Private versus Public Information in Coordination Problems[*]

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1. Introduction

Many economic problems are naturally modelled as a game of incomplete information, where a player's payoff depends on his own action, the actions of others, and some unknown economic fundamentals. Rational behavior in such environments clearly depends on a player's beliefs about economic fundamentals; but it also depends on "higher order beliefs," i.e., players' beliefs about other players beliefs, players' beliefs about other players' beliefs about other players' beliefs, and so on. While this dependence has been understood for a long time, it is only recently that a small theoretical literature in game theory has examined in more detail how the equilibria of incomplete information games vary with such higher order beliefs.[1]

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[1]Rubinstein [1989], Monderer and Samet [1989], Carlsson and van Damme [1993a], Morris, Rob and Shin [1995], Kajii and Morris [1997].

Conventional wisdom holds that such higher order beliefs may play an important role in some economic phenomena. It is reported that, following the Asian crisis, some of the investors who withdrew their capital from Brazil, did so not because they overestimated the economic linkages between Asia and Brazil, but because they thought others might do so. It is reported that negative shocks to some hedge fund portfolios led some lenders to make excessive margin calls and seek to liquidate collateral, not because they thought the fund was insolvent, but because they thought that other lenders might think so, and they had an incentive to get out early. It is reported that apparently irrelevant news about the economy leads some firms to reduce their investments (and thus to recession), not because they think that the news is relevant, but because they think others may think so. It is reported that some investors are currently paying inflated prices for internet stocks not because they believe that future dividends will be high enough to justify those prices, but because they believe that others believe so, and therefore there are short run speculative profits to be made.

Applied modelers have taken such reports very seriously. Yet the resulting models have typically managed to avoid modelling higher order beliefs. How? If economic fundamentals permit multiple equilibria and economic players observe some payoff-irrelevant public signals ("sunspots"), then players might happen to switch behavior contingent on the sunspot signal. Such sunspot shifts are used in the academic literature to "explain" currency crises (e.g., Obstfeld [1986]), bank runs (e.g., Diamond and Dybvig [1983]), macroeconomic recession (e.g., Cooper [1999]) and other phenomena. Such sunspot explanations are apparently intended as proxies for the true higher order beliefs explanations. After all, it is common knowledge in the *models* that the sunspots are payoff-irrelevant (as are, presumably, literal sunspots). Yet the informal discussion accompanying such sunspot explanations invariably appeal to "sunspots" that the modeler may believe are payoff-irrelevant but for which there is clearly not common knowledge of their irrelevance among economic agents (e.g., the Asian crisis for Brazil).

While such sunspot proxy explanations are insightful in some contexts, we believe that it is both important and feasible to develop richer models that permit a substantive role for higher order beliefs. Our purpose in this paper is to illustrate this methodological claim, using a simple example.[2] Consider the problem of two players who stand to gain from co-ordinating their investment: each player has a greater incentive to invest if the other player does so. Each player observes

[2]In doing so, we are attempting to clarify some of our earlier work (Morris and Shin [1998a]) when we analyzed a more complicated model of currency crises. By abstracting from the particular application, we hope to bring out our methodoligical points more clearly. We are pursuing these issues further in models of currency crises (Morris and Shin [1998b] and Corsetti, Morris and Shin [1999]) and pricing debt with liquidity risk (Morris and Shin [1999]).

both a public signal and a private signal about the gains from investment (the "economic fundamentals"), where both signals are normally distributed. Allowing each player to observe such simple two dimensional signals allows a non-trivial role for higher order beliefs. If I observe a low public signal and a high private signal about fundamentals, I have a medium expectation about fundamentals but I expect you to be more pessimistic than me; if I observe a high public signal and a low private signal about fundamentals, I may have the identical medium expectation about fundamentals but now I expect you to be more optimistic than me.

If public signals are sufficiently informative relative to private signals, then strategic complementarities lead to the existence of multiple equilibria. But if private signals are sufficiently informative relative to public signals, then there is a unique equilibrium, and we can explore the role of higher order beliefs in determining individual behavior. In particular, for any given (first order) expectation that a player has about economic fundamentals, he will invest only if the public signal exceeds some threshold. The public signal is having an extra effect (in addition to its influence on the player's first order expectation of fundamentals) because it conveys information in equilibrium about the likelihood of the other player investing. In this sense, public signals play the role of sunspots in the standard multiple equilibrium - sunspot story. But the theory endogenously pins down which public event serves as the co-ordinating sunspot.

There is a closed form characterization for the unique equilibrium, and this can be used to illustrate the tractability of the approach and the intuitiveness of the conclusions. In particular, we quantify a "publicity effect" that measures the extra strategic role that public signals play (relative to private signals observed by all players) and identify when it will be most important. We identify observational implications of the model (assuming public but not private signals are observed by the econometrician). We characterize the unique equilibrium as the precision of public and private signals goes to infinite. We show how to perform (intuitive) comparative static and policy analysis in this model (a particular difficulty for the multiple equilibrium - sunspot approach). And we note how the analysis easily extends to many players.

Essentially this example was analyzed in the introduction of Carlsson and van Damme [1993a] for the case where economic fundamentals and signals are distributed uniformly. They obtained a closed form solution for the unique equilibrium. Their main result was that, under some weak regularity conditions on the probability distribution over fundamentals and signals, there is a unique equilibrium as the noise in the signals becomes sufficiently small, and the risk dominant equilibrium of the underlying game is played. If we let the precision of private signals tend to infinite in our model, we replicate their limiting results in our very special setting. But by obtaining a closed form solution for a case with multidimensional signals, we are able to explore how higher order beliefs influence outcomes away from the limit they study.

2. A Coordination Game with Public and Private Signals

Two players are playing the following symmetric game:

	INVEST	NOT INVEST
INVEST (I) (Attack / Deposit)	θ, θ	$\theta - 1, 0$
NOT INVEST (N) (Do Nothing / Withdraw)	$0, \theta - 1$	$0, 0$

Three stylized interpretations of these payoffs explain the connection to some of the literatures that we discussed in the introduction:

1. Investment Game. Two players are deciding whether to invest. There is a safe action (not invest); there is a risky action (invest) which gives a higher payoff if the other player invests.

2. Currency Crisis. Two players are deciding whether to attack a currency. There is a safe action (do nothing); there is a risky action (attack by selling the foreign currency short) which gives a higher payoff if the other player attacks.

3. Bank Run. Two players are deciding whether to leave their deposits in a bank. There is a safe action (withdraw); there is a risky action (deposit) which gives a higher payoff if the other player deposits.

If there were complete information about θ, there would be three cases to consider.

- If $\theta > 1$, each player has a dominant strategy to invest.
- If $\theta \in [0, 1]$, there are two pure strategy Nash equilibria: both invest and both not invest.
- If $\theta < 0$, each player has a dominant strategy to not invest.

But there is incomplete information. Parameter θ is normally distributed with mean y and precision α (i.e., variance is $\frac{1}{\alpha}$ and standard deviation is $\frac{1}{\sqrt{\alpha}}$). The mean y is publicly observed. In addition, each player observes a private signal $x_i = \theta + \varepsilon_i$. Each ε_i is distributed normally with mean 0 and variance $\frac{1}{\beta}$. Thus each player i observes a public signal $y \in \mathbf{R}$ and a private signal $x_i \in \mathbf{R}$.

Although this is a very simple and natural information structure, it turns out that it is rich enough to allow a substantive role for higher order uncertainty. Let us start by briefly describing why this is the case. Write $\overline{\theta}_i \equiv \frac{\alpha y + \beta x_i}{\alpha + \beta}$ for player i's expectation of θ. Thus player 1's posterior on θ is normally distributed with mean $\overline{\theta}_i$ and precision $\alpha + \beta$. Since ε_2 is distributed independently, player 1's posterior on x_2 is normally distributed with mean $\overline{\theta}_i$ and precision $\frac{\beta(\alpha+\beta)}{\alpha+2\beta}$ $\left(= 1 \middle/ \left(\frac{1}{\alpha+\beta} + \frac{1}{\beta}\right)\right)$. Observe then that player 1's expectation of player 2's expectation of θ is:

$$\begin{aligned}
E_1\left(\overline{\theta}_2 \middle| x_1\right) &= E_1\left(\frac{\alpha y + \beta x_2}{\alpha + \beta} \middle| x_1\right) \\
&= \frac{\alpha y + \beta E_1\left(x_2 | x_1\right)}{\alpha + \beta} \\
&= \frac{\alpha y + \beta \overline{\theta}_1}{\alpha + \beta} \\
&= \frac{\alpha y + \beta\left(\frac{\alpha y + \beta x_1}{\alpha + \beta}\right)}{\alpha + \beta} \\
&= \frac{\left(\alpha^2 + 2\alpha\beta\right) y + \beta^2 x_1}{(\alpha + \beta)^2} \\
&= \overline{\theta}_1 + \frac{\alpha\beta}{(\alpha + \beta)^2}\left(y - x_1\right)
\end{aligned}$$

