Addendum to: An Elementary Theory of Global Supply Chains

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Abstract

This addendum provides the proofs of all propositions in Section 6 of our main paper.

1 Coordination Costs

1.1 Assumptions

Compared to the model described in Section 2 of our main paper, we assume that trade in intermediate goods is subject to coordination costs $\tau \in (0,1)$. If the production of a unit u of the final good in a given country involves n international transactions—i.e. export and import at stages $0 < S_1^u \leq S_2^u \leq ... \leq S_n^u < S$—then the final good is defect free with probability $(1-\tau)^n \in (0,1)$. The case considered in Section 2 corresponds to the limit as coordination costs τ go to zero. Upon completion of each unit of the final good, we assume that consumers perfectly observe whether the unit is defect free or not. A unit of the final good with a defect has zero price. Like in Section 2, we assume that the (defect-free) final good is freely traded and we use it as our numeraire. Finally, we assume that firms perfectly observe all international transactions so that two units of the same intermediate good s may, in principle, command two different prices if their production requires a different number of international transactions.

1.2 Pattern of International Specialization

Let $c^u(s)$ denote the country in which stage s has been performed for the production of a given unit u. Using this notation, the pattern of international specialization can be characterized as follows.

Proposition 1 [Coordination Costs] *In any competitive equilibrium, the allocation of stages to countries, $c^u : \mathcal{S} \to \mathcal{C}$, is increasing in s for all $u \in \left[0, \sum_{c\in\mathcal{C}} Q_c(S)\right]$.*

Proof. We proceed in two steps.

Step 1: *In any competitive equilibrium, c^u is a step-function for all $u \in \left[0, \sum_{c\in\mathcal{C}} Q_c(S)\right]$.*

We proceed by contradiction. Suppose that c^u is not a step-function. In this case, the number of international transactions associated with the production of that particular unit must be infinite, which is incompatible with zero profits.

Step 2: *In any competitive equilibrium, c^u is increasing in s for all $u \in \left[0, \sum_{c\in\mathcal{C}} Q_c(S)\right]$.*

Let us start by computing the expected cost of producing a given unit u. By Step 1, we know that that there must be a sequence of countries $\{c_1^u, ..., c_n^u\}$ and stages $\{S_1^u, ..., S_n^u\}$ such that: (i) $c_k^u \neq c_{k+1}^u$ for all $k = 1,..,n-1$; (ii) $S_0^u = 0 < S_1^u < ... < S_{n-1}^u < S_n^u = S$; and (iii) $c^u(s) = c_k^u$ for all $s \in \left(S_{k-1}^u, S_k^u\right)$. Let p_k^u denote the cost of producing that unit up to stage S_k^u. By the same logic as in the proof of Lemma 2 of our main paper, we know that $\{p_1^u, ..., p_n^u\}$ satisfies

$$p_k^u = e^{\lambda_{c_k^u} N_k^u} p_{k-1}^u + \left(e^{\lambda_{c_k^u} N_k^u} - 1\right)\left(w_{c_k^u}/\lambda_{c_k^u}\right),$$

where $N_k^u \equiv S_k^u - S_{k-1}^u$. Using the previous expression and iterating, it is then easy to check that the expected cost of producing unit u is given by

$$p_n^u = \left\{\sum_{k=1}^n \left[\prod_{k'>k} e^{\lambda_{c_{k'}^u} N_{k'}^u}\right]\left(e^{\lambda_{c_k^u} N_k^u} - 1\right)\left(w_{c_k^u}/\lambda_{c_k^u}\right)\right\} \Big/ (1-\tau)^n .$$

The rest of our proof proceeds by contradiction. Suppose that c^u is not weakly increasing. Then there must exist $k_0 \in \{1, ..., n-1\}$ and $c > c'$ such that $c^u_{k_0} = c$ and $c^u_{k_0+1} = c'$. Suppose, in addition, that $N^u_{k_0} = S^u_{k_0} - S^u_{k_0-1} \leq S^u_{k_0+1} - S^u_{k_0} = N^u_{k_0+1}$. The other case can be treated in a similar manner. Now consider an alternative allocation of stages to countries, $\widetilde{c}^u$, in which country c' produces all stages from $S^u_{k_0-1}$ to $S^u_{k_0}$ and c produces all stages from $S^u_{k_0+1} - N^u_{k_0}$ to $S^u_{k_0+1}$. The rest of the allocation of stages to countries is the same as in c^u. Let $\widetilde{p}^u_n$ denote the expect unit cost of the final good associated with $\widetilde{c}^u$:

$$\begin{aligned}\widetilde{p}^u_n = \Big\{ \textstyle\sum_{k=1}^{k_0-1} & \left[\left(\prod_{k'>k} e^{\lambda_{c^u_{k'}} N^u_{k'}}\right)\right] \left(e^{\lambda_{c^u_k} N^u_k} - 1\right) \left(w_{c^u_k}/\lambda_{c^u_k}\right) \\ & + \left[e^{\lambda_c N^u_{k_0}} \left(\prod_{k'>k_0+1} e^{\lambda_{c^u_{k'}} N^u_{k'}}\right)\right] \left(e^{\lambda_{c'} N^u_{k_0+1}} - 1\right) (w_{c'}/\lambda_{c'}) \\ & + \left[\prod_{k'>k_0+1} e^{\lambda_{c^u_{k'}} N^u_{k'}}\right] \left(e^{\lambda_c N^u_{k_0}} - 1\right) (w_c/\lambda_c) \\ & + \textstyle\sum_{k>k_0+1}^{n} \left[\prod_{k'>k} e^{\lambda_{c^u_{k'}} N^u_{k'}}\right] \left(e^{\lambda_{c^u_k} N^u_k} - 1\right) \left(w_{c^u_k}/\lambda_{c^u_k}\right) \Big\} \Big/ (1-\tau)^{\widetilde{n}},\end{aligned}$$

where, by construction, we must have $\widetilde{n} \leq n$. Under perfect competition, we know that unit costs of production must be minimized. Thus we must have $\widetilde{p}^u_n \geq p^u_n$. Since $\widetilde{n} \leq n$, this further requires

