\left\{j:g_{j1}\left(t_j\right)>\overline{h}_j^*\left(g_{j2}\left(t_j\right)\right)\right\}$ unless $g_{j1}\left(t_j\right)=\overline{h}_j^*\left(g_{j2}\left(t_j\right)\right)$ for some j. Thus

$$\begin{aligned}&\sum_{t_{-i},\theta}\pi_i\left(g_i^{-i}\left(\underline{h}_i^*\left(\psi_i^*\right)+c,\psi_i^*\right)\right)\left[t_{-i},\theta\right]\lambda_i^1\left(\left\{j:g_{j1}\left(t_j\right)>\overline{h}_j^*\left(g_{j2}\left(t_j\right)\right)\right\}\right)\\\geq&\sum_{t_{-i},\theta}\pi_i\left(g_i^{-i}\left(\underline{h}_i^*\left(\psi_i^*\right)+c,\psi_i^*\right)\right)\left[t_{-i},\theta\right]\lambda_i^1\left(\left\{j:g_{j1}\left(t_j\right)\geq\underline{h}_j^*\left(g_{j2}\left(t_j\right)+c\right)\right\}\right)-\eta\end{aligned}$$

and so

$$\left\{\begin{array}{l}\sum\limits_{t_{-i},\theta}\pi_i\left(g_i^{-i}\left(\overline{h}_i^*\left(\psi_i^*\right),\psi_i^*\right)\right)\left[t_{-i},\theta\right]\lambda_i^1\left(\left\{j:g_{j1}\left(t_j\right)>\overline{h}_j^*\left(g_{j2}\left(t_j\right)\right)\right\}\right)\\+\sum\limits_{t_{-i},\theta}\pi_i\left(g_i^{-i}\left(\overline{h}_i^*\left(\psi_i^*\right),\psi_i^*\right)\right)\left[t_{-i},\theta\right]\lambda_i^2\left(\theta\right)\end{array}\right\}$$
$$\geq\ c\varepsilon-\eta\left(\lambda_i^1\left(\mathcal{I}/\left\{i\right\}\right)-\lambda_i^1\left(\varnothing\right)\right)$$

For sufficiently small η, the right hand side is strictly positive, contradicting our assumption that

$$0\in R_i^*\left(\boldsymbol{\lambda},g_i^{-i}\left(\overline{h}_i^*\left(\psi_i^*\right),\psi_i^*\right)\right).$$

■

This proof appeals to the "translation" argument of contradiction of Frankel, Morris and Pauzner (FMP) (2003). In FMP, there were one dimensional continuous types but many (countable or continuum) actions for each player. Here we have a binary action game, but allow multidimensional discrete signals and not restricting to the standard noisy information structure of global games. Oury (2005) gives results for multidimensional global games (with multidimensional actions and signals) in the standard noise framework. In the next section, we present a simple example of multidimensional signal binary action game with continuous signals.

6 Multidimensional Example

Global game applications have typically focussed on games that are symmetric across players with one dimensional signals. Here, we sketch a simple example that is asymmetric across players and allows for multiple dimensional signals. It thus illustrates how the logic of proposition 14 could be useful in applications. For simplicity, we simply present a continuous signals example. At some cost of tractability, we could discretize the example so that 14 applied.

Two players must decide whether to "invest" or "not invest". The cost of investing is either $\overline{p}$ or $\underline{p}$, where $0 < \underline{p} < \overline{p} < 1$. Player i is high cost with probability λ_i. The return to investing is 1 if (i) $\theta > \overline{\theta}$ or (ii) if $\theta \geq \underline{\theta}$ and the other player invests; otherwise the return to investing is 0. Let θ be uniformly distributed on the real line. Player i observes a signal $x_i = \theta + \varepsilon_i$, where each ε_i is independently distributed with full support density f. Thus $\varepsilon_1 - \varepsilon_2$ and $\varepsilon_2 - \varepsilon_1$ are both distributed with the same symmetric density, h, where

$$h(\eta) = \int_{\varepsilon_1=-\infty}^{\infty} f(\varepsilon_1) f(\varepsilon_1 - \eta) \, d\varepsilon_1$$

Write H for the corresponding c.d.f. Assume that $\underline{\theta} < \overline{\theta}$ and that $\overline{\theta} - \underline{\theta}$ is very large.