Thus player 1 expects player 2 to be more optimistic than himself, if the public signal exceeds his private signal; player 1 expects player 2 to be more pessimistic than himself, if his private signal exceeds the public signal. As we will see in the analysis that follows, if each player is expecting the other to be just a little bit more pessimistic than himself, risky co-ordination becomes very difficult to maintain.[3]

We now analyze the equilibria of this game. For now, fix the public signal y. A strategy for player i is a function specifying an action for each possible private

[3] More generally, write $\widetilde{\theta}^k$ for player 1's kth order expectation of θ. Now $\widetilde{\theta}^1 = E_1(\theta) = \frac{\alpha y + \beta x_1}{\alpha + \beta}$, $\widetilde{\theta}^2 = E_1\left(E_2(\theta)\right) = \frac{\left((\alpha+\beta)^2 - \beta^2\right) y + \beta^2 x_1}{(\alpha+\beta)^2}$, $\widetilde{\theta}^3 = E_1\left(E_2\left(E_1(\theta)\right)\right) = \frac{\left((\alpha+\beta)^3 - \beta^3\right) y + \beta^3 x_1}{(\alpha+\beta)^3}$ and $\widetilde{\theta}^k = \frac{\left((\alpha+\beta)^k - \beta^k\right) y + \beta^k x_1}{(\alpha+\beta)^k}$. Thus as $k \to \infty$, $\widetilde{\theta}^k \to y$.

signal; a natural kind of strategy we might consider is one where a player takes the risky action only if he observes a private signal above some cutoff point, $\widehat{x}$:

$$s_i(x_i) = \begin{cases} \texttt{Invest}, \text{ if } x_i > \widehat{x} \\ \texttt{NotInvest}, \text{ if } x_i \leq \widehat{x} \end{cases}$$

We will refer to this strategy as the switching strategy around $\widehat{x}$. Suppose that player 1 thought that player 2 was following such a "switching" strategy. Player 1's expected payoff would be

$$\begin{aligned} u(x_1, \widehat{x}) &= E(\theta\,|x_1) - \Pr(x_2 \leq \widehat{x}\,|x_1) \\ &= \frac{\alpha y + \beta x_1}{\alpha + \beta} - \Phi\left(\sqrt{\left(\frac{\beta(\alpha+\beta)}{\alpha+2\beta}\right)}\left(\widehat{x} - \frac{\alpha y + \beta x_1}{\alpha+\beta}\right)\right). \end{aligned}$$

Notice that this expression is continuous and strictly increasing in x_1. Thus player 1's (essentially) unique best response is to follow a switching strategy, investing if $u(x_1, \widehat{x}) > 0$ and not investing otherwise.[4] Thus we have a symmetric switching strategy equilibrium, with both players choosing switching point $\widehat{x}$, exactly if $u(\widehat{x}, \widehat{x}) = 0$. In fact, invest is a rationalizable action only if x exceeds the lowest equilibrium switching point and not invest is rationalizable action only if x is less than the higher equilibrium switching point.

Proposition 2.1. *If $u(\widehat{x}, \widehat{x}) = 0$, then there is a symmetric switching strategy equilibrium around $\widehat{x}$. If $\underline{x}$ and $\overline{x}$ are, respectively, the smallest and largest solutions to the equation $u(\widehat{x}, \widehat{x}) = 0$, and s_i is a strategy that survives iterated deletion of dominated strategies, then $s_i(x_i) =$ `Not Invest` if $x_i < \underline{x}$ and $s_i(x_i) =$ `Invest` if $x_i > \overline{x}$.*

Corollary 2.2. *If $\widehat{x}$ is the unique solution to the equation $u(\widehat{x}, \widehat{x}) = 0$, then the switching strategy around $\widehat{x}$ is the unique rationalizable strategy for either player.*

The idea of the proof is the following. It can be shown a strategy s_i survives k rounds of iterated deletion of strictly dominated strategies if and only if $s_i(x_i) =$ `Not Invest` if $x < \underline{x}_k$ and $s_i(x_i) =$ `Invest` if $x > \overline{x}_k$. This is true for $k = 0$, setting $\underline{x}_0 = -\infty$ and $\overline{x}_0 = -\infty$, and one can verify inductively that it holds for all k. But if $\underline{x}_k < \underline{x}$, we know that $u(\underline{x}_k, \underline{x}_k) < 0$. Thus if a player expected his opponent to follow a trigger strategy around $\underline{x}_k$ and observed signal $\underline{x}_k$, he would a strictly negative payoff to investing. This implies that $\underline{x}_{k+1} > \underline{x}_k$. It turns out that $\underline{x}_k \uparrow \underline{x}$ and $\overline{x}_k \downarrow \overline{x}$. A full proof is given in appendix B.

[4] There will be exactly one signal for which both investing and not investing will be best responses. The tie-breaking rule used in such instances has no effect on our analysis.

It is useful to re-write the equation $u(\widehat{x},\widehat{x}) = 0$ in terms of $\overline{\theta} = \frac{\alpha y + \beta \widehat{x}}{\alpha + \beta}$, the expected value of θ given private signal $\widehat{x}$ and public signal y. We also make explicit the dependence on the public signal. Now $\widehat{x} = \frac{(\alpha+\beta)\overline{\theta} - \alpha y}{\beta} = \overline{\theta} + \frac{\alpha}{\beta}\left(\overline{\theta} - y\right)$, so substituting into the equation $u(\widehat{x},\widehat{x}) = 0$ (and making the dependence on the public signal y explicit) and we have equilibrium condition:

$$\begin{aligned} \widetilde{u}\left(\overline{\theta}, y\right) &\equiv \overline{\theta} - \Phi\left(\sqrt{\left(\frac{\beta(\alpha+\beta)}{\alpha+2\beta}\right)}\left(\frac{\alpha}{\beta}\left(\overline{\theta} - y\right)\right)\right) \qquad (2.1) \\ &= \overline{\theta} - \Phi\left(\sqrt{\gamma}\left(\overline{\theta} - y\right)\right) \\ &= 0 \end{aligned}$$

where $\gamma = \widetilde{\gamma}(\alpha,\beta) = \frac{\alpha^2}{\beta}\left(\frac{\alpha+\beta}{\alpha+2\beta}\right)$. Note that $\widetilde{u}\left(\overline{\theta}, y\right)$ represents the expected utility of investing for a player whose expectation of θ is $\overline{\theta}$, if he expects his opponent to follow a $\overline{\theta}$ switching strategy.

Figures 1 through 4 plot the function $\widetilde{u}\left(\overline{\theta}, \frac{1}{2}\right)$ for $\gamma = 1000$, 10, 5 and $\frac{1}{1000}$, respectively. The intuition here is the following. If public information is large (i.e., $\alpha \gg \beta$ and thus γ is large), then players with $\overline{\theta}$ less than $y = \frac{1}{2}$ confidently expect that their opponent will have observed a higher signal, and therefore will be investing. Thus his expected utility is (about) $\overline{\theta}$. But as $\overline{\theta}$ moves above $y = \frac{1}{2}$, he rapidly becomes confident that his opponent has observed a lower signal and will not be investing. Thus his expected utility drops rapidly, around y, to (about) $\overline{\theta} - 1$. But if public information is small (i.e., $\alpha \ll \beta$ and γ is small), then players with $\overline{\theta}$ not too far above or below $y = \frac{1}{2}$ attach probability (about) $\frac{1}{2}$ to their opponent observing a higher signal. Thus his expected utility is (about) $\overline{\theta} - \frac{1}{2}$.

There is unique equilibrium if, for all y, there is a unique value of $\overline{\theta}$ solving $\overline{\theta} - \Phi\left(\sqrt{\gamma}\left(\overline{\theta} - y\right)\right) = 0$; there are multiple equilibria if, for some y, there are multiple values of $\overline{\theta}$ solving $\overline{\theta} - \Phi\left(\sqrt{\gamma}\left(\overline{\theta} - y\right)\right) = 0$. We can characterize analytically when there is a unique equilibrium.

Proposition 2.3. *There is a unique equilibrium if and only* $\widetilde{\gamma}(\alpha,\beta) \leq 2\pi$.

Figure 5 plots in (α,β)-space the uniqueness and non-uniqueness regions.

Proof. We first show that there is a unique $\overline{\theta}$ solving $\overline{\theta} - \Phi\left(\sqrt{\gamma}\left(\overline{\theta} - y\right)\right) = 0$ for all y if $\widetilde{\gamma}(\alpha,\beta) \leq 2\pi$. If $\gamma \leq 2\pi$, then $\frac{d\widetilde{u}}{d\overline{\theta}} = 1 - \sqrt{\gamma}\phi\left(\gamma\left(\overline{\theta} - y\right)\right) \geq 1 - \sqrt{\frac{\gamma}{2\pi}} \geq 0$. If $\gamma < 2\pi$, then the inequality is strict and there can be at most one solution. If $\gamma = 2\pi$, then the only point when $\frac{d\widetilde{u}}{d\overline{\theta}} = 0$ is when $\overline{\theta} = y$. At this point, $\frac{d^2\widetilde{u}}{d\overline{\theta}^2} = 0$ but $\frac{d^3\widetilde{u}}{d\overline{\theta}^3} > 0$, so again there is a unique solution.