$$\begin{aligned}& \left[e^{\lambda_c N^u_{k_0}} \left(\prod_{k'>k_0+1} e^{\lambda_{c^u_{k'}} N^u_{k'}}\right)\right] \left(e^{\lambda_{c'} N^u_{k_0+1}} - 1\right) (w_{c'}/\lambda_{c'}) + \left[\prod_{k'>k_0+1} e^{\lambda_{c^u_{k'}} N^u_{k'}}\right] \left(e^{\lambda_c N^u_{k_0}} - 1\right) (w_c/\lambda_c) \\ & \geq \left[\prod_{k'>k_0} e^{\lambda_{c^u_{k'}} N^u_{k'}}\right] \left(e^{\lambda_c N^u_{k_0}} - 1\right) (w_c/\lambda_c) + \left[\prod_{k'>k_0+1} e^{\lambda_{c^u_{k'}} N^u_{k'}}\right] \left(e^{\lambda_{c'} N^u_{k_0+1}} - 1\right) (w_{c'}/\lambda_{c'}),\end{aligned}$$

which simplifies into $w_{c'} \geq (\lambda_{c'}/\lambda_c)\, w_c > w_c$. But if the previous inequality holds, then starting from the allocation c^u, the cost of producing unit u could be strictly lowered by performing all tasks in $\left(S^u_{k_0-1}, S^u_{k_0+1}\right)$ in country c: it reduces the rate of mistakes, it reduces wages, and it reduces the number of international transactions. Since unit costs must be minimized in any competitive equilibrium, we have established, by contradiction, that c^u is increasing in s for all $u \in \left[0, \sum_{c \in \mathcal{C}} Q_c(S)\right]$. ■

2 Simultaneous versus Sequential Production

2.1 Assumptions

Compared to the model described in Section 2 of our main paper, we assume that there are multiple supply chains, indexed by $n \in \mathcal{N} \equiv \{1, ..., N\}$, each associated with the production of a part. We allow supply chains to differ in terms of their complexity, S^n, but we require failure rates to be constant across chains and given by λ_c. Parts are ordered such that S^n is weakly increasing in n. Parts are assembled into a unique final good using labor. Formally, the output Y_c of the final good in country c is given by

$$Y_c = F\left(X^1_c, ..., X^N_c, A_c\right),$$

where $F(\cdot)$ is a production function with constant returns to scale, X^n_c is the amount of part n used in the production of the final good in country c, and $A_c \leq L_c$ corresponds to the amount of labor used for assembly in country c.

2.2 Pattern of International Specialization

In this generalized version of our model, the pattern of international specialization still takes a very simple form, as the next proposition demonstrates.

Proposition 1 [Simultaneous Production] *In any free trade equilibrium, there exists a sequence of stages* $S_0 \equiv 0 \leq S_1 \leq ... \leq S_C = S^N$ *such that for all* $n \in \mathcal{N}$, $s \in (0, S^n]$, *and* $c \in \mathcal{C}$, $Q_c^n(s) > 0$ *if and only if* $s \in (S_{c-1}, S_c]$. *Furthermore, if country c is engaged in parts production,* $A_c < L_c$, *then all countries* $c' > c$ *are only involved in parts production,* $A_{c'} = 0$.

Proof. We proceed in two steps.

Step 1: *In any free trade equilibrium, if* $A_c < L_c$, *then* $A_{c'} = 0$ *for all* $c' > c$.

We proceed by contradiction. Suppose that there exist $c' > c$ such that $A_{c'} > 0$ and $A_c < L_c$. Since the production function for the final good, F, is identical across countries, zero profits in the final good sector and $A_{c'} > 0$ require $w_c \geq w_{c'}$. Since $\lambda_{c'} < \lambda_c$, the unit cost of production of any intermediate good in any chain is then lower in country c' than in country c, thereby contradicting $A_c < L_c$.

Step 2: *In any free trade equilibrium, there exists a sequence of stages* $S_0 \equiv 0 \leq S_1 \leq ... \leq S_C = S$ *such that for all* $n \in \mathcal{N}$, $s \in (0, S^n]$, *and* $c \in \mathcal{C}$, $Q_c^n(s) > 0$ *if and only if* $s \in (S_{c-1}, S_c]$.

Using the same argument as in Proposition 1 of our main paper, one can easily show that for all $n \in \mathcal{N}$, there exist $S_0^n \equiv 0 \leq S_1^n \leq ... \leq S_C^n = S^n$ such that for all $s \in (0, S^n]$ and $c \in \mathcal{C}$, $Q_c^n(s) > 0$ if and only if $s \in (S_{c-1}^n, S_c^n]$. Compared to our main paper, the only difference is that some countries might not participate in chain n. Let us first show that (i) for all n, $n' \in \mathcal{N}$ and $c \in \mathcal{C}$, if $S_c^n < S^n$ and $S_c^{n'} < S^{n'}$, then $S_c^n = S_c^{n'}$. We proceed by contradiction. Suppose that there exist n, n', c such that $S_c^n < S^n$, $S_c^{n'} < S^{n'}$ and $S_c^n \neq S_c^{n'}$. Let $c_1 \equiv \inf \left\{ c \in \mathcal{C} | S_c^n \neq S_c^{n'} \right\}$. Suppose, without loss of generality, that $S_{c_1}^n < S_{c_1}^{n'}$. Let $c_2 \equiv \inf \left\{ c' > c_1 | S_{c'}^n > S_{c_1}^n \right\}$. Since $S_{c_1}^n < S^n$, c_2 is well-defined. Let $\epsilon \equiv \min \left(S_{c_1}^{n'} - S_{c_1}^n, S_{c_2}^n - S_{c_1}^n \right)$. By construction, c_1 produces all intermediate goods $s \in (S_{c_1}^n, S_{c_1}^n + \epsilon]$ in chain n', whereas c_2 produces all intermediate goods $s \in (S_{c_1}^n, S_{c_1}^n + \epsilon]$ in chain n. Thus for all $\varepsilon \in (0, \epsilon]$, the zero-profit condition—condition (2) in our main paper—implies

$$\begin{aligned}
p^{n'}\left(S_{c_1}^n + \varepsilon\right) &= e^{\lambda_{c_1}\varepsilon} p^{n'}\left(S_{c_1}^n\right) + \left(e^{\lambda_{c_1}\varepsilon} - 1\right)\left(w_{c_1}/\lambda_{c_1}\right), \\
p^{n}\left(S_{c_1}^n + \varepsilon\right) &= e^{\lambda_{c_2}\varepsilon} p^{n}\left(S_{c_1}^n\right) + \left(e^{\lambda_{c_2}\varepsilon} - 1\right)\left(w_{c_2}/\lambda_{c_2}\right), \\
p^{n'}\left(S_{c_1}^n + \varepsilon\right) &\leq e^{\lambda_{c_2}\varepsilon} p^{n'}\left(S_{c_1}^n\right) + \left(e^{\lambda_{c_2}\varepsilon} - 1\right)\left(w_{c_2}/\lambda_{c_2}\right), \\
p^{n}\left(S_{c_1}^n + \varepsilon\right) &\leq e^{\lambda_{c_1}\varepsilon} p^{n}\left(S_{c_1}^n\right) + \left(e^{\lambda_{c_1}\varepsilon} - 1\right)\left(w_{c_1}/\lambda_{c_1}\right),
\end{aligned}$$

