Information and Interaction

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Three Rationalizations Motivation: Idiosyncratic Shocks and Aggregate Volatility

- Suppose each agent in a large population receives an idiosyncratic shock to his productivity. Will these shocks give rise to aggregate volatility of output or will they wash out by a law of large numbers?

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- They might be hard to distinguish
- Natural to combine networks and information to think about interaction

Networks and Information

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- Consider a network game with quadratic payoffs (linear best response) and normally distributed payoff shocks
- For a fixed network game, characterize what can happen for all information structures at once

This Talk

1. characterization of all outcomes and "robust predictions / information design" agenda

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3. three rationalizations (an example of an application)

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2. one dimensional signals (an example of a particular information structure)
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4. networks and information: alternative approaches

Game

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$$u_i\left(a, \theta_i\right) = \left(\sum_{j \neq i} \gamma_{ij} a_j + \theta_i\right) a_i + \frac{1}{2} \gamma_{ii} a_i^2$$

and where θ_i is agent's i "payoff type" and $\gamma_{ii} < 0$.

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  and where θ_i is agent's i "payoff type" and $\gamma_{ii} < 0$.
- so agent i will have linear best response, choosing a_i to satisfy

$$\forall i \in N, \quad \mathbb{E}[\theta_i + \sum_{j=1}^{N} \gamma_{ij} a_j | a_i] = 0. \tag{1}$$

  or

$$a_i = -\frac{1}{\gamma_{ii}} \left(\theta_i + \sum_{j \neq i} \gamma_{ij} a_j\right)$$

Game

- the strategic interaction is characterized by the parameters $\{\gamma_{ij}\}_{i,k \in N}$, which are represented by:

$$\Gamma = \begin{pmatrix} \gamma_{11} & \cdots & \gamma_{1N} \\ \vdots & \ddots & \vdots \\ \gamma_{N1} & \cdots & \gamma_{NN} \end{pmatrix}$$

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- payoff types are jointly normally distributed:

$$\begin{pmatrix} \theta_1 \\ \vdots \\ \theta_N \end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix} \mu_{\theta_1} \\ \vdots \\ \mu_{\theta_N} \end{pmatrix}, \Sigma_{\theta\theta} \right),$$

where $\Sigma_{\theta\theta}$ is an arbitrary positive definite matrix.

Information Structure

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Information Structure

- Traditional approach: solve for some particular information structures
  1. complete information
  2. solve for class of one dimensional signals
  3. solve for general signals
- Different (sometimes better?) approach: solve for what could happen for all (normal) information structures

Bayes correlated equilibrium

Definition

An outcome (joint distribution of $(\theta_1, ..., \theta_N, a_1, ..., a_N)$) form a Bayes correlated equilibrium if the marginal distribution $(\theta_1, ..., \theta_N)$ over payoff states coincides with the common prior and:

$$\forall i, \forall a_i, \quad \mathbb{E}_\mu[\theta_i + \sum_{j=1}^{N} \gamma_{ij} a_j | a_i] = 0$$

Theorem

An outcome arises as the Bayes Nash equilibrium of the game with some information structure if and only if it is a Bayes correlated equilibrium

- no reference to information structures, just restrictions on the set of random variables corresponding to obedience constraints

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- no reference to information structures, just restrictions on the set of random variables corresponding to obedience constraints
- true for arbitrary games: Bergemann and Morris (2016); for symmetric linear best response games: Bergemann-Morris (2013) and Bergemann-Heumann-Morris (2015)

Bayes correlated equilibrium

Theorem

A joint distribution μ of variables $(\theta_1, ..., \theta_N, a_1, ..., a_N)$ forms a normal Bayes correlated equilibrium if and only if (1) the mean vector of actions satisfies:

$$\begin{pmatrix} \mu_{a_1} \\ \vdots \\ \mu_{a_N} \end{pmatrix} = -\Gamma^{-1} \cdot \begin{pmatrix} \mu_{\theta_1} \\ \vdots \\ \mu_{\theta_N} \end{pmatrix}; \quad (2)$$

(2) the variance of individual actions satisfies:

$$\begin{pmatrix} \sigma_{a_1} \\ \vdots \\ \sigma_{a_N} \end{pmatrix} = -(P_{aa} \circ \Gamma)^{-1} \cdot \begin{pmatrix} \sigma_{\theta_1} corr(\theta_1, a_1) \\ \vdots \\ \sigma_{\theta_N} corr(\theta_N, a_N) \end{pmatrix}; \quad (3)$$

(3) the correlation matrix $corr(\theta_1, ..., \theta_N, a_1, ..., a_N)$ is positive semi-definite..