We now show that if $\widetilde{\gamma}(\alpha,\beta) > 2\pi$, there are multiple (in fact, three) values of $\overline{\theta}$ solving $\overline{\theta} - \Phi\left(\sqrt{\gamma}\left(\overline{\theta} - \frac{1}{2}\right)\right) = 0$. One solution to this equation is $\overline{\theta} = \frac{1}{2}$ (since

$\widetilde{u}\left(\frac{1}{2},\frac{1}{2}\right)=0$ for any γ). But if $\gamma>2\pi$, then $\frac{d\widetilde{u}\left(\frac{1}{2},\frac{1}{2}\right)}{d\overline{\theta}}=1-\gamma\phi\left(0\right)=1-\sqrt{\frac{\gamma}{2\pi}}<0$. But $\widetilde{u}\left(\overline{\theta},y\right)$ is continuous in $\overline{\theta}$, tends to $-\infty$ as $\overline{\theta}\rightarrow-\infty$, and tends to ∞ as $\overline{\theta}\rightarrow\infty$. So there are at least three solutions. ■

3. Properties of the Unique Equilibrium

Throughout the remainder of the paper, we assume that there is a unique equilibrium, i.e., that $\widetilde{\gamma}\left(\alpha,\beta\right)\leq2\pi$. Under this assumption, we can invert the equilibrium condition (2.1) to show in $\left(\overline{\theta},y\right)$ space what the unique equilibrium looks like

$$y=h_{\gamma}\left(\overline{\theta}\right)=\overline{\theta}-\frac{1}{\sqrt{\gamma}}\Phi^{-1}\left(\overline{\theta}\right) \tag{3.1}$$

Figures 6 and 7 plot this for $\gamma=5$ and $\gamma=\frac{1}{1000}$. The picture has an elementary intuition. If $\overline{\theta}<0$, it is optimal to not invest (independent of the public signal). If $\overline{\theta}>1$, it is optimal to invest (independent of the public signal). But if $0<\overline{\theta}<1$, there is a trade-off. The higher y is (for a given $\overline{\theta}$), the more likely it is that the other player will invest. Thus if $0<\overline{\theta}<1$, the player will always invest for sufficiently high y, and not invest for sufficiently low y. This implies in particular that changing y has a larger impact on a player's action than changing his private signal (controlling for the informativeness of the signals). We next turn to examining this "publicity" effect.

3.1. The Publicity Effect

To explore the strategic impact of public information, we examine how much a player's private signal must adjust to compensate for a given change in the public signal. Equation (3.1) can be written as

$$\frac{\alpha y+\beta x}{\alpha+\beta}-\Phi\left(\sqrt{\gamma}\left(\frac{\alpha y+\beta x}{\alpha+\beta}-y\right)\right)=0.$$

Totally differentiating with respect to y gives

$$\frac{dx}{dy}=-\frac{\frac{\alpha}{\beta}+\sqrt{\gamma}\phi\left(\cdot\right)}{1-\sqrt{\gamma}\phi\left(\cdot\right)}.$$

This measures how much the private signal would have to change to compensate for a change in the public signal (and still leave the player indifferent between investing or not investing). We can similarly see how much the private signal would have to change to compensate for a change in the public signal, if there was no strategic effect. Totally differentiating

$$\overline{\theta} = \frac{\alpha y + \beta x}{\alpha + \beta} = k$$

we obtain

$$\frac{dx}{dy} = -\frac{\alpha}{\beta}$$

The *publicity multiplier* is the ratio of these two:

$$\zeta = \frac{1 + \frac{\beta}{\alpha}\sqrt{\gamma}\phi\left(\sqrt{\gamma}\left(\frac{\alpha y + \beta \widehat{x}(y)}{\alpha+\beta} - y\right)\right)}{1 - \sqrt{\gamma}\phi\left(\sqrt{\gamma}\left(\frac{\alpha y + \beta \widehat{x}(y)}{\alpha+\beta} - y\right)\right)} = \frac{1 + \frac{\beta}{\alpha}\sqrt{\gamma}\phi\left(\sqrt{\gamma}\Phi^{-1}\left(\overline{\theta}\right)\right)}{1 - \sqrt{\gamma}\phi\left(\left(\Phi^{-1}\left(\overline{\theta}\right)\right)\right)}$$

Thus suppose a player's expectation of θ is $\overline{\theta}$ and he is has observed a public signal that makes him indifferent between investing and not investing; ζ measures the extra clout of public information relative to private information at that point. Notice that (for any given α and β) it is maximized when $\overline{\theta} = \frac{1}{2}$, and thus the critical public signal $y = \frac{1}{2}$. Thus it is precisely when there is no conflict between private and public signals that the multiplier has its biggest effect. Here the publicity multiplier equals

$$\zeta^* = \frac{1 + \frac{\beta}{\alpha}\sqrt{\frac{\gamma}{2\pi}}}{1 - \sqrt{\frac{\gamma}{2\pi}}}$$

Notice that when private information is very accurate relative to private information (i.e., $\beta \to 0$), the publicity multiplier is very small. The multiplier is biggest just before we hit the multiplicity zone of the parameter space (i.e., when $\gamma \approx \sqrt{2\pi}$).

There is plentiful anecdotal evidence that in settings where co-ordination is important, public signals play a role in co-ordinating outcomes that exceeds the information content of those announcements. For example, financial markets apparently "overreact" to announcements from the Federal Reserve Board and public announcements in general. If market participants are concerned about the reaction of other participants to the news, the "overreaction" may be rational and determined by the type of equilibrium logic of our example. Further evidence for this is briefings on market conditions by key players in financial markets using conference calls with hundreds of participants. Such public briefings have a larger impact on the market than bilateral briefings with the same information, because they automatically convey to participants not only information about market conditions but also valuable information about the beliefs of the other participants.

Urban renewal also has a co-ordination aspect. Private firms' incentives to invest in a run down neighborhood depend partly on exogenous characteristics of the neighborhood, but also depend to a great extent on whether other firms are investing. A well publicized investment in the neighborhood might be expected to have an apparently disproportionate effect on the probability of ending in the good equilibrium. The willingness of public authorities to subsidize football stadiums and conference centers is consistent with this view.

An indirect econometric test of the publicity effect is performed by Chwe [1998]. Chwe observes that the per viewer price of advertising during the Super Bowl is exceptionally high (i.e., the price of advertising increases more than linearly in the number of viewers). The premium price is explained by the fact that any information conveyed by those advertisements becomes not merely known to the wide audience, but also common knowledge among them. The value of this common knowledge to advertisers should depend on whether there is a significant co-ordination problem in consumers' decisions whether to purchase the product. Chwe makes some plausible ex ante guesses about when co-ordination is an important issue because of network externalities (e.g., the Apple Macintosh) or social consumption (e.g., beer); and when it is not (e.g., batteries). He then confirms econometrically that it is the advertisers of co-ordination goods who pay a premium for large audiences.

3.2. Limiting Behavior

An interesting special case of the model arises as signals become more informative. First, consider what happens when we increase the precision of private signals, while holding the precision of public signals fixed (i.e., let $\beta \to \infty$ for fixed α). In this case, $\widetilde{\gamma}(\alpha,\beta) = \frac{\alpha^2}{\beta}\left(\frac{\alpha+\beta}{\alpha+2\beta}\right) \to 0$ and

$$h_{\widetilde{\gamma}(\alpha,\beta)}\left(\overline{\theta}\right) \to \begin{cases} -\infty, \text{ if } \overline{\theta} > \frac{1}{2} \\ \frac{1}{2}, \text{ if } \overline{\theta} = \frac{1}{2} \\ \infty, \text{ if } \overline{\theta} < \frac{1}{2} \end{cases}$$

This result says that as private signals become more informative, a player's decision to invest depends only on whether his expected value of $\overline{\theta}$ exceeds $\frac{1}{2}$. In fact, Carlsson and van Damme's [1993a] showed quite generally that in two player, two action games, as private information dominates public information, the risk dominant action must be played. The intuition is that as private signals become more informative, the prior looks locally uniform. Thus for any signal a player observes, he attaches probability of around $\frac{1}{2}$ to the other player observing a lower signal. If a player attaches probability $\frac{1}{2}$ to other investing, and $\overline{\theta} = \frac{1}{2}$, then he is indifferent between investing and not investing.

If we increase the precision of public signals, while holding the precision of private signals fixed (i.e., let $\alpha \to \infty$ for fixed β), then we clearly exit the unique equilibrium zone.[5] But we can examine what happens to the unique equilibrium as the precision of both signals tends to infinite in such a way that uniqueness is maintained. Specifically, let $\alpha \to \infty$ and let $\frac{\beta}{\alpha^2} \to c$, where $c > \frac{1}{4\pi}$. In this case,

$$\widetilde{\gamma}(\alpha, \beta) \to \frac{\alpha^2}{c\alpha^2}\left(\frac{\alpha + c\alpha^2}{\alpha + 2c\alpha^2}\right) \to \frac{1}{2c} < 2\pi.$$

Thus

$$h_{\widetilde{\gamma}(\alpha,\beta)}\left(\overline{\theta}\right) \to \overline{\theta} - \left(\sqrt{2c}\right)\Phi^{-1}\left(\overline{\theta}\right).$$

This result says that *even though the public signal becomes irrelevant to a player's expected value of θ in the limit*, it continues to have a large impact on the outcome. For example, suppose $c = 1$ and $y = \frac{1}{3}$ (i.e., public information looks bad). Each player i will invest only if $\overline{\theta}_i \geq 0.7$, i.e., they will be very conservative. This is true even as they players ignore y (i.e., $\overline{\theta}_i \to x_i$ for each player i).