where $p^n(\cdot)$ and $p^{n'}(\cdot)$ represent the price of intermediate goods in chains n and n', respectively. By definition of c_1, we know that $p^n\left(S_{c_1}^n\right) = p^{n'}\left(S_{c_1}^n\right) \equiv p_c$. Thus the four previous conditions imply

$$e^{\lambda_{c_1}\varepsilon}\left(p_c + w_{c_1}/\lambda_{c_1}\right) - e^{\lambda_{c_2}\varepsilon}\left(p_c + w_{c_2}/\lambda_{c_2}\right) + \left(w_{c_2}/\lambda_{c_2}\right) - \left(w_{c_1}/\lambda_{c_1}\right) = 0.$$

Since the previous equation holds for all $\varepsilon \in (0, \epsilon]$, $\lambda_{c_1} \neq \lambda_{c_2}$ implies $p_c + w_{c_1}/\lambda_{c_1} = p_c + w_{c_2}/\lambda_{c_2} = 0$, which contradicts w_{c_1},$w_{c_2} > 0$. Using the exact same argument, one can also show that (ii) for all n, $n' \in \mathcal{N}$ and $c \in \mathcal{C}$, if $S_c^n = S^n$ and $S_c^{n'} < S^{n'}$, then $S_c^{n'} \geq S^n$. To conclude, let $S^c \equiv \max_{n \in \mathcal{N}} \{S_c^n\}$. Given the definition of S_c^n and properties (i) and (ii), for all $n \in N$, $s \in (0, S^n]$, and $c \in C$, $Q_c^n(s) > 0$ if and only if $s \in (S_{c-1}, S_c]$. Our proposition follows directly from Steps 1 and 2. ∎

3 Imperfect Observability of Mistakes

3.1 Assumptions

Like in Section 2 of our main paper, mistakes occur at a constant rate λ_c in country c. When a mistake occurs on a unit u of intermediate good at stage s, any intermediate good produced after stage s using unit u is also defective and the associated final good is worthless. Our only point of departure from our benchmark model is that mistakes are imperfectly observed. Specifically, we assume that if a mistake occurs in the production of intermediate good s, it can only be observed at that stage with probability $\beta_c \in [0,1]$, where country c is the country in which intermediate good s is produced. The location in which different stages associated with a given unit have been performed is public information. All markets are perfectly competitive and all goods are freely traded. Thus different units of a given intermediate good s produced at different locations may command a different price $p(s,\theta)$ depending on their "quality," i.e., the commonly known probability $\theta \in [0,1]$ that they are are defect free. Of course, units which are known to have a defect have zero price, $p(s,0) = 0$.

In this environment, if a firm from country c combines $q[s,\theta(s)]$ units of intermediate good s with quality $\theta(s)$ with $q[s,\theta(s)]\,ds$ units of labor, its output at stage $s+ds$ is given by

$$q[s+ds,\theta(s+ds)] = (1-\beta_c\lambda_c ds)\, q[s,\theta(s)]\,, \tag{1}$$

where the quality at stage $s+ds$ can be computed using Bayes' rule:

$$\theta(s+ds) = \frac{1-\lambda_c ds}{1-\beta_c\lambda_c ds}\theta(s)\,.$$

Since ds is infinitesimal, a first-order Taylor expansion implies

$$\theta(s+ds) = [1-(1-\beta_c)\,\lambda_c ds]\,\theta(s)\,. \tag{2}$$

For simplicity, we restrict ourselves to "symmetric" free trade equilibria in which all units of the final good are produced in the same manner. Thus each stage of production is performed in only one country, which implies that the quality $\theta(s)$ at a given stage s is the same for all units in equilibrium.

3.2 Pattern of International Specialization

Before deriving the pattern of international specialization in this more general environment, note that the zero-profit condition in country c requires

$$p[s+ds,\theta(s+ds)]\,q[s+ds,\theta(s+ds)] \leq p[s,\theta(s)]\,q[s,\theta(s)] + w_c q[s,\theta(s)]\,ds,$$

with equality if $Q_c(s') > 0$ for all $s' \in (s, s+ds]$. Using equations (1) and (2), we can rearrange the previous condition as

$$p\left[s+ds, \left[1-\left(1-\beta_c\right)\lambda_c ds\right]\theta\left(s\right)\right] \leq p\left[s, \theta\left(s\right)\right]\left(1+\beta_c\lambda_c ds\right) + w_c ds, \tag{3}$$

with equality if $Q_c(s') > 0$ for all $s' \in (s, s+ds]$. Building on this new zero profit condition, the next proposition demonstrates how both β_c and λ_c shape the pattern of international specialization.

Proposition 1 [Imperfect Observability of Mistakes] *Suppose that $\beta_c\lambda_c$ is strictly decreasing in c and λ_c is weakly decreasing in c. Then in any symmetric free trade equilibrium, there exists a sequence of stages $S_0 \equiv 0 < S_1 < ... < S_C = S$ such that for all $s \in \mathcal{S}$ and $c \in \mathcal{C}$, $Q_c(s) > 0$ if and only if $s \in (S_{c-1}, S_c]$.*

Proof. In any symmetric free trade equilibrium, each stage of production is performed in the same country for all units of the final good. Let $c(s) \in \mathcal{C}$ denote the country performing stage s in equilibrium. Similarly, let $w^*(s) \equiv w^*_{c(s)}$, $\beta^*(s) \equiv \beta_{c(s)}$, and $\lambda^*(s) \equiv \lambda_{c(s)}$ denote the wage, the probability of observing a mistake, and the rate of mistakes, respectively, in the country performing stage s in equilibrium. Note that $w^*(s)$, $\beta^*(s)$, and $\lambda^*(s)$ are measurable since if a firm produces intermediate good s, then it necessarily produces a positive measure of intermediate goods around that stage.

Our proof proceeds in three steps.

Step 1: *For any pair of stages $s_1 > s_2$, if the quality at stage s_2 is equal to $\theta(s_2)$, then the quality $\theta(s_1)$ at stage s_1 satisfies*

$$\theta\left(s_1\right) = e_2^{-\int_{s_2}^{s_1}\left(1-\beta^*(s)\right)\lambda^*(s)ds}\theta\left(s_2\right).$$

This directly follows from equation (2) and the definitions of $\beta^*(s)$ and $\lambda^*(s)$.