A pure strategy for player i is a pair $s_i = (\underline{s}_i, \overline{s}_i)$, where $\underline{s}_i, \overline{s}_i : \mathbb{R} \to \{\text{invest,not invest}\}$ and $\underline{s}_i(x_i)$ $(\overline{s}_i(x_i))$ is player i's action if his cost is low (high). Strategy s_i is a $x_i^* = (\underline{x}_i^*, \overline{x}_i^*)$ threshold strategy for player i if

$$\underline{s}_i(x_i) = \begin{cases} \text{invest, if } x_i \geq \underline{x}_i^* \\ \text{not invest, if } x_i < \underline{x}_i^* \end{cases}$$

and

$$\overline{s}_i(x_i) = \begin{cases} \text{invest, if } x_i \geq \overline{x}_i^* \\ \text{not invest, if } x_i < \overline{x}_i^* \end{cases}$$

Proposition 15 *There is an essentially unique equilibrium of this game in threshold strategies. There exists δ (independent of $\underline{\theta}$ and $\overline{\theta}$) such that, if*

$$\lambda_1 \overline{p} + \left(1 - \lambda_1 \underline{p}\right) + \lambda_2 \overline{p} + \left(1 - \lambda_2 \underline{p}\right) < 1$$

player i invests $x_i \geq \underline{\theta} + \delta$; and if

$$\lambda_1 \overline{p} + \left(1 - \lambda_1 \underline{p}\right) + \lambda_2 \overline{p} + \left(1 - \lambda_2 \underline{p}\right) > 1$$

player i does not invest $x_i \leq \overline{\theta} - \delta$.

Thus the proposition identifies whether or investment occurs in (most of) the interval $\left[\underline{\theta}, \overline{\theta}\right]$ where there are multiple equilibria under complete information.

Proof. Now suppose that there exist a threshold strategy profile characterized by $x_1^* = (\underline{x}_1^*, \overline{x}_1^*)$ and $x_2^* = (\underline{x}_2^*, \overline{x}_2^*)$, with $\underline{\theta} \ll \underline{x}_i^* \leq \overline{x}_i^* \ll \overline{\theta}$ such that the best response to the strategy profile is to invest more. Then the action "invest" will infect the intermediate region. Does there exist such a threshold strategy profile? Assume specifically that the expected payoff to investing for the marginal signal is c. Then we must have:

$$\begin{aligned}
\lambda_2 \Pr\left(x_2 \geq \overline{x}_2^* | x_1 = \underline{x}_1^*\right) + (1 - \lambda_2) \Pr\left(x_2 \geq \underline{x}_2^* | x_1 = \underline{x}_1^*\right) - \underline{p} &= c \\
\lambda_2 \Pr\left(x_2 \geq \overline{x}_2^* | x_1 = \overline{x}_1^*\right) + (1 - \lambda_2) \Pr\left(x_2 \geq \underline{x}_2^* | x_1 = \overline{x}_1^*\right) - \overline{p} &= c \\
\lambda_1 \Pr\left(x_1 \geq \overline{x}_1^* | x_2 = \underline{x}_2^*\right) + (1 - \lambda_1) \Pr\left(x_1 \geq \underline{x}_1^* | x_2 = \underline{x}_2^*\right) - \underline{p} &= c \\
\lambda_1 \Pr\left(x_1 \geq \overline{x}_1^* | x_2 = \overline{x}_2^*\right) + (1 - \lambda_1) \Pr\left(x_1 \geq \underline{x}_1^* | x_2 = \overline{x}_2^*\right) - \overline{p} &= c
\end{aligned}$$

Now substitute in that

$$\Pr\left(x_j \geq \widehat{x}_j^* | x_i = \widehat{x}_i^*\right) = \Pr\left(x_i - x_j = \varepsilon_i - \varepsilon_j \leq \widehat{x}_i^* - \widehat{x}_j^*\right) = H\left(\widehat{x}_i^* - \widehat{x}_j^*\right)$$

so

$$\begin{aligned}
\lambda_2 H\left(\underline{x}_1^* - \overline{x}_2^*\right) + (1 - \lambda_2) H\left(\underline{x}_1^* - \underline{x}_2^*\right) - \underline{p} &= c \qquad (17) \\
\lambda_2 H\left(\overline{x}_1^* - \overline{x}_2^*\right) + (1 - \lambda_2) H\left(\overline{x}_1^* - \underline{x}_2^*\right) - \overline{p} &= c \qquad (18) \\
\lambda_1 H\left(\underline{x}_2^* - \overline{x}_1^*\right) + (1 - \lambda_1) H\left(\underline{x}_2^* - \underline{x}_1^*\right) - \underline{p} &= c \qquad (19) \\
\lambda_1 H\left(\overline{x}_2^* - \overline{x}_1^*\right) + (1 - \lambda_1) H\left(\overline{x}_2^* - \underline{x}_1^*\right) - \overline{p} &= c \qquad (20)
\end{aligned}$$