The intuition for this result is the following. Suppose public information looks bad ($y < \frac{1}{2}$). If each player's private information is much more accurate than the public signal, they will mostly ignore y in forming their own expectation of θ. But each will nonetheless expect the other to have observed a somewhat worse signal than themselves. This pessimism about the other's signal makes it very hard to support an investment equilibrium.

3.3. Empirical Implications?

Private information models are always hard to test because, by assumption, the private information is not observable. However, we can hope to come up with suggestive indirect tests. We discussed earlier the implications that markets might appear to react excessively to public information.

This approach also has an important empirical prediction concerning the distribution of the payoff arising from the action of invest. This empirical prediction has some implications for our understanding of market risk, and of the formal analysis of "Value at Risk" (we are examining this issue in more detail in Morris and Shin [1999]). When prices are sensitive to the flow volume of trade - such as when a large number of investors try to sell their holdings simultaneously - there is

[5] For sufficiently large α, either action is rationalizable as long as $y \in (0,1)$ and $\overline{\theta} \in (0,1)$. If either $\overline{\theta} \geq 1$ or $\overline{\theta} > 0$ and $y \geq 1$, then only invest is rationalizable. If either $\overline{\theta} \leq 0$ or $\overline{\theta} < 1$ and $y \leq 0$, then only not invest is rationalizable.

an element of coordination in the trading strategies of investors. The significance of this coordination element is increasing in the degree to which prices are more sensitive to trades, and in the degree to which traders are leveraged. When traders are highly leveraged, the potential losses arising from a sudden price change induces them to act more sensitively to the selling of others. The recent episodes of turbulence in the financial markets (such as the sudden fall in the dollar against the yen on October 8th 1998) are difficult to explain except by reference to such whiplash effects.

The action to invest can be interpreted as the action of going long on a particular asset. If the horizon of the investor is short (as is the case for most active traders), the payoff to holding the asset can be represented as the sum of θ, the payoff arising from the asset in the absence of selling pressure, minus the proportion of investors who sell (or by some function of this proportion). In the unique equilibrium, in the limit when the noise become small ($\beta \to \infty$), the payoff function becomes approaches

$$\begin{cases} \theta & \text{if} \quad \theta \geq \frac{1}{2} \\ \theta - 1 & \text{if} \quad \theta < \frac{1}{2} \end{cases}$$

This implies that the ex ante distribution of the payoff to holding the asset (conditional on the public signal) can be written as

$$F(\theta) = \begin{cases} \Phi\left(\sqrt{\alpha}\left(\theta - (y-1)\right)\right) & \text{if} \quad \theta \leq -\frac{1}{2} \\ \Phi\left(\sqrt{\alpha}\left(\frac{1}{2} - y\right)\right) & \text{if} \quad -\frac{1}{2} \leq \theta \leq \frac{1}{2} \\ \Phi\left(\sqrt{\alpha}\left(\theta - y\right)\right) & \text{if} \quad \theta \geq \frac{1}{2} \end{cases}$$

and the associated density function is given by the bi-modal density:

$$f(\theta) = \begin{cases} \phi\left(\sqrt{\alpha}\left(\theta - (y-1)\right)\right) & \text{if} \quad \theta < -\frac{1}{2} \\ 0 & \text{if} \quad -\frac{1}{2} \leq \theta < \frac{1}{2} \\ \phi\left(\sqrt{\alpha}\left(\theta - y\right)\right) & \text{if} \quad \theta \geq \frac{1}{2} \end{cases}$$

The striking feature of the density f is that it is bi-modal, and the shape of the density is very far from the smooth, bell-shaped distribution of payoffs routinely seen in textbooks of finance.[6] The potential losses will be underestimated if the investor believes that the payoff is normally distributed.

The analysis of "market risk" has emerged as one of the most important topics in finance. However, the current state of the art (such as J.P. Morgan's RiskMetrics) relies on payoffs which are assumed to be normal, and which are calibrated from a window of observations from the recent past. It is perhaps no accident that

[6] If we were not looking at the limit as $\beta \to \infty$, the distribution would be smoothed out, but the bi-modal feature would be maintained.

otherwise sophisticated banks and financial institutions incurred large losses during the turbulent summer and autumn of 1998. When coordination is an element of the payoff to holding an asset, the payoff distribution arising from the asset is very unlikely to be normally distributed, even if the underlying fundamentals are normally distributed.

This observation also sheds light on the debate on the "fat tails" of observed asset returns. It is well known that, although the distribution of returns over long horizons show fat tails relative to the normal distribution, the "fatness" of the tail varies over time. During relatively quiet times, the fatness of the tail is not as pronounced as it is during more turbulent episodes in the market. Such observations are very suggestive of the sort of effects we have alluded to here.

3.4. Higher Order Beliefs and Bounded Rationality

A standard response to suggestions that higher order beliefs (and especially very high order beliefs) might play an important role is economic analysis is that reasoning to such high levels is beyond the capacity of most decision makers. It is implied that ignoring high levels of beliefs might represent a good model of bounded rationality.

The model studied in this paper illustrates what we think is wrong with this view. First, although second order beliefs were important, the equilibrium analysis is simple and the intuition behind the resulting equilibria is straightforward. It is not clear why restricting the notion of rationality is especially compelling here.

But second, we want to argue that very high order beliefs *are* the driving force behind this example, yet the analysis suggests a very simple and intuitive (boundedly rational?) heuristic for behavior in the face of such very high order uncertainty. To develop this theme, we informally introduce some ideas from the higher order beliefs literature (see the references in footnote 1). An event is said to be common knowledge if everyone knows it, everyone knows that everyone knows it, and so on. Monderer and Samet [1989] introduced the following natural generalization of common knowledge: for some number $p \in [0, 1]$, an event is *common p-belief* if everyone believes it with probability at least p, everyone believes with probability p that everyone believes it with probability p, and so on. This "hierarchical" definition of common p-belief is complex but is luckily equivalent to a more tractable "fixed point" definition. An event is *p-evident* if, whenever it is true, everyone believes it with probability at least p. Monderer and Samet showed that an event E is common p-belief at state ω if and only if there is a p-evident event F such that ω is an element of F and event E is believed with probability at least p whenever F is true.

We can illustrate these concepts in our example. Fix a public signal y. Describe a state of the world by the triple $\left(\theta,\overline{\theta}_1,\overline{\theta}_2\right)$ consisting of the state of fundamentals, player 1's expectation of fundamentals and player 2's expectation of fundamentals. We will be interesting in events of the form "both players' expectation of fundamentals is at least α," i.e., events of the form $E_\alpha = \left\{\left(\theta,\overline{\theta}_1,\overline{\theta}_2\right) \in \mathbf{R}^3 : \overline{\theta}_1 \geq \alpha \text{ and } \overline{\theta}_2 \geq \alpha\right\}$. When is the event E_α p-evident? We need to check that whenever $\overline{\theta}_1 \geq \alpha$, the probability player 1 assigns to $\overline{\theta}_2 \geq \alpha$ is at least p (and vice-versa). Since this probability is increasing in $\overline{\theta}_1$, it is enough to check this for $\overline{\theta}_1 = \alpha$. But $\Pr\left(\overline{\theta}_2 \geq \alpha \middle| \overline{\theta}_1 = \alpha\right) = 1 - \Phi\left(\sqrt{\gamma}\left(\alpha - y\right)\right)$ (this calculation was a key component of our equilibrium calculations). Thus event E_α is p-evident if and only if $\Phi\left(\sqrt{\gamma}\left(\alpha - y\right)\right) \leq 1-p$. So the fixed point characterization of common p-belief implies that event E_0 is common p-belief at state (θ,β,β) if and only if there exists $0 \leq \alpha \leq \beta$ such that $\Phi\left(\sqrt{\gamma}\left(\alpha - y\right)\right) \leq 1 - p$. But the equilibrium condition (3.1) tells us that the unique equilibrium switching point $\overline{\theta}$ uniquely solves $\Phi\left(\sqrt{\gamma}\left(\overline{\theta} - y\right)\right) = \overline{\theta}$. Thus we have the following alternative characterization of equilibrium:

- A player with expectation $\overline{\theta}$ invests in the unique equilibrium if and only if it is common $\left(1-\overline{\theta}\right)$-belief that each player's expectation of θ exceeds 0.[7]

The connection between common p-belief and equilibria of co-ordination games with incomplete information is quite general. The insight is that if you only take a strategically risky action (i.e., one where it is costly if your opponent fails to co-ordinate) if there exists a p-evident event that you can both co-ordinate your actions on. The more strategically risky the action (i.e., in this example, the lower $\overline{\theta}$) the greater degree of p-evidence your require. Events that are p-evident for high p can be though of as approximately public: whenever they occur, there is a large probability that each player observes them.