Step 2: *For any stage s_0, if the quality at stage s_0 is equal to $\theta(s_0)$, then $p[s_0, \theta(s_0)]$ satisfies*

$$p\left[s_0, \theta\left(s_0\right)\right] = e^{-\int_{s_0}^{S}\lambda^*(s)ds}\theta\left(s_0\right) - W\left(s_0\right),$$

where $W(s_0) > 0$ is strictly decreasing in s_0.

By Step 1 we know that if $\theta(s_0)$ is the quality at stage s_0, then for any stage $s > s_0$, quality is given by

$$\theta\left(s\right) = e^{-\int_{s_0}^{s}\left(1-\beta^*(s)\right)\lambda^*(s)ds}\theta\left(s_0\right). \tag{4}$$

For all $s \geq s_0$, with a slight abuse of notation, let $p(s) \equiv p[s, \theta(s)]$. Using condition (3), one can show that

$$\frac{dp\left(s\right)}{ds} = \beta\left(s\right)\lambda\left(s\right)p\left(s\right) + w^*\left(s\right), \text{ for all } s \in \left[s_0, S\right]. \tag{5}$$

By choice of numeraire, we know that $p[S, 1] = 1$. By non-arbitrage, if $\theta(S)$ is the quality at stage S, then

$$p\left(S\right) = \theta\left(S\right) = e^{-\int_{s_0}^{S}\left(1-\beta^*(s)\right)\lambda^*(s)ds}\theta\left(s_0\right), \tag{6}$$

where the second equality follows from equation (4) evaluated at $s = S$. Solving the differential equation (5) with terminal condition (6), we obtain at $s = s_0$

$$p(s_0) = e^{-\int_{s_0}^{S} \lambda^*(s) ds} \theta(s_0) - W(s_0), \tag{7}$$

where

$$W(s_0) = \int_{s_0}^{S} e^{-\int_{s_0}^{s} \beta^*(s') \lambda^*(s') ds'} w^*(s) \, ds > 0. \tag{8}$$

Differentiating we get

$$W'(s_0) = -w^*(s_0) + \beta^*(s_0) \lambda^*(s_0) \int_{s_0}^{S} e^{-\int_{s_0}^{s} \beta^*(s') \lambda^*(s') ds'} w^*(s) \, ds,$$

which can be rearranged as

$$W'(s_0) = \beta^*(s_0) \lambda^*(s_0) \left\{ -\frac{w^*(s_0)}{\beta^*(s_0) \lambda^*(s_0)} + \int_{s_0}^{S} e^{-\int_{s_0}^{s} \beta^*(s') \lambda^*(s') ds'} w^*(s) \, ds \right\}.$$

Since along the optimal path the expected unit cost of production is minimized, we must have, in particular,

$$\int_{s_0}^{S} e^{-\int_{s_0}^{s} \beta^*(s') \lambda^*(s') ds'} w^*(s) \, ds \leq w^*(s_0) \int_{s_0}^{S} e^{-\beta^*(s_0) \lambda^*(s_0)(s - s_0)} ds$$

where

$$w^*(s_0) \int_{s_0}^{S} e^{-\beta^*(s_0) \lambda^*(s_0)(s - s_0)} ds < w^*(s_0) \int_{s_0}^{\infty} e^{-\beta^*(s_0) \lambda^*(s_0)(s - s_0)} ds = \frac{w^*(s_0)}{\beta^*(s_0) \lambda^*(s_0)}.$$

Combining the three previous expressions, we obtain $W'(s_0) < 0$.

Step 3: *If $c_2 > c_1$ and $Q_{c_1}(s_1) > 0$, then $Q_{c_2}(s_2) = 0$ for all $s_2 < s_1$.*

We proceed by contradiction. Consider a symmetric free trade equilibrium with two countries, $c_2 > c_1$, and two stages, $s_1 > s_2 > 0$, such that $Q_{c_1}(s_1) > 0$ and $Q_{c_2}(s_2) > 0$. Country c_1 produces all intermediate goods $s \in \Delta(s_1)$, whereas country c_2 produces all intermediate goods $s \in \Delta(s_2)$. Let $\theta^*(s)$ and $p^*(s) \equiv p[s, \theta^*(s)]$ denote the quality and the price of intermediate good s, respectively. Condition (3) implies

$$\begin{aligned}
p^*(s_1) &= \left(1 + \beta_{c_1} \lambda_{c_1} ds\right) p^*(s_1 - ds) + w_{c_1} ds, \\
p^*(s_2) &= \left(1 + \beta_{c_2} \lambda_{c_2} ds\right) p^*(s_2 - ds) + w_{c_2} ds, \\
p\left\{s_1, \left[1 - \left(1 - \beta_{c_2}\right) \lambda_{c_2} ds\right] \theta_1^*\right\} &\leq \left(1 + \beta_{c_2} \lambda_{c_2} ds\right) p^*(s_1 - ds) + w_{c_2} ds, \\
p\left\{s_2, \left[1 - \left(1 - \beta_{c_1}\right) \lambda_{c_1} ds\right] \theta_2^*\right\} &\leq \left(1 + \beta_{c_1} \lambda_{c_1} ds\right) p^*(s_2 - ds) + w_{c_1} ds,
\end{aligned}$$

with $\theta_1^* \equiv \theta^*(s_1 - ds)$ and $\theta_2^* \equiv \theta^*(s_2 - ds)$. The previous conditions imply

$$\begin{aligned} & p\left\{s_1, \left[1-\left(1-\beta_{c_2}\right)\lambda_{c_2}ds\right]\theta_1^*\right\} + p\left\{s_2, \left[1-\left(1-\beta_{c_1}\right)\lambda_{c_1}ds\right]\theta_2^*\right\} \\ & \qquad \leq p^*(s_1) + p^*(s_2) + \left(\beta_{c_2}\lambda_{c_2}ds - \beta_{c_1}\lambda_{c_1}ds\right)\left[p^*(s_1 - ds) - p^*(s_2 - ds)\right]. \end{aligned}$$