Now observe that since h is symmetric by construction, $H(-x) = 1 - H(x)$. Thus (19) and (20) can be re-written as

$$\begin{aligned} \lambda_1 \left(1 - H\left(\overline{x}_1^* - \underline{x}_2^*\right)\right) + \left(1 - \lambda_1\right)\left(1 - H\left(\underline{x}_1^* - \underline{x}_2^*\right)\right) - \underline{p} &= c && (21) \\ \lambda_1 \left(1 - H\left(\overline{x}_1^* - \overline{x}_2^*\right)\right) + \left(1 - \lambda_1\right)\left(1 - H\left(\underline{x}_1^* - \overline{x}_2^*\right)\right) - \overline{p} &= c && (22) \end{aligned}$$

Now multiplying equations (17), (18), (21) and (22) by $1 - \lambda_1, \lambda_1, 1 - \lambda_2$ and λ_2 respectively, we get:

$$\begin{aligned} \left(1 - \lambda_1\right)\lambda_2 H\left(\underline{x}_1^* - \overline{x}_2^*\right) + \left(1 - \lambda_1\right)\left(1 - \lambda_2\right) H\left(\underline{x}_1^* - \underline{x}_2^*\right) - \left(1 - \lambda_1\right)\underline{p} &= \left(1 - \lambda_1\right)c && (23) \\ \lambda_1 \lambda_2 H\left(\overline{x}_1^* - \overline{x}_2^*\right) + \lambda_1\left(1 - \lambda_2\right) H\left(\overline{x}_1^* - \underline{x}_2^*\right) - \lambda_1 \overline{p} &= \lambda_1 c && (24) \\ \lambda_1\left(1 - \lambda_2\right)\left(1 - H\left(\overline{x}_1^* - \underline{x}_2^*\right)\right) + \left(1 - \lambda_1\right)\left(1 - \lambda_2\right)\left(1 - H\left(\underline{x}_1^* - \underline{x}_2^*\right)\right) - \left(1 - \lambda_2\right)\underline{p} &= \left(1 - \lambda_2\right)c && (25) \\ \lambda_1 \lambda_2 \left(1 - H\left(\overline{x}_1^* - \overline{x}_2^*\right)\right) + \left(1 - \lambda_1\right)\lambda_2\left(1 - H\left(\underline{x}_1^* - \overline{x}_2^*\right)\right) - \lambda_2 \overline{p} &= \lambda_2 c && (26) \end{aligned}$$

Now summing equations (23), (24), (25) and (26), we obtain:

$$1 - \left(\lambda_1 \overline{p} + \left(1 - \lambda_1 \underline{p}\right)\right) - \left(\lambda_2 \overline{p} + \left(1 - \lambda_2 \underline{p}\right)\right) = 2c$$

or

$$c = \frac{1}{2}\left(1 - \left(\lambda_1 \overline{p} + \left(1 - \lambda_1\right)\underline{p}\right) - \left(\lambda_2 \overline{p} + \left(1 - \lambda_2 \underline{p}\right)\right)\right)$$

Thus for any H, investment invades if and only if the average expected cost of investment is at most $\frac{1}{2}$. ■

7 Concluding Remarks

This paper has aimed at achieving the following objectives. First, we have presented a global game analysis where we dispense with talk of "noisy signals", and instead deal with type spaces directly. Second, with our framework, we have been able to characterize the higher-order beliefs that allow the global game argument to "work". Essentially, the property that matters is the stationarity of beliefs with respect to the ordering of types in the region where the players do not have dominant actions. Finally, by characterizing the beliefs and higher order beliefs that are necessary and sufficient for an action to be rationalizable, we have shed light on precisely what kind of departure from common knowledge is underpinning play in global games.

By focusing on the underlying belief foundations as the basis for play in global games, we have taken a step away from the practice of identifying the global game approach as being tied to a particular formalism of noisy signals with public and private information. To the extent that multiplicity is restored in some cases, it is because one or more of the stationarity of beliefs with respect to types is violated. Reorienting the questions in this way tells us whether the particular context is a plausible setting for analysis. The new perspective can therefore be illuminating in helping applied researchers to settle questions of when uniqueness may be a reasonable outcome.

