Advanced Economic Growth, Problem Set 1

This problem set is due on or before the recitation on Friday, September 21

   Please answer the following questions:

Exercise 1 Consider a finite horizon continuous time maximization problem, where the objective function is

$$W\left(x\left(t\right),y\left(t\right)\right) = \int_0^{t_1} f\left(t, x\left(t\right), y\left(t\right)\right) dt$$

with $x\left(0\right) = x_0$ and $t_1 < \infty$, and the constraint equation is

$$\dot{x}\left(t\right) = g\left(t, x\left(t\right), y\left(t\right)\right).$$

Imagine that t_1 is also a choice variable.

1. Show that $W\left(x\left(t\right), y\left(t\right)\right)$ can be written as

$$W\left(x\left(t\right), y\left(t\right)\right) = \int_0^{t_1}\left[H\left(t, x\left(t\right), y\left(t\right)\right) + \dot{\lambda}\left(t\right) x\left(t\right)\right] dt - \lambda\left(t_1\right) x\left(t_1\right) + \lambda\left(0\right) x_0,$$

   where $H\left(t, x, y\right) \equiv f\left(t, x\left(t\right), y\left(t\right)\right) + \lambda\left(t\right) g\left(t, x\left(t\right), y\left(t\right)\right)$ is the Hamiltonian and $\lambda\left(t\right)$ is the costate variable.

2. Now suppose that the pair $\left(\hat{x}\left(t\right), \hat{y}\left(t\right)\right)$ together with terminal date $\hat{t}_1$ constitutes an optimal solution for this problem. Consider the following class of variations:

$$\begin{aligned}
y\left(t, \varepsilon\right) &= \hat{y}\left(t\right) + \varepsilon\eta\left(t\right) \text{ for } t \in \left[0, \hat{t}_1\right] \text{ and } y\left(t, \varepsilon\right) = \hat{y}\left(\hat{t}_1\right) + \varepsilon\eta\left(t\right) \text{ for } t \in \left[\hat{t}_1, \hat{t}_1 + \varepsilon\Delta t\right], \\
t_1 &= \hat{t}_1 + \varepsilon\Delta t.
\end{aligned}$$

   Denote the corresponding path of the state variable by $x\left(t, \varepsilon\right)$. Evaluate $W\left(x\left(t, \varepsilon\right), y\left(t, \varepsilon\right)\right)$ at this variation. Explain why this variation is feasible for ε small enough.

3. Show that for a feasible variation,

$$\begin{aligned}
\left.\frac{dW\left(x\left(t, \varepsilon\right), y\left(t, \varepsilon\right)\right)}{d\varepsilon}\right|_{\varepsilon=0} &= \int_0^{\hat{t}_1}\left[H_x\left(t, \hat{x}\left(t\right), \hat{y}\left(t\right)\right) + \dot{\lambda}\left(t\right)\right]\frac{\partial x\left(t, \varepsilon\right)}{\partial\varepsilon} dt \\
&\quad + \int_0^{\hat{t}_1} H_y\left(t, \hat{x}\left(t\right), \hat{y}\left(t\right)\right)\eta\left(t\right) dt \\
&\quad + H\left(\hat{t}_1, \hat{x}\left(\hat{t}_1\right), \hat{y}\left(\hat{t}_1\right)\right)\Delta t - \lambda\left(\hat{t}_1\right)\frac{\partial x\left(\hat{t}_1, \varepsilon\right)}{\partial\varepsilon}.
\end{aligned}$$

4. Explain why the previous expression has to be equal to 0.

5. Now taking the limit as $\hat{t}_1 \to \infty$, derive the weaker form of the transversality condition (7.45) in Chapter 7 in *Introduction to Modern Economic Growth*.

6. What are the advantages and disadvantages of this method of derivation relative to that used in the proof of Theorem 7.13 in Chapter 7 in *Introduction to Modern Economic Growth*.

Exercise 2 This exercise contains an alternative derivation of the stationary form of HJB equation, (7.41), in Chapter 7 in *Introduction to Modern Economic Growth*. Consider the discounted infinite-horizon problem, with $f\left(t, x\left(t\right), y\left(t\right)\right) = \exp\left(-\rho t\right) f\left(x\left(t\right), y\left(t\right)\right)$, and $g\left(t, x\left(t\right), y\left(t\right)\right) = g\left(x\left(t\right), y\left(t\right)\right)$.