By Step 2, we know that $p[s, \theta(s)]$ must be differentiable in $\theta(s)$. Since ds is infinitesimal, a first-order Taylor approximation of the left handside leads to

$$\begin{aligned} & p\left\{s_1, \left[1-\left(1-\beta_{c_2}\right)\lambda_{c_2}ds\right]\theta_1^*\right\} + p\left\{s_2, \left[1-\left(1-\beta_{c_1}\right)\lambda_{c_1}ds\right]\theta_2^*\right\} \\ & \qquad = p^*(s_1) - \left[\left(1-\beta_{c_2}\right)\lambda_{c_2} - \left(1-\beta_{c_1}\right)\lambda_{c_1}\right]\frac{\partial p(s_1, \theta_1^*)}{\partial \theta}\theta_1^* ds \\ & \qquad\qquad + p^*(s_2) - \left[\left(1-\beta_{c_1}\right)\lambda_{c_1} - \left(1-\beta_{c_2}\right)\lambda_{c_2}\right]\frac{\partial p(s_2, \theta_2^*)}{\partial \theta}\theta_2^* ds. \end{aligned}$$

Combining the two previous expressions we obtain

$$\begin{aligned} & \left[\left(1-\beta_{c_2}\right)\lambda_{c_2} - \left(1-\beta_{c_1}\right)\lambda_{c_1}\right]\left[\theta_2^*\frac{\partial p(s_2, \theta_2^*)}{\partial \theta} - \theta_1^*\frac{\partial p(s_1, \theta_1^*)}{\partial \theta}\right] \\ \leq \quad & \left(\beta_{c_2}\lambda_{c_2} - \beta_{c_1}\lambda_{c_1}\right)\left[p^*(s_1 - ds) - p^*(s_2 - ds)\right]. \end{aligned} \tag{9}$$

By Step 2, we know that

$$\begin{aligned} & \left(\beta_{c_2}\lambda_{c_2} - \beta_{c_1}\lambda_{c_1}\right)\left[p^*(s_1 - ds) - p^*(s_2 - ds)\right] \\ & \qquad = \left(\beta_{c_2}\lambda_{c_2} - \beta_{c_1}\lambda_{c_1}\right)\left[e^{-\int_{s_1-ds}^{S}\lambda^*(s)ds}\theta_1^* - e^{-\int_{s_2-ds}^{S}\lambda^*(s)ds}\theta_2^* - W(s_1) + W(s_2)\right]. \end{aligned}$$

Since $W(\cdot)$ is strictly decreasing, $s_1 > s_2$, and $\beta_{c_2}\lambda_{c_2} - \beta_{c_1}\lambda_{c_1} < 0$, the previous equality implies

$$\begin{aligned} & \left(\beta_{c_2}\lambda_{c_2} - \beta_{c_1}\lambda_{c_1}\right)\left[p^*(s_1 - ds) - p^*(s_2 - ds)\right] \\ & \qquad < \left(\beta_{c_2}\lambda_{c_2} - \beta_{c_1}\lambda_{c_1}\right)\left[e^{-\int_{s_1-ds}^{S}\lambda^*(s)ds}\theta_1^* - e^{-\int_{s_2-ds}^{S}\lambda^*(s)ds}\theta_2^*\right]. \end{aligned}$$

Using Step 1, we can rearrange the previous expression as

$$\begin{aligned} & \left(\beta_{c_2}\lambda_{c_2} - \beta_{c_1}\lambda_{c_1}\right)\left[p^*(s_1 - ds) - p^*(s_2 - ds)\right] \\ < \quad & \left(\beta_{c_2}\lambda_{c_2} - \beta_{c_1}\lambda_{c_1}\right)\theta_2^* e^{-\int_{s_2-ds}^{S}\lambda^*(s)ds}\left[e^{\int_{s_2-ds}^{s_1-ds}\beta^*(s)\lambda^*(s)ds} - 1\right]. \end{aligned} \tag{10}$$

By Steps 1 and 2, we also know that

$$\theta_2^* p_\theta(s_2, \theta_2^*) - \theta_1^* p_\theta(s_1, \theta_1^*) = \theta_2^* e^{-\int_{s_2}^{S}\lambda^*(s)ds}\left(1 - e^{\int_{s_2}^{s_1}\beta^*(s)\lambda^*(s)ds}\right). \tag{11}$$

Combining inequalities (9) and (10) with equation (11) and using the fact that ds is infinitesimal, we obtain after simplifications

$$\left(1-\beta_{c_1}\right)\lambda_{c_1}-\left(1-\beta_{c_2}\right)\lambda_{c_2}<\left(\beta_{c_2}\lambda_{c_2}-\beta_{c_1}\lambda_{c_1}\right),$$

which contradicts $\lambda_{c_1}\geq\lambda_{c_2}$. The final part of the proof is identical to the argument in the proof of Proposition 1 in our main paper and omitted. ■

4 General Production Functions

4.1 Assumptions

Compared to the model described in Section 2 of our main paper, we assume that, in any country c and at any stage s, the sequential production process corresponds to the limit, when δs goes to zero, of the following CES production function:

$$q(s+\delta s)=e^{-\lambda_c(s)\delta s}\left\{(1-\delta s)q(s)^{\frac{\sigma-1}{\sigma}}+\delta s\left[l(s)/\delta s\right]^{\frac{\sigma-1}{\sigma}}\right\}^{\frac{\sigma}{\sigma-1}}, \tag{12}$$

where $\sigma\geq 0$ denotes the elasticity of substitution at all stages of production and $\lambda_c(s)\in\mathbb{R}$ is a measure of country c's total factor productivity at stage s. In this environment, the first-order conditions associated with profit maximization of a firm in country c are given by

$$\begin{aligned} p(s+\delta s)e^{-\lambda_c(s)\delta s}\left\{(1-\delta s)q(s)^{\frac{\sigma-1}{\sigma}}+\delta s\left[l(s)/\delta s\right]^{\frac{\sigma-1}{\sigma}}\right\}^{\frac{1}{\sigma-1}}(1-\delta s)q(s)^{-\frac{1}{\sigma}} &\leq p(s),\\ p(s+\delta s)e^{-\lambda_c(s)\Delta s}\left\{(1-\delta s)l(s)^{\frac{\sigma-1}{\sigma}}+\delta s\left[l(s)/\delta s\right]^{\frac{\sigma-1}{\sigma}}\right\}\left(\frac{l(s)}{\delta s}\right)^{-\frac{1}{\sigma}} &\leq w_c, \end{aligned}$$

where the two previous inequalities hold with equality if $q(s)$, $l(s)>0$. In this situation, optimal labor demand at stage s is given by

$$l(s)=q(s)\,\delta s\left(\frac{p(s)}{w_c(1-\delta s)}\right)^{\sigma}. \tag{13}$$

Combining equations (12) and (13), we obtain

$$q(s+\delta s)=e^{-\lambda_c(s)\delta s}q(s)\left(1-\delta s+\delta s\left(\frac{w_c(1-\delta s)}{p(s)}\right)^{1-\sigma}\right)^{\frac{\sigma}{\sigma-1}}.$$

Using a first-order Taylor approximation when δs goes to ds infinitesimal, this can be rearranged as

$$q(s+ds)=\left\{1-\left[\lambda_c(s)-\frac{\sigma}{1-\sigma}\left(1-\left(w_c/p(s)\right)^{1-\sigma}\right)\right]ds\right\}q(s), \tag{14}$$

as argued in Section 6.4 of our main paper.