Bayes correlated equilibrium

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- the set of feasible correlation matrices are independent of the interaction matrix and depend only on correlation matrix $P_{\theta\theta}$.
- the variance of actions depends on the correlations of actions P_{aa} which is arbitrary and the interaction matrix.

Proof

Equilibrium:

$$\forall i \in N, \quad \mathbb{E}[\theta_i + \sum_{j=1}^{N} \gamma_{ij} a_j | a_i] = 0. \tag{4}$$

Taking expectations:

$$\forall i \in N, \quad \mu_{\theta_i} + \sum_{j=1}^{N} \gamma_{ij} \mu_{a_j} = 0. \tag{5}$$

(2) is matrix representation of (5).

Proof

Multiplying (4) by a_i, taking expectations:

$$\forall i \in N, \quad \mathbb{E}[\theta_i a_i] + \sum_{j=1}^{N} \gamma_{ij} \mathbb{E}[a_i a_j] = 0.$$

rewrite as:

$$\forall i \in N, \quad cov(\theta_i, a_i) + \mu_{\theta_i} \mu_{a_i} + \sum_{j=1}^{N} \gamma_{ij} (cov(a_i, a_j) + \mu_{a_i} \mu_{a_j}) = 0.$$

Using (5):

$$\forall i \in N, \quad cov(\theta_i, a_i) + \sum_{j=1}^{N} \gamma_{ij} cov(a_i, a_j) = 0.$$

Proof

By definition of a covariance, we have:

$$\forall i \in N, \quad \rho_{\theta_i, a_i}\sigma_{\theta_i}\sigma_{a_i} + \sum_{j=1}^{N} \gamma_{ij}\rho_{a_i a_j}\sigma_{a_j}\sigma_{a_i} = 0. \tag{6}$$

(3) is the matrix representation of (6).

One Dimensional Signals

- agent $i \in N$ observes signal i, with:

$$\begin{pmatrix} \theta_1 \\ \vdots \\ \theta_N \\ s_1 \\ \vdots \\ s_N \end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix} \mu_{\theta_1} \\ \vdots \\ \mu_{\theta_N} \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} \Sigma_{\theta\theta} & \Sigma_{s\theta} \\ \Sigma_{\theta s} & \Sigma_{ss} \end{pmatrix} \right).$$

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- this completely determines the information structure.
- normalize so that $\Sigma_{ss} = P_{ss}$.

One Dimensional Signals

we look equilibria in linear strategies defined by (α_i^*, β_i^*), such that:

$$a_i^* = \alpha_i^* s_i + \beta_i^*.$$

Proposition (Characterization for One Dimensional Signals: Strategy)

The coefficients $(\alpha_1^*, ..., \alpha_N^*)$ *and* $(\beta_1^*, ..., \beta_N^*)$ *form a linear Bayes Nash equilibrium if and only if:*

$$\begin{pmatrix} \beta_1^* \\ \vdots \\ \beta_N^* \end{pmatrix} = -\Gamma^{-1} \cdot \begin{pmatrix} \mu_{\theta_1} \\ \vdots \\ \mu_{\theta_N} \end{pmatrix} \tag{7}$$

and

$$\begin{pmatrix} \alpha_1^* \\ \vdots \\ \alpha_N^* \end{pmatrix} = -(P_{ss} \circ \Gamma)^{-1} \cdot \begin{pmatrix} cov(\theta_1, s_1) \\ \vdots \\ cov(\theta_N, s_N) \end{pmatrix}. \tag{8}$$

One Dimensional Signals

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One Dimensional Signals

- the constant term is independent of the information structure.
- the response of an agent to his own signal depends on the information structure and the interaction matrix.

One Dimensional Signals

we now characterize the outcomes of an equilibrium when agents receive one dimensional signals.

Proposition (Characterization for One Dimensional Signals: Outcomes)

The joint distribution of actions and payoff states $(\theta_1, \ldots, \theta_N, a_1, \ldots, a_N)$ *in the outcome of the Bayes Nash equilibrium is given by:*

1. *The first moments are given by:*

$$\begin{pmatrix} \mu_{a_1} \\ \vdots \\ \mu_{a_N} \end{pmatrix} = -\Gamma^{-1} \cdot \begin{pmatrix} \mu_{\theta_1} \\ \vdots \\ \mu_{\theta_N} \end{pmatrix}$$

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- does not affect the correlation between agents actions.
- the correlation between actions is determined by the correlations of signals.