References

[1] Angeletos, G.-M., C. Hellwig and A. Pavan (2006). "Signalling in a Global Game: Coordination and Policy Traps," *Journal of Political Economy* 114, 452-484.

[2] ——— (2007). "Dynamic Global Games of Regime Change: Learning, Multiplicity and Timing of Attacks" *Econometrica*, 75, 711-756

[3] Angeletos, G-M, and I. Werning (2006). "Crises and Prices: Information Aggregation, Multiplicity and Volatility," *American Economic Review,* 96, 1720-1736

[4] Atkeson, A. (2001). "Discussion of Morris and Shin," *NBER Macroeconomics Annual 2000*, 161-164.

[5] Aumann, R. (1976). "Agreeing to Disagree," *Annals of Statistics* 4, 1236-1239.

[6] Carlsson, H. and E. van Damme (1993). "Global Games and Equilibrium Selection," *Econometrica* 61, 989-1018.

[7] Dekel, Fudenberg and Morris (2007) "Interim Correlated Rationalizability" *Theoretical Economics* 2, 15-40.

[8] Frankel, D., S. Morris and A. Pauzner (2003). "Equilibrium Selection in Global Games with Strategic Complementarities, *Journal of Economic Theory* 108, 1-44.

[9] Hellwig, C. (2002). "Public Information, Private Information and the Multiplicity of Equilibria in Coordination Games," *Journal of Economic Theory* 107, 191-222.

[10] Hellwig, C., A. Mukherji and A. Tsyvinski (2006). "Self-Fulfilling Currency Crises: the Role of Interest Rates" *American Economic Review,* 95

[11] Izmalkov, S. and M. Yildiz (2006) "Investor Sentiments." working paper, MIT

[12] Kurz (2006). "Private Information and Diverse Beliefs: How Different?" working paper, Stanford University

[13] Mason, R. and A. Valentinyi (2006). "The Existence and Uniqueness of Monotone Pure Strategy Equilibrium."

[14] Milgrom, P. and J. Roberts (1990). "Rationalizability, Learning and Equilibrium in Games with Strategic Complementarities," *Econometrica* 58, 1255-1277.

[15] Monderer, D. and D. Samet (1989). "Approximating Common Knowledge with Common Beliefs," *Games and Economic Behavior* 1, 170-190.

[16] ——— (1998). "Unique Equilibrium in a Model of Self-Fulfilling Currency Attacks," *American Economic Review* 88, 587-597.

[17] ——— (2003). "Global Games: Theory and Applications," in *Advances in Economics and Econometrics (Proceedings of the Eighth World Congress of the Econometric Society)*, edited by M. Dewatripont, L. Hansen and S. Turnovsky. Cambridge, England: Cambridge University Press (2003), 56-114.

[18] ——— (2004). "Coordination Risk and the Price of Debt," *European Economic Review* 48, 133-153.

[19] ——— (2005c). "Heterogeneity and Uniqueness in Interaction Games," in *The Economy as an Evolving Complex System III*, edited by L. Blume and S. Durlauf. Santa Fe Institute Studies in the Sciences of Complexity. New York: Oxford University Press, 207-242.

[20] Oury, M. (2005). "Multidimensional Global Games," HEC Paris.

[21] Plantin, G., H. Sapra and H. S. Shin (2006) "Marking to Market: Panacea or Pandora's Box?" forthcoming in the *Journal of Accounting Research*.

[22] Sims, C. (2005a). "Comment on 'Central Bank Transparency and the Signal Value of Prices'" *Brookings Papers on Economic Activity* 2, 51-58.

[23] ——— (2005b). "Rational Inattention: A Research Agenda," available at http://sims.princeton.edu/RIplus

[24] Svensson, L. (2006) "'The Social Value of Public Information: Morris and Shin (2002) is Actually Pro Transparency, Not Con," *American Economic Review* 96, 448-452.

[25] Van Zandt, T. and X. Vives (2007) "Monotone Equilibria in Bayesian Games with Strategic Complementarities" *Journal of Economic Theory* 134, 339–360

[26] Vives, X. (1990). "Nash Equilibrium with Strategic Complementarities," *Journal of Mathematical Economics* 19, 305-321.

[27] Weinstein, J. and M. Yildiz (2007) "A Structure Theorem for Rationalizability with Application to Robust Predictions of Refinements" *Econometrica*, 75(2), 365-400

[28] Woodford, M. (2005). "Central-Bank Communication and Policy Effectiveness," in *The Greenspan Era: Lessons for the Future*, Kansas City: Federal Reserve Bank of Kansas City.