4.2 Pattern of International Specialization

Before deriving the pattern of international specialization in this more general environment, note that the zero-profit condition in country c requires

$$p(s+ds) \leq \frac{1}{Q_c(s+ds)}\left[Q_c\left(s\right)p(s) + w_c L_c(s)\right],$$

with equality if $Q_c\left(s'\right) > 0$ for all $s' \in (s, s+ds]$. Combining the previous condition with equations (13) and (14) and using again a first-order Taylor approximation when δs goes to ds infinitesimal, we get

$$p(s+ds) \leq \left\{1 + \left[\lambda_c\left(s\right) - \frac{\sigma}{1-\sigma}\left(1 - \left(\frac{w_c}{p(s)}\right)^{1-\sigma}\right)\right] ds\right\} p(s) + w_c ds \left(\frac{p(s)}{w_c}\right)^{\sigma}, \tag{15}$$

with equality if $Q_c\left(s'\right) > 0$ for all $s' \in (s, s+ds]$. Building on this new zero-profit condition, we now provide sufficient conditions under which the pattern of international specialization can still be described as in Proposition 1 of our main paper.

Proposition 1 [General Production Function] *Suppose that $\sigma < 1$ and that $\lambda_c(s)$ is strictly decreasing in c, differentiable in s with either $\lambda_c'\left(s\right) > 0$ or $\lambda_c'\left(s\right) = 0$ for all s, and weakly submodular in (s,c). Then in any free trade equilibrium, there exists a sequence of stages $S_0 \equiv 0 < S_1 < ... < S_C = S$ such that for all $s \in \mathcal{S}$ and $c \in \mathcal{C}$, $Q_c\left(s\right) > 0$ if and only if $s \in (S_{c-1}, S_c]$.*

Proof. Recall that if a firm in country c produces intermediate good s, then it necessarily produces a measure $\Delta > 0$ of intermediate goods around that stage. Specifically, there exists an $s_\Delta < s \leq s_\Delta + \Delta$ such that $Q_c\left(s'\right) > 0$ for all $s' \in (s_\Delta, s_\Delta + \Delta]$. Throughout this proof we use the same notation as in the proof of Proposition 1 of our main paper and define $\Delta\left(s\right) \equiv (s_\Delta, s_\Delta + \Delta)$, for some s_Δ satisfying the previous conditions. The local properties that follow do not depend on which exact s_Δ we choose. We proceed in four steps.

Step 1: *$p\left(\cdot\right)$ is continuous.*

Consider a stage $s_0 \in (0, S]$. Good market clearing conditions require at least one country, call it c_0, to produce intermediate good s_0. By assumption, $Q_{c_0}\left(s\right) > 0$ for all $s \in \Delta\left(s_0\right)$. By condition (15), we therefore have, for all $s \in \Delta\left(s_0\right)$,

$$p\left(s\right) = \left\{1 + \left[\lambda_{c_0}\left(s - ds\right) - \frac{\sigma}{1-\sigma}\left(1 - \left(\frac{w_{c_0}}{p(s-ds)}\right)^{1-\sigma}\right)\right] ds\right\} p(s-ds) + w_{c_0} ds \left(\frac{p(s-ds)}{w_c}\right)^{\sigma},$$

which implies

$$\frac{dp(s)}{ds} = \left\{\lambda_{c_0}\left(s\right) - \frac{\sigma}{1-\sigma} + \frac{1}{1-\sigma}\left(\frac{w_{c_0}}{p(s)}\right)^{1-\sigma}\right\} p(s). \tag{16}$$

Thus $p\left(\cdot\right)$ is piecewise differentiable over $(0, S]$, and in turn, continuous almost everywhere. To conclude let us show that p cannot have any jumps. We proceed by contradiction. Suppose that there exists $s_0 \in (0, S)$ such that $p\left(s_0^+\right) \neq p\left(s_0^-\right)$. Then there must exist $c_0 \neq c_1$ such that firms in country c_0

produce intermediate good s_0 and sell it to firms in country c_1. If $p\left(s_0^+\right) > p\left(s_0^-\right)$, then

$$p\left(s_0^+\right) > \left\{1 + \left[\lambda_{c_0}\left(s_0 - ds\right) - \frac{\sigma}{1-\sigma}\left(1 - \left(\frac{w_{c_0}}{p(s_0 - ds)}\right)^{1-\sigma}\right)\right] ds\right\} p(s_0 - ds) + w_{c_0} ds \left(\frac{p(s_0 - ds)}{w_c}\right)^{\sigma},$$

which violates condition (15). If instead $p\left(s_0^+\right) < p\left(s_0^-\right)$, then

$$p\left(s_0 + ds\right) > \left\{1 + \left[\lambda_{c_1}\left(s_0^-\right) - \frac{\sigma}{1-\sigma}\left(1 - \left(\frac{w_{c_1}}{p(s_0^-)}\right)^{1-\sigma}\right)\right] ds\right\} p(s_0^-) + w_{c_1} ds \left(\frac{p(s_0^-)}{w_c}\right)^{\sigma},$$

which again violates condition (15).

Step 2: *If $c_2 > c_1$, then $w_{c_2} > w_{c_1}$.*

In a free trade equilibrium, factor market clearing conditions require country c_1 to produce at least one intermediate good in $(0, S)$, call it s_1. By assumption, this requires $Q_{c_1}(s) > 0$ for all $s \in \Delta(s_1)$. Thus condition (15) implies

$$\begin{aligned} p\left(s_1\right) &= p(s_1 - ds) + \left(\lambda_{c_1}\left(s_1\right) - \frac{\sigma}{1-\sigma}\right) p(s_1 - ds)ds + \frac{1}{1-\sigma}\left(\frac{w_{c_1}}{p(s_1 - ds)}\right)^{1-\sigma} p(s_1 - ds)ds, \\ p\left(s_1\right) &\leq p(s_1 - ds) + \left(\lambda_{c_2}\left(s_1\right) - \frac{\sigma}{1-\sigma}\right) p(s_1 - ds)ds + \frac{1}{1-\sigma}\left(\frac{w_{c_2}}{p(s_1 - ds)}\right)^{\sigma-1} p(s_1 - ds)ds, \end{aligned}$$

Since $\lambda_{c_2}\left(s_1\right) < \lambda_{c_1}\left(s_1\right)$, these equation imply $w_{c_2} > w_{c_1}$.