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- how do these have different implications on the outcome of a game?
- consider some joint distribution of actions $(a_1, ..., a_N)$.
- what models would allow us to rationalize this joint distribution of actions as the outcome of a Bayes Nash equilibrium?

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1. Agents have complete information, no strategic interactions but heterogenous payoff shocks.
2. Agents have complete information, independent payoff shocks but heterogenous strategic interactions.
3. Agents have no strategic interactions, independent payoff shocks but incomplete information.

Three Rationalizations

Let Σ_{aa} be a variance/covariance matrix of actions, then Σ_{aa} is the outcome of a Bayes Nash equilibrium in the following models:

1. Agents do not interact between each other $(\Gamma = \mathbb{I})$, agents have complete information and the distribution over types is given by:
$$\Sigma_{\theta\theta} = \Sigma_{aa}. \tag{9}$$

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$$\Sigma_{\theta\theta} = \Sigma_{aa}. \tag{9}$$
2. Agents have complete information, types are independently distributed with a variance of 1 $(\Sigma_{\theta\theta} = \mathbb{I})$, the interaction matrix is given by any solution to:
$$\Gamma = \Sigma_{aa}^{-1/2}, \tag{10}$$
such that Γ is negative semi-definite.

Three Rationalizations

3. Agents do not interact between each other $(\Gamma = \mathbb{I})$, types are independently distributed $(P_{\theta\theta} = \mathbb{I})$, agents receive one dimensional signals of the form:

$$\begin{pmatrix} s_1 \\ \vdots \\ s_N \end{pmatrix} = P_{aa}^{1/2} \begin{pmatrix} \frac{\theta_1}{\sigma_{\theta_1}} \\ \vdots \\ \frac{\theta_N}{\sigma_{\theta_N}} \end{pmatrix},$$

where elements of the diagonal of $P_{aa}^{1/2}$ are positive, and the variance of payoff shocks satisfies:

$$\forall i \in N, \quad \sigma_{\theta_i} = \frac{\sigma_{a_i}}{corr(s_i, \theta_i)}.$$

Three Rationalizations: Example

Let

$$\Sigma_{aa} = \begin{pmatrix} 6.5 & 1.77 & 1.77 \\ 1.77 & 3.75 & 2.75 \\ 1.77 & 2.75 & 3.75 \end{pmatrix}; \tag{11}$$

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$$\Gamma = \Sigma_{aa}^{-1/2} = \begin{pmatrix} -0.42 & 0.06 & 0.06 \\ 0.06 & -0.71 & 0.29 \\ 0.06 & 0.29 & -0.71 \end{pmatrix} \tag{12}$$

Three Rationalizations: Example

3. Agents do not interact between each other $(\Gamma = \mathbb{I})$, types are independently distributed $(P_{\theta\theta} = \mathbb{I})$, agents receive one dimensional signals of the form:

$$\begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix} = \begin{pmatrix} 0.97 & 0.15 & 0.15 \\ 0.15 & 0.90 & 0.39 \\ 0.15 & 0.39 & 0.90 \end{pmatrix} \begin{pmatrix} \frac{\theta_1}{\sigma_{\theta_1}} \\ \frac{\theta_2}{\sigma_{\theta_2}} \\ \frac{\theta_3}{\sigma_{\theta_3}} \end{pmatrix}$$

and

$$\begin{pmatrix} \sigma_{\theta_1} \\ \sigma_{\theta_2} \\ \sigma_{\theta_3} \end{pmatrix} = \begin{pmatrix} 6.66 \\ 4.13 \\ 4.13 \end{pmatrix}$$

Networks and Incomplete Information: Unified Analysis

1. A number of recent papers unify networks and incomplete information: Calvo-Argmengol, Marti and Prat (2015), de Marti and Zenou (2015), Blume, Brock, Durlauf and Jayaraman (2015), Golub and Morris (2017), Lambert, Martini and Ostrovsky (2017)

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2. Mean actions are pinned down by network centrality under the common prior assumption, information only relevant for second moments; our BCE approach makes this point in a stark way; Golub and Morris 2017 show that this is not true without the common prior assumption.

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2. Mean actions are pinned down by network centrality under the common prior assumption, information only relevant for second moments; our BCE approach makes this point in a stark way; Golub and Morris 2017 show that this is not true without the common prior assumption.
3. Even more unified analysis if we interpret each signal of each player as a separate player (cf, agent normal form); see Morris (1997), Morris (2000), Golub and Morris (2017), Lambert, Martini and Ostrovsky (2017).