Thus there is a very simple boundedly rational heuristic for behavior in co-ordination problems with payoff uncertainty. Only take strategically risky actions when there is some almost public event indicating that the actions make sense. If some new piece of information arrives that others might interpret as bad news, do not take the strategically risky action. This heuristic is in the spirit of the fully rational model, it is consistent with unique equilibrium and it seems to capture better than multiple equilibrium-sunspots models the phenomena they seek to explain.

[7] When there is not a unique equilibrium (i.e., if $\widetilde{\gamma}\left(\alpha,\beta\right) > 2\pi$)), investing is rationalizable if and only if it is common $\left(1-\overline{\theta}\right)$-belief that each player's expectation of θ exceeds 0; not investing is rationalizable if and only if it is common $\overline{\theta}$-belief that each player's expectation of θ is less than 1. Thus $\widetilde{\gamma}\left(\alpha,\beta\right) \leq 2\pi$ is a sufficient condition for the events "it is common $\left(1-\overline{\theta}\right)$-belief that each player's expectation of θ exceeds 0" and "it is common $\overline{\theta}$-belief that each player's expectation of θ is less than 1" to be essentially disjoint.

4. Extensions

4.1. Comparative Statics and Public Policy

A major weakness of multiple equilibrium - sunspot models is that comparative statics and thus policy analysis cannot be performed. We will briefly illustrate how our model can be extended to deliver natural comparative static and policy conclusions. Suppose we introduce a parameter $z > 0$ representing the cost of misco-ordination (our original game was the case where $z = 1$)

	INVEST	NOT INVEST
INVEST (I) (Attack / Deposit)	θ, θ	$\theta - z, 0$
NOT INVEST (N) (Do Nothing / Withdraw)	$0, \theta - z$	$0, 0$

There are multiple equilibria of the complete information game if $\theta \in [0, z]$. For large z, we get the possibility that an inefficient equilibrium is played (both not invest) for very high values of θ. One interpretation is that these bad equilibria are the outcome of a negative externality (in not investing, I don't take into account the cost it imposes on you). A natural policy response would be to impose a small tax τ on this action. Arguments in favor of taxing capital outflows to prevent currency crises can be understood in these terms. The payoff matrix now becomes:

	INVEST	NOT INVEST
INVEST (I) (Attack / Deposit)	θ, θ	$\theta - z, -\tau$
NOT INVEST (N) (Do Nothing / Withdraw)	$-\tau, \theta - z$	$-\tau, -\tau$

But now there are multiple equilibria of the complete information game if $\theta \in [-\tau, z - \tau]$. The tax does have the positive impact of ruling out the bad equilibrium (both not invest) for values of θ in the interval $(z - \tau, z]$. But the powerful intuition underlying such policy proposals is the hope that such a tax will reduce the probability of an attack for all values of θ. But the complete information analysis simply cannot deliver that conclusion. We conclude by demonstrating the comparative statics of policy in this simple example.

So consider the incomplete information environment described earlier (parameterized by α and β). The analysis proceeds essentially as before. Uniqueness now requires that $z^2\widetilde{\gamma}(\alpha,\beta) \leq 2\pi$. The equilibrium condition becomes

$$\tau + \overline{\theta} = z\Phi\left(\sqrt{\gamma}\left(\overline{\theta} - y\right)\right).$$

Assuming the uniqueness condition is satisfied with strict inequality, we get that

$$\frac{d\overline{\theta}}{d\tau} = -\frac{1}{1 - z\sqrt{\gamma}\phi\left(\sqrt{\gamma}\left(\overline{\theta} - y\right)\right)} < 0,$$

i.e., that increasing the tax lowers the critical expectation about θ at which players invest. Thus the intuitive comparative statics hold globally.

4.2. Many Players

The above analysis easily extends to the many player case, and we briefly sketch how this works (there is a more detailed treatment in appendix A). Suppose now there are K players and the payoff to not investing 0, and the payoff to investing is $\theta - 1 + f(\pi)$, where π is the proportion of $K-1$ opponents investing and $f : [0,1] \to [0,1]$ is increasing with $f(0) = 0$ and $f(1) = 1$. As before, θ is drawn from normal distribution with mean y and precision α and each player i observes a signal $x_i = \theta + \varepsilon_i$, where each ε_i is drawn from a normal distribution with mean 0 and precision β. If $K = 2$, this reduces to exactly the model we have been analyzing before.

Suppose player 1 has observed public signal y and private signal x_1, giving him an expectation of fundamentals of $\overline{\theta}_1$. His posterior over the private signals of the other players, $(x_2, \ldots.., x_K)$ are distributed normally with some correlation. Writing $H(m,n,\rho,c)$ for the probability that at least m out of n normally distributed random variables are less than or equal to c if each of the random variables has mean 0 and variance 1, and the correlation coefficient between any two random variables is ρ, it turns out that the probability that at least k out of players 2 through K observing a signal less than or equal to x_1 is

$$H\left(k, K-1, \frac{\beta}{\alpha + 2\beta}, \frac{\alpha}{\sqrt{\beta}}\sqrt{\frac{\alpha+\beta}{\alpha+2\beta}}\left(\overline{\theta}_1 - y\right)\right)$$

Thus if player 1 has expectation $\overline{\theta}$ and expects others to follow switching strategies with switching point $\overline{\theta}$, his expected payoff is

$$\widetilde{U}\left(\overline{\theta}\right) = \overline{\theta} - \sum_{k=1}^{K-1}\left[\begin{array}{c} f\left(\frac{K-k}{K-1}\right) \\ -f\left(\frac{K-k-1}{K-1}\right)\end{array}\right]\left(H\left(k, K-1, \frac{\beta}{\alpha+2\beta}, \frac{\alpha}{\sqrt{\beta}}\sqrt{\frac{\alpha+\beta}{\alpha+2\beta}}\left(\overline{\theta}_1 - y\right)\right)\right) \tag{4.1}$$

As before, there is a unique equilibrium if and only if $\widetilde{U}\left(\overline{\theta}\right)=0$ has a unique solution (the argument is exactly that for the two player case reported in proposition 2.1 and proved in appendix B). Using this equation, it is possible to show for any K, there exists constants $\underline{\kappa}$ and $\overline{\kappa}$ such that there is a unique equilibrium if $\widetilde{\gamma}(\alpha,\beta)=\frac{\alpha}{\sqrt{\beta}}\sqrt{\frac{\alpha+\beta}{\alpha+2\beta}}\leq\underline{\kappa}$ (independent of the shape of f); and there are multiple equilibria if $\widetilde{\gamma}(\alpha,\beta)\geq\overline{\kappa}$. Also, as $\beta\to 0$, it is possible to show that $\widetilde{U}\left(\overline{\theta}\right)\to\overline{\theta}-1+\frac{1}{K}\sum_{k=0}^{K-1}f\left(\frac{k}{K-1}\right)$. Thus an individual invests only if $\overline{\theta}\geq 1-\frac{1}{K}\sum_{k=0}^{K-1}f\left(\frac{k}{K-1}\right)$.[8]

Thus adding many players does not change the qualitative conclusions of the analysis. A clean illustration of this point is obtained by considering the special case the function f is linear, i.e., $f(\pi)=\pi$. In this case, one may verify that (for any K) the equation (4.1) reduces to $\widetilde{U}\left(\overline{\theta}\right)=\overline{\theta}-\Phi\left(\sqrt{\gamma}\left(\overline{\theta}-y\right)\right)$, i.e., exactly the condition we obtained in the two player case.

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Appendix A: the K player case

There are $K \geq 2$ players, the payoff to not investing is 0, the payoff to investing is $\theta - 1 + f(\pi)$, where π is the proportion of $K-1$ opponents investing and $f : [0,1] \to [0,1]$ is increasing with $f(0) = 0$ and $f(1) = 1$; θ is drawn from normal distribution with mean y and precision α and each player i observes a signal $x_i = \theta + \varepsilon_i$, where each ε_i is drawn from a normal distribution with mean 0 and precision β.