Step 3: *If $s_2 > s_1$, then $p(s_2) > p(s_1)$.*

We first show that if for any $s_0 \in (0, S)$, firms in country c_0 produce all stages $s \in (s_0 - \Delta, s_0]$ and firms in country c_1 produce all stages $s \in (s_0, s_0 + \Delta)$, then $dp(s_0^-)/ds = dp(s_0^+)/ds$. Since $p\left(\cdot\right)$ is continuous, we know from equation (16) that

$$\begin{aligned} \frac{dp(s_0^-)}{ds} &= \left\{\lambda_{c_0}\left(s_0\right) - \frac{\sigma}{1-\sigma} + \frac{1}{1-\sigma}\left(\frac{w_{c_0}}{p(s_0)}\right)^{1-\sigma}\right\} p(s_0), \\ \frac{dp(s_0^+)}{ds} &= \left\{\lambda_{c_1}\left(s_0\right) - \frac{\sigma}{1-\sigma} + \frac{1}{1-\sigma}\left(\frac{w_{c_1}}{p(s_0)}\right)^{1-\sigma}\right\} p(s_0). \end{aligned}$$

Thus, we need to show that that

$$\begin{aligned} & p(s_0)\left\{\lambda_{c_0}\left(s_0\right) - \frac{\sigma}{1-\sigma} + \frac{1}{1-\sigma}\left(\frac{w_{c_0}}{p(s_0)}\right)^{1-\sigma}\right\} \\ = & \left\{\lambda_{c_1}\left(s_0\right) - \frac{\sigma}{1-\sigma} + \frac{1}{1-\sigma}\left(\frac{w_{c_1}}{p(s_0)}\right)^{1-\sigma}\right\} p(s_0). \end{aligned} \tag{17}$$

From condition (15), we know that

$$p(s_0+ds) - \left\{1 + \left[\lambda_{c_1}(s_0) - \frac{\sigma}{1-\sigma} + \frac{1}{1-\sigma}\left(\frac{w_{c_1}}{p(s_0)}\right)^{1-\sigma}\right] ds\right\} p(s_0)$$
$$\geq p(s_0+ds) - \left\{1 + \left[\lambda_{c_0}(s_0) - \frac{\sigma}{1-\sigma} + \frac{1}{1-\sigma}\left(\frac{w_{c_0}}{p(s_0)}\right)^{1-\sigma}\right] ds\right\} p(s_0).$$

and

$$p(s_0) - \left\{1 + \left[\lambda_{c_0}(s_0-ds) - \frac{\sigma}{1-\sigma} + \frac{1}{1-\sigma}\left(\frac{w_{c_0}}{p(s_0-ds)}\right)^{1-\sigma}\right] ds\right\} p(s_0-ds)$$
$$\geq p(s_0) - \left\{1 + \left[\lambda_{c_1}(s_0-ds) - \frac{\sigma}{1-\sigma} + \frac{1}{1-\sigma}\left(\frac{w_{c_1}}{p(s_0-ds)}\right)^{1-\sigma}\right] ds\right\} p(s_0-ds).$$

After simplifications, the two previous inequalities can be rearranged as

$$p(s_0)\left[\lambda_{c_0}(s_0) - \frac{\sigma}{1-\sigma} + \frac{1}{1-\sigma}\left(\frac{w_{c_0}}{p(s_0)}\right)^{1-\sigma} - \lambda_{c_1}(s_0) + \frac{\sigma}{1-\sigma} - \frac{1}{1-\sigma}\left(\frac{w_{c_1}}{p(s_0)}\right)^{1-\sigma}\right]$$
$$\geq 0 \geq p(s_0-ds)\left[\lambda_{c_0}(s_0-ds) - \frac{\sigma}{1-\sigma} + \frac{1}{1-\sigma}\left(\frac{w_{c_0}}{p(s_0-ds)}\right)^{1-\sigma}\right.$$
$$\left. -\lambda_{c_1}(s_0-ds) + \frac{\sigma}{1-\sigma} - \frac{1}{1-\sigma}\left(\frac{w_{c_1}}{p(s_0-ds)}\right)^{1-\sigma}\right].$$

Since $p(\cdot)$, $\lambda_{c_0}(\cdot)$ and $\lambda_{c_1}(\cdot)$ are continuous, and since ds is infinitesimal, the above chain of inequalities yields equation (17).

Let us consider separately the two cases: (*i*) $\lambda_c'(s) = 0$ for all s and (*ii*) $\lambda_c'(s) > 0$ for all s. We start with case (*i*). In this case we can write $\lambda_{c_0}(s) \equiv \lambda_{c_0}$. Let us show that if country c_0 produces all stages $s \in (s_0, s_0+\Delta]$ for any $s_0 \in [0, S-\Delta]$ and $dp(s_0^+)/ds > 0$, then for $s \in (s_0, s_0+\Delta)$ we have $dp(s)/ds > 0$ and $dp((s_0+\Delta)^-)/ds > 0$. Let $x_{c_0}(s) \equiv p(s)/w_{c_0}$. Since country c_0 produces all stages $s \in (s_0, s_0+\Delta]$, we know that $p(\cdot)$ is a solution of equation (16) over that interval. Thus $x_{c_0}(\cdot)$ is continuously differentiable over $(s_0, s_0+\Delta)$ and its derivative satisfies:

$$\frac{dx_{c_0}(s)}{ds} = \left(\lambda_{c_0} - \frac{\sigma}{1-\sigma}\right) x_{c_0}(s) + \frac{1}{1-\sigma} x_{c_0}^{\sigma}(s). \tag{18}$$

By equation (18), $dx_c(s)/ds = 0$ implies that either $x_c(s) = 0$ or $x_c(s) = (\sigma + \lambda_c(\sigma-1))^{\frac{1}{\sigma-1}}$ (if the latter is well-defined). Yet the constant functions $x_c(s) = 0$ or $x_c(s) = (\sigma + \lambda_{c_0}(\sigma-1))^{\frac{1}{\sigma-1}}$ (if the latter is well-defined) are solutions to the differential equation (18). Therefore, by the Cauchy-Lipschitz Theorem, if a solution to this differential equation, $x_c(\cdot)$, is such that $dx_c(s)/ds$ is zero for some $s \in (s_0, s_0+\Delta)$, then $x_c(\cdot)$ must be constant over $(s_0, s_0+\Delta)$. This in turn implies that $dx_{c_0}(s)/ds$ for $s \in (s_0, s_0+\Delta)$ and $dx_{c_0}((s_0+\Delta)^-)/ds$ all have the same sign as $dx_{c_0}(s_0^+)/ds$. Since $dx_{c_0}(s)/ds = (dp(s)/ds)/w_{c_0}$, the same property holds for dp/ds.