Suppose that the admissible pair $(\hat{x}(t), \hat{y}(t))$ is optimal starting at $t = 0$ with initial condition $x(0) = x_0$. Let

$$V(x(0)) = \int_0^\infty \exp(-\rho t) f((\hat{x}(t), \hat{y}(t))) \, dt,$$

which is well defined. [You may want to verify this for bonus points]. Now write

$$V(x(0)) = f((\hat{x}(t), \hat{y}(t))) \Delta t + o(\Delta t) + \int_{\Delta t}^\infty \exp(-\rho t) f((\hat{x}(t), \hat{y}(t))) \, dt,$$

where $o(\Delta t)$ denotes second-order terms that satisfy $\lim_{\Delta t \to 0} o(\Delta t) / \Delta t = 0$. Explain why this equation can be written as

$$V(x(0)) = f((\hat{x}(t), \hat{y}(t))) \Delta t + o(\Delta t) + \exp(-\rho \Delta t) V(x(\Delta t)).$$

Now subtract $V(x(\Delta t))$ from both sides and divide both sides by Δt to obtain

$$\frac{V(x(0)) - V(x(\Delta t))}{\Delta t} = f((\hat{x}(t), \hat{y}(t))) + \frac{o(\Delta t)}{\Delta t} + \frac{\exp(-\rho \Delta t) - 1}{\Delta t} V(x(\Delta t)).$$

Show that taking the limit as $\Delta t \to 0$ gives the HJB equation.

Exercise 3 Consider the following optimal growth model without discounting:

$$\max \int_0^\infty [u(c(t)) - u(c^*)] \, dt$$

subject to

$$\dot{k}(t) = f(k(t)) - c(t) - \delta k(t)$$

with initial condition $k(0) > 0$, and c^* defined as the golden rule consumption level

$$c^* = f(k^*) - \delta k^*$$

where k^* is the golden rule capital-labor ratio given by $f'(k^*) = \delta$.

1. Set up the Hamiltonian for this problem with costate variable $\lambda(t)$.

2. Characterize the solution to this optimal growth program.

3. Show that the standard transversality condition that $\lim_{t \to \infty} \lambda(t) k(t) = 0$ is not satisfied at the optimal solution. Explain why this is the case.

Exercise 4 Consider a world economy consisting of $j = 1, ..., M$ economies. Suppose that each of those are closed and have access to the same production and R&D technology as described in the base lab equipment model with expanding input varieties. The only differences across countries are in the size of labor force, L_j, productivity of R&D, η_j, and discount rate ρ_j. You may also want to assume that one unit of R&D expenditure costs ζ_j units of final good in country j, and this varies across countries. There are no technological exchange across countries.

1. Formally define the "world equilibrium" in which each country is in equilibrium.

2. Characterize the world equilibrium. Show that in the world equilibrium, each country will grow at a constant rate starting at $t = 0$.

3. Show that except in "knife-edge" cases, output in each country will grow at a different long-run rate. Characterize the growth rate of each country.

4. Now consider the effects of policy and taxes on long-run income per capita differences. Show that, in the model discussed in this exercise, arbitrarily small differences in policy or discount factors across countries will lead to "infinitely large" differences in long-run income per capita. Does the environment in this exercise provide a satisfactory model for the study of long-run income per capita differences across countries? If yes, please elaborate how such a model can be mapped to reality. If not, explain which features of the model appear unsatisfactory to you and how you would want to change them.

Exercise 5 Consider the baseline expanding input variety with the lab equipment technology, but with one difference. A firm that invents a new machine receives a patent, which expires at the Poisson rate ι. Once the patent expires, that machine is produced competitively and is supplied to final good producers at marginal cost.

1. Characterize the equilibrium in this case and show how the equilibrium growth rate depends on ι. [Hint: notice that there will be two different types of machines, supplied at different prices].

2. What is the value of ι that maximizes the equilibrium rate of economic growth?

3. Show that a policy of $\iota = 0$ does not necessarily maximize social welfare at time $t = 0$.

Exercise 6 In the baseline Schumpeterian growth model, modify the production function of the final good sector to

$$Y(t) = \frac{1}{1 - \beta} \left[\int_0^1 q(\nu, t)^{\zeta_1} x(\nu, t \mid q)^{1 - \beta} d\nu \right] L^\beta.$$

Suppose also that producing one unit of an intermediate good of quality q costs ψq^{ζ_2} and that one unit of final good devoted to research on the machine line with quality q generates a flow rate of innovation equal to η / q^{ζ_3}. Characterize the equilibrium of this economy and determine what combinations of the parameters ζ_1, ζ_2 and ζ_3 will ensure balanced growth.