Suppose player 1 has observed public signal y and private signal x; what probability does he assign to the event that at least k out of players 2 through K observing a signal less than or equal to x? To answer this question, consider the following linear transformation of $(x_2, ..., x_K)$:

$$z_j = \sqrt{\beta}\sqrt{\frac{\alpha+\beta}{\alpha+2\beta}}\left(x_j - \frac{\alpha y + \beta x}{\alpha+\beta}\right) = \sqrt{\beta}\sqrt{\frac{\alpha+\beta}{\alpha+2\beta}}\left(\theta + \varepsilon_i - \frac{\alpha y + \beta x}{\alpha+\beta}\right)$$

Now for player 1,

$$\begin{pmatrix} \theta \\ \varepsilon_2 \\ \varepsilon_3 \\ \cdot \\ \varepsilon_N \end{pmatrix} \sim N\left(\begin{pmatrix} \frac{\alpha y+\beta x}{\alpha+\beta} \\ 0 \\ 0 \\ \cdot \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{1}{\alpha+\beta} & 0 & 0 & \cdot & 0 \\ 0 & \frac{1}{\beta} & 0 & \cdot & 0 \\ 0 & 0 & \frac{1}{\beta} & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \frac{1}{\beta} \end{pmatrix}\right)$$

Now

$$\mathbf{z} = \begin{pmatrix} z_2 \\ z_3 \\ \cdot \\ z_N \end{pmatrix} = A \begin{pmatrix} \theta \\ \varepsilon_2 \\ \varepsilon_3 \\ \cdot \\ \varepsilon_N \end{pmatrix} - \sqrt{\beta}\sqrt{\frac{\alpha+\beta}{\alpha+2\beta}}\left(\frac{\alpha y+\beta x}{\alpha+\beta}\right)\begin{pmatrix} 1 \\ 1 \\ \cdot \\ 1 \end{pmatrix}$$

where

$$A = \sqrt{\beta}\sqrt{\frac{\alpha+\beta}{\alpha+2\beta}}\begin{pmatrix} 1 & 1 & 0 & \cdot & 0 \\ 1 & 0 & 1 & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & 0 & 0 & \cdot & 1 \end{pmatrix}$$

Thus

$$\begin{aligned} \mathbf{z} &\sim N\left(A\begin{pmatrix} \frac{\alpha y+\beta x}{\alpha+\beta} \\ 0 \\ 0 \\ \cdot \\ 0 \end{pmatrix} - \sqrt{\beta}\sqrt{\frac{\alpha+\beta}{\alpha+2\beta}}\left(\frac{\alpha y+\beta x}{\alpha+\beta}\right)\begin{pmatrix} 1 \\ 1 \\ \cdot \\ 1 \end{pmatrix}, A\begin{pmatrix} \frac{1}{\alpha+\beta} & 0 & 0 & \cdot & 0 \\ 0 & \frac{1}{\beta} & 0 & \cdot & 0 \\ 0 & 0 & \frac{1}{\beta} & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \frac{1}{\beta} \end{pmatrix}A'\right) \\ &= N\left(\begin{pmatrix} 0 \\ 0 \\ 0 \\ \cdot \\ 0 \end{pmatrix}, \frac{\beta(\alpha+\beta)}{\alpha+2\beta}\begin{pmatrix} \frac{1}{\alpha+\beta}+\frac{1}{\beta} & \frac{1}{\alpha+\beta} & \frac{1}{\alpha+\beta} & \cdot & \frac{1}{\alpha+\beta} \\ \frac{1}{\alpha+\beta} & \frac{1}{\alpha+\beta}+\frac{1}{\beta} & \frac{1}{\alpha+\beta} & \cdot & \frac{1}{\alpha+\beta} \\ \frac{1}{\alpha+\beta} & \frac{1}{\alpha+\beta} & \frac{1}{\alpha+\beta}+\frac{1}{\beta} & \cdot & \frac{1}{\alpha+\beta} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \frac{1}{\alpha+\beta} & \frac{1}{\alpha+\beta} & \frac{1}{\alpha+\beta} & \cdot & \frac{1}{\alpha+\beta}+\frac{1}{\beta} \end{pmatrix}\right) \\ &= N\left(\begin{pmatrix} 0 \\ 0 \\ 0 \\ \cdot \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & \rho & \rho & \cdot & \rho \\ \rho & 1 & \rho & \cdot & \rho \\ \rho & 0 & 1 & \cdot & \rho \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \rho & \rho & \rho & \cdot & 1 \end{pmatrix}\right) \end{aligned}$$

where $\rho = \frac{\beta}{\alpha+2\beta}$.

Writing $\overline{\theta}$ for $\frac{\alpha y+\beta x}{\alpha+\beta}$, observe that $x=\overline{\theta}+\frac{\alpha}{\beta}\left(\overline{\theta}-y\right)$, so $x_j \geq x$ if and only if

$$z_j=\sqrt{\beta}\sqrt{\frac{\alpha+\beta}{\alpha+2\beta}}\left(x_j-\overline{\theta}\right)\geq\sqrt{\beta}\sqrt{\frac{\alpha+\beta}{\alpha+2\beta}}\left(\frac{\alpha}{\beta}\right)\left(\overline{\theta}-y\right)=\frac{\alpha}{\sqrt{\beta}}\sqrt{\frac{\alpha+\beta}{\alpha+2\beta}}\left(\overline{\theta}-y\right)$$

Now write $H(m,n,\rho,c)$ for the probability that at least m out of n normally distributed random variables are less than or equal to c if each of the random variables has mean 0 and variance 1, and the correlation coefficient between any two random variables is ρ. It turns out that the probability that at least k out of players 2 through K observing a signal less than or equal to x is

$$H\left(k,K-1,\frac{\beta}{\alpha+2\beta},\frac{\alpha}{\sqrt{\beta}}\sqrt{\frac{\alpha+\beta}{\alpha+2\beta}}\left(\overline{\theta}-y\right)\right)$$

As in the two person case, the key equation is the payoff to investing for a player with posterior $\overline{\theta}$, if he expects others to follow the $\overline{\theta}$ switching strategy. This equals

$$\widetilde{U}\left(\overline{\theta}\right)=\overline{\theta}-\sum_{k=1}^{K-1}\left[\begin{array}{c} f\left(\frac{K-k}{K-1}\right) \\ -f\left(\frac{K-k-1}{K-1}\right)\end{array}\right]\left(H\left(k,K-1,\frac{\beta}{\alpha+2\beta},\frac{\alpha}{\sqrt{\beta}}\sqrt{\frac{\alpha+\beta}{\alpha+2\beta}}\left(\overline{\theta}-y\right)\right)\right) \tag{4.2}$$

There is a unique equilibrium (for given K, α, β and f) if and only if the equation $\widetilde{U}\left(\overline{\theta}\right)=0$ has a unique solution. But observe that

$$\frac{d\widetilde{U}}{d\overline{\theta}}=1-\frac{\alpha}{\sqrt{\beta}}\sqrt{\frac{\alpha+\beta}{\alpha+2\beta}}\sum_{k=1}^{K-1}\left[\begin{array}{c} f\left(\frac{K-k}{K-1}\right) \\ -f\left(\frac{K-k-1}{K-1}\right)\end{array}\right]\left(\frac{dH}{dc}\left(k,K-1,\frac{\beta}{\alpha+2\beta},\frac{\alpha}{\sqrt{\beta}}\sqrt{\frac{\alpha+\beta}{\alpha+2\beta}}\left(\overline{\theta}_1-y\right)\right)\right)$$

Observe that $H(m,n,\rho,c)$ is continuous in ρ and c and strictly increasing in c for all $m=1,..,n$; thus the following are well defined:

$$\begin{aligned}\underline{c}(n) &= \inf_{\rho\in[\mathbf{0},1]}\ \min_{m=1,\ldots,n}\left|\frac{dH(m,n,\rho,0)}{dc}\right| \\ \overline{c}(n) &= \sup_{\rho\in[\mathbf{0},1]}\ \max_{m=1,\ldots,n}\ \max_{c\in\mathbf{R}}\left|\frac{dH(m,n,\rho,c)}{dc}\right|\end{aligned}$$

Now writing as before, $\widetilde{\gamma}(\alpha,\beta)=\frac{\alpha}{\sqrt{\beta}}\sqrt{\frac{\alpha+\beta}{\alpha+2\beta}}$, we have:

Lemma 4.1. *Fix K. If $\widetilde{\gamma}(\alpha,\beta)\leq\frac{1}{\overline{c}(K)}$, then there is a unique equilibrium (for any f); if $\widetilde{\gamma}(\alpha,\beta)\geq\frac{1}{\underline{c}(K)}$, then there are multiple equilibria (for any f).*

Thus β must be of the order of α^2 for uniqueness, independent of the shape of f and the number of players.

We can also examine what happens as $\beta \to 0$. First, note that $H\left(m, n, \frac{1}{2}, 0\right) = \frac{m+1}{n+1}$ for all m and n (David and Six [1971]). This implies that as $\beta \to 0$, and thus $\frac{\beta}{\alpha+2\beta} \to \frac{1}{2}$, equation (4.2) becomes

$$\widetilde{U}\left(\overline{\theta}\right) = \overline{\theta} - 1 + \frac{1}{K} \sum_{k=1}^{K-1} f\left(\frac{k}{K-1}\right)$$

Thus in the unique equilibrium, investment occurs only if $\overline{\theta} \geq 1 - \frac{1}{K} \sum_{k=1}^{K-1} f\left(\frac{k}{K-1}\right)$.

Appendix B: Proof of Proposition 2.1

We will provide a proof of proposition 2.1. The same proof also works for the many player case described in section 4.2 and appendix A. The payoffs in our game exhibit strategic complementarities. The following features of our model echo the general results concerning games with strategic complementarities obtained by Milgrom and Roberts [1990].

- There is a "smallest" and "largest" equilibrium, corresponding to the smallest and largest solutions to the equation $u(x, x) = 0$.
- Any strategy other than those lying between the smallest and largest equilibrium strategies can be eliminated by iterated deletion of dominated strategies. Thus, if $\underline{x}$ and $\bar{x}$ are, respectively, the smallest and largest solutions to $u(x, x) = 0$, then rationalizability removes all indeterminacy in a player's strategy except for the interval $[\underline{x}, \bar{x}]$.
- If there is a unique solution to $u(x, x) = 0$, then there is a unique equilibrium, and this is obtained as the uniquely rationalizable strategy.

We now turn to the proof of the main lemma. Begin with the function $u(x, \hat{x})$. It can be seen to satisfy the following three properties.

Monotonicity. u is strictly increasing in its first argument, and is strictly decreasing in its second argument.

Continuity. u is continuous.

Crossing. For any $\hat{x} \in \mathbb{R} \cup \{-\infty, \infty\}$, there is x such that $u(x, \hat{x}) = 0$.

By appealing to these features, we can define two sequences of real numbers. First, define the sequence

$$\underline{x}^1, \underline{x}^2, \cdots, \underline{x}^k, \cdots \tag{4.3}$$

as the solutions to the equations:

$$\begin{aligned} u\left(\underline{x}^1, -\infty\right) &= 0 \\ u\left(\underline{x}^2, \underline{x}^1\right) &= 0 \\ &\vdots \\ u\left(\underline{x}^{k+1}, \underline{x}^k\right) &= 0 \\ &\vdots \end{aligned}$$

In an analogous way, we define the sequence

$$\bar{x}^1, \bar{x}^2, \cdots, \bar{x}^k, \cdots \tag{4.4}$$

as the solutions to the equations:

$$\begin{aligned} u\left(\bar{x}^1, \infty\right) &= 0 \\ u\left(\bar{x}^2, \bar{x}^1\right) &= 0 \\ &\vdots \\ u\left(\bar{x}^{k+1}, \bar{x}^k\right) &= 0 \\ &\vdots \end{aligned}$$

We can then prove:

Lemma A1. Let x solve $u(x, x) = 0$. Then

$$\underline{x}^1 < \underline{x}^2 < \cdots < \underline{x}^k < \cdots < x \tag{4.5}$$

$$\bar{x}^1 > \bar{x}^2 > \cdots > \bar{x}^k > \cdots > x \tag{4.6}$$

Moreover, if $\underline{x}$ and $\bar{x}$ are, respectively, the smallest and largest solutions to $u(x, x) = 0$, then

$$\underline{x} = \lim_{k \to \infty} \underline{x}^k \quad \text{and} \quad \bar{x} = \lim_{k \to \infty} \bar{x}^k. \tag{4.7}$$

Proof. Since $u(\underline{x}^1, -\infty) = u(\underline{x}^2, \underline{x}^1) = 0$, monotonicity implies $\underline{x}^1 < \underline{x}^2$. Thus, suppose $\underline{x}^{k-1} < \underline{x}^k$. Since $u\left(\underline{x}^k, \underline{x}^{k-1}\right) = u\left(\underline{x}^{k+1}, \underline{x}^k\right) = 0$, monotonicity implies $\underline{x}^k < \underline{x}^{k+1}$. Finally, since $u(x, x) = 0 = u\left(\underline{x}^{k+1}, \underline{x}^k\right)$, and $\underline{x}^k < \underline{x}^{k+1}$, monotonicity implies that $\underline{x}^k < x$. Thus, $\underline{x}^1 < \underline{x}^2 < \cdots < \underline{x}^k < \cdots < x$. An exactly analogous argument shows that $\bar{x}^1 > \bar{x}^2 > \cdots > \bar{x}^k > \cdots > x$. Now, suppose $\underline{x}$ is the smallest solution to $u(x,x) = 0$. By (4.5) and the monotonicity of u, $\underline{x}$ is the smallest upper bound for the sequence $\{\underline{x}^k\}$. Since $\{\underline{x}^k\}$ is an increasing, bounded sequence, it converges to its smallest upper bound. Thus $\underline{x} = \lim_{k\to\infty} \underline{x}^k$. Analogously, if $\bar{x}$ is the largest solution to $u(x,x) = 0$, then (4.6) and monotonicity of u implies that $\bar{x} = \lim_{k\to\infty} \bar{x}^k$. This proves lemma A1.

Lemma A2. If s is a strategy which survives k rounds of iterated deletion of interim dominated strategies, then

$$s(x) = \begin{cases} \texttt{Not Invest} & \text{if } \; x < \underline{x}^k \\ \texttt{Invest} & \text{if } \; x > \bar{x}^k \end{cases} \tag{4.8}$$

The argument is as follows. Let s_{-i} be the strategy profile used by player i's opponent, and denote by $\tilde{u}^i(x, s_{-i})$ the expected payoff to i of `Invest` conditional on x when his opponent's strategy profile is given by s_{-i}. The incidence of `Invest` is maximized when everyone invests irrespective of the signal. Conversely, the incidence of `Invest` is *minimized* when everyone plays `Not Invest` irrespective of the signal. The expected payoff to `Invest` increases with the incidence of investment by others. Thus, for any x and any s_{-i},

$$u(x,\infty) \le \tilde{u}^i(x, s_{-i}) \le u(x,-\infty) \tag{4.9}$$

From the definition of $\underline{x}^1$ and monotonicity,

$$x < \underline{x}^1 \Longrightarrow \text{for any } s_{-i},\ \tilde{u}^i(x, s_{-i}) \le u(x,-\infty) < u\left(\underline{x}^1, -\infty\right) = 0. \tag{4.10}$$

In other words, $x < \underline{x}^1$implies that conditional on x, `Invest` is strictly dominated by `Not Invest`. Similarly, from the definition of $\bar{x}^1$ and monotonicity,

$$x > \bar{x}^1 \Longrightarrow \text{for any } s_{-i},\ \tilde{u}^i(x, s_{-i}) \ge u(x,\infty) > u\left(\bar{x}^1, \infty\right) = 0. \tag{4.11}$$

In other words, $x > \bar{x}^1$ implies that `Not Invest` is strictly dominated by `Invest`. Thus, if strategy s_i survives the initial round of deletion of dominated strategies,

$$s_i(x) = \begin{cases} \texttt{Not Invest} & \text{if } \; x < \underline{x}^1 \\ \texttt{Invest} & \text{if } \; x > \bar{x}^1 \end{cases} \tag{4.12}$$

so that (4.8) holds for $k = 1$.

For the inductive step, suppose that (4.8) holds for k, and denote by U^k the set of strategies which satisfy (4.8) for k. We must now show that, if player i faces a strategy profile from U^k, then any strategy which is not in U^{k+1} is dominated. Thus, suppose that player i believes that he faces a strategy s_{-i} from U^k. Given this, the incidence of `Invest` is maximized when s_{-i} is the$\bar{x}^k$-switching strategy, and the incidence of `Invest` is *minimized* when s_{-i} is the $\underline{x}^k$-switching strategy. The payoff to `Invest` increases with the incidence of investment by others. Thus, for any x and any strategy s_{-i} from U^k,

$$u\left(x, \bar{x}^k\right) \leq \tilde{u}^i\left(x, s_{-i}\right) \leq u\left(x, \underline{x}^k\right). \tag{4.13}$$

From the definition of $\underline{x}^k$ and monotonicity, we have the following implication for any strategy s_{-i} drawn from U^k.

$$x < \underline{x}^{k+1} \Longrightarrow \tilde{u}^i\left(x, s_{-i}\right) \leq u\left(x, \underline{x}^k\right) < u\left(\underline{x}^{k+1}, \underline{x}^k\right) = 0. \tag{4.14}$$

In other words, when $x < \underline{x}^k$ and when all others are using strategies from U^k, `Invest` is strictly dominated by `Not Invest`. Similarly, from the definition of $\bar{x}^k$ and monotonicity, we have the following implication for any strategy s_{-i} from U^k.

$$x > \bar{x}^{k+1} \Longrightarrow \ \tilde{u}^i\left(x, s_{-i}\right) \geq u\left(x, \bar{x}^k\right) > u\left(\bar{x}^{k+1}, \bar{x}^k\right) = 0. \tag{4.15}$$

In other words, when $x > \bar{x}^{k+1}$ and all others are using strategies from U^k, `Not Invest` is strictly dominated by `Invest`. Thus, if strategy s_i survives $k+1$ rounds of iterated deletion of dominated strategies,

$$s_i\left(x\right) = \begin{cases} \texttt{Not Invest} & \text{if} \quad x < \underline{x}^{k+1} \\ \texttt{Invest} & \text{if} \quad x > \bar{x}^{k+1} \end{cases} \tag{4.16}$$

This proves lemma A2.

With these preliminary results, we can complete the proof of the main lemma. First, let us show that if x solves $u\left(x, x\right) = 0$, then there is an equilibrium in switching strategies around x. Since $u\left(x, x\right) = 0$, if everyone else is using the x-switching strategy, the payoff to `Invest` given x is the same as that for `Not Invest`. Since u is strictly increasing in its first argument,

$$x_* < x < x^* \Longleftrightarrow u\left(x_*, x\right) < 0 < u\left(x^*, x\right)$$

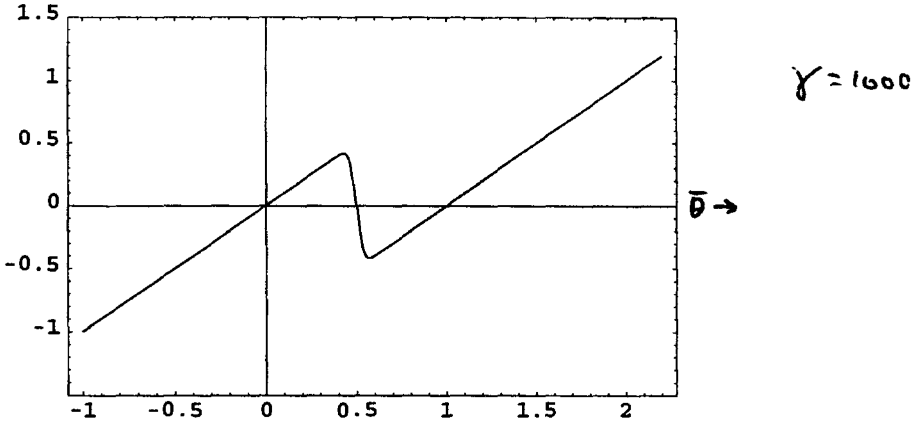
so that the x-switching strategy is the strict best reply.

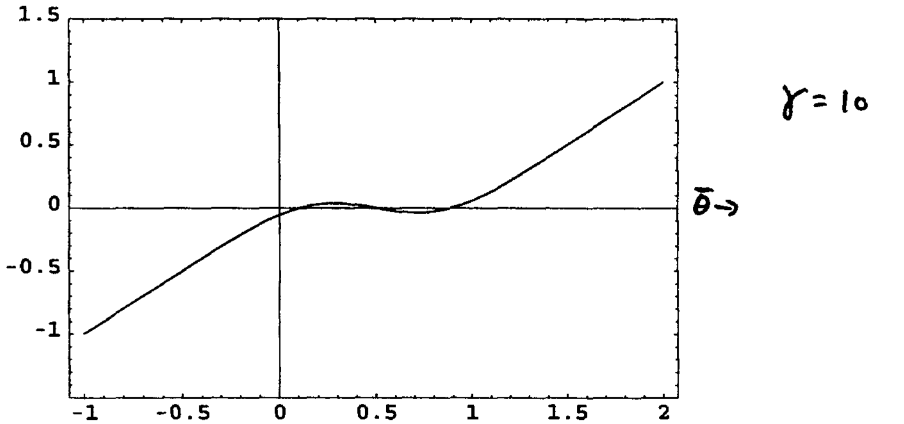
Finally, if x is the unique solution to $u\left(x, x\right) = 0$, then from Lemma A1, we know that

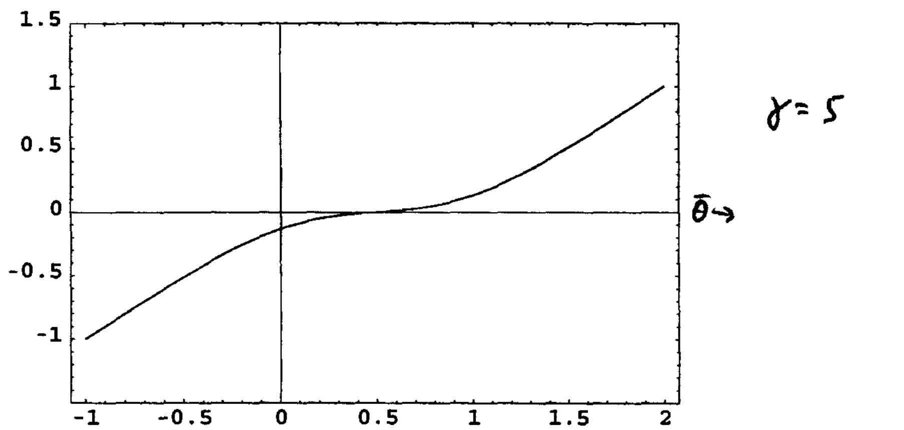
$$x = \lim_{k \to \infty} \underline{x}^k = \lim_{k \to \infty} \bar{x}^k \tag{4.17}$$

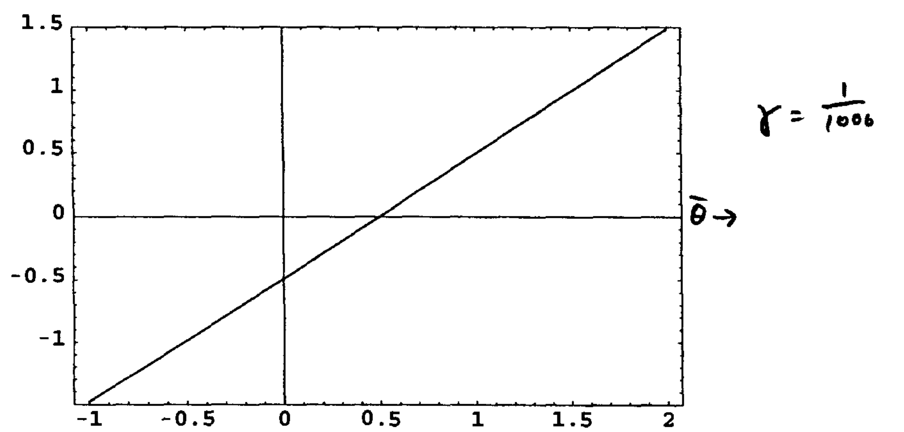
so that the only strategy which survives the iterated deletion of dominated strategies is the x-switching strategy.

FIGURES FOR "PRIVATE VERSUS PUBLIC INFORMATION IN COORDINATION PROBLEMS" BY MORRIS + SHIN









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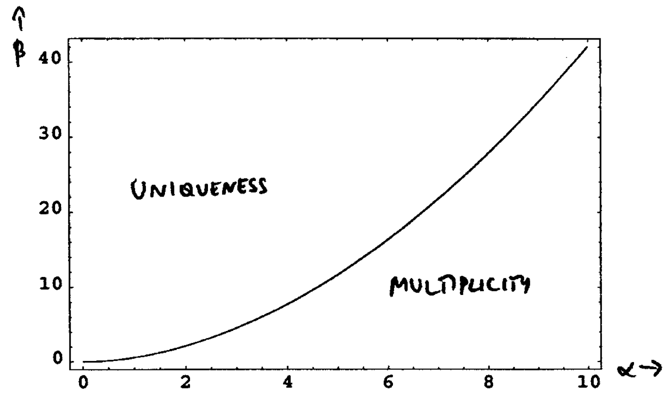
β
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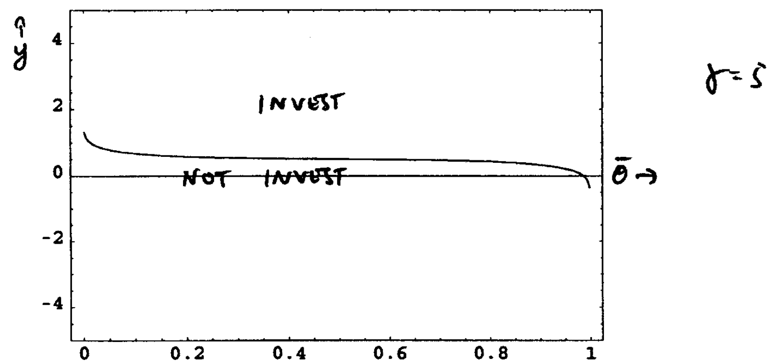
20
10
0
UNIQUENESS
MULTIPLICITY
0
2
4
6
8
10
α

6.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0
-2
-4
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NOT INVEST
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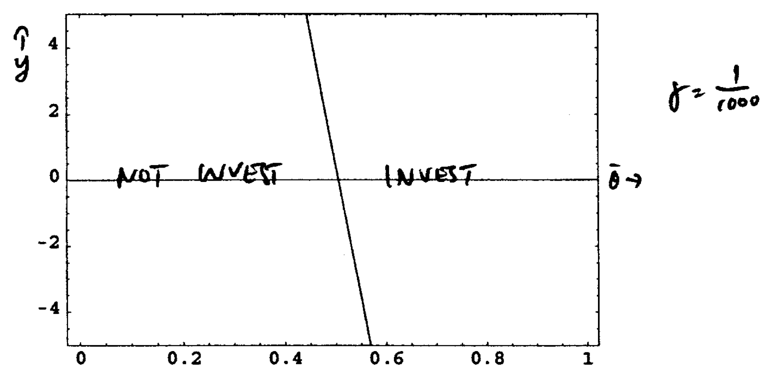
0.2
0.4
0.6
0.8
1
θ̄
γ = 5

7.


y
4
2
0
-2
-4
NOT INVEST
INVEST
0
0.2
0.4
0.6
0.8
1
θ̄
γ = 1/1000