Let us turn to case (*ii*). When $\lambda_c'(s) > 0$ for all s, we show here that if country c_0 produces all stages $s \in (s_0, s_0 + \Delta]$ for any $s_0 \in [0, S - \Delta]$ and $dp(s_0^+)/ds \geq 0$, then $dp((s_0 + \Delta)^-)/ds \geq 0$ and either $dp(s)/ds = 0$ for all $s \in (s_0, s_0 + \Delta)$ or $dp(s)/ds \geq 0$ for all $s \in (s_0, s_0 + \Delta)$ with $dp(s)/ds > 0$ for almost all $s \in (s_0, s_0 + \Delta)$. Again, let $x_{c_0}(s) \equiv p(s)/w_{c_0}$. Since country c_0 produces all stages $s \in (s_0, s_0 + \Delta]$, we know that $p(\cdot)$ is a solution of equation (16) over that interval. Thus $x_{c_0}(\cdot)$ is continuously differentiable over $(s_0, s_0 + \Delta)$ and its derivative satisfies (18). By equation (18), $dx_{c_0}(s)/ds = 0$ implies that either $x_{c_0}(s) = 0$ or $x_{c_0}(s) = [\sigma + \lambda_{c_0}(s)(\sigma - 1)]^{\frac{1}{\sigma-1}}$ (if the latter is well-defined). If $x_{c_0}(s) = 0$, then the same argument as in case (*i*) implies that $x_{c_0}(\cdot)$ must be constant over $(s_0, s_0 + \Delta)$. Since $dx_{c_0}(s)/ds = (dp(s)/ds)/w_{c_0}$, this implies $dp((s_0 + \Delta)^-)/ds \geq 0$ and $dp(s)/ds = 0$ for all $s \in (s_0, s_0 + \Delta)$. If instead $x_{c_0}(s) = [\sigma + \lambda_{c_0}(s)(\sigma - 1)]^{\frac{1}{\sigma-1}} > 0$, then differentiating equation (18), we get:

$$\frac{d^2 x_{c_0}(s)}{ds^2} = \lambda_{c_0}'(s) x_{c_0}(s) + \left[\lambda_{c_0}(s) - \frac{\sigma}{1-\sigma}\right] \frac{dx_{c_0}(s)}{ds} + \frac{\sigma}{1-\sigma} x_{c_0}^{\sigma-1}(s) \frac{dx_{c_0}(s)}{ds}.$$

Since $\lambda_{c_0}'(s) > 0$, $dx_{c_0}(s)/ds = 0$ and $x_{c_0}(s) > 0$ imply $d^2 x_{c_0}(s)/ds^2 > 0$. Therefore, dx_{c_0}/ds can never go from being positive to being negative: if $dx(s_0^+)/ds \geq 0$, then $dx((s_0+\Delta)^-)/ds \geq 0$, $dx(s)/ds \geq 0$ for all $s \in (s_0, s_0+\Delta)$, and $dx(s)/ds > 0$ for almost all $s \in (s_0, s_0+\Delta)$. Since $dx_{c_0}(s)/ds = (dp(s)/ds)/w_{c_0}$, the same property holds for dp/ds, which establishes our claim.

To conclude, note that Step 2 and the assumption that it takes ε workers in any country to produce good 0 imply $p(0) = \varepsilon w_1$. When ε is sufficiently close to 0, equation (18) implies that $dp(0^+)/ds > 0$, as the term in $x_{c_0}^{\sigma}(s)$ dominates when $\sigma < 1$. At this point, we have established that dp/ds is continuous and never changes sign. Moreover, since $dp(0^+)/ds > 0$, $dp(s)/ds$ can only be zero on a set of measure zero. Therefore, p is strictly increasing over $(0, S]$.

Step 4: *If* $c_2 > c_1$ *and* $Q_{c_1}(s_1) > 0$*, then* $Q_{c_2}(s) = 0$ *for all* $s < s_1$*.*

We proceed by contradiction. Suppose that there exist two countries, $c_2 > c_1$, and two intermediate goods, $s_1 > s_2 > 0$, such that c_1 produces s_1 and c_2 produces s_2. By assumption, c_1 produces all intermediate goods $s \in \Delta(s_1)$, whereas c_2 produces all intermediate goods $s \in \Delta(s_2)$. Thus condition (15) implies

$$\begin{aligned}
p(s_1) &= p(s_1 - ds) + \left(\lambda_{c_1}(s_1) - \frac{\sigma}{1-\sigma}\right) p(s_1 - ds) ds + \frac{1}{1-\sigma} \left(\frac{w_{c_1}}{p(s_1 - ds)}\right)^{1-\sigma} p(s_1 - ds) ds, \\
p(s_2) &= p(s_2 - ds) + \left(\lambda_{c_2}(s_2) - \frac{\sigma}{1-\sigma}\right) p(s_2 - ds) ds + \frac{1}{1-\sigma} \left(\frac{w_{c_2}}{p(s_2 - ds)}\right)^{1-\sigma} p(s_2 - ds) ds, \\
p(s_1) &\leq p(s_1 - ds) + \left(\lambda_{c_2}(s_1) - \frac{\sigma}{1-\sigma}\right) p(s_1 - ds) ds + \frac{1}{1-\sigma} \left(\frac{w_{c_2}}{p(s_1 - ds)}\right)^{1-\sigma} p(s_1 - ds) ds, \\
p(s_2) &\leq p(s_2 - ds) + \left(\lambda_{c_1}(s_2) - \frac{\sigma}{1-\sigma}\right) p(s_2 - ds) ds + \frac{1}{1-\sigma} \left(\frac{w_{c_1}}{p(s_2 - ds)}\right)^{1-\sigma} p(s_2 - ds) ds.
\end{aligned}$$

Combining the four previous expressions and rearranging, we get

$$\lambda_{c_2}(s_2) + \lambda_{c_1}(s_1) - \lambda_{c_1}(s_2) - \lambda_{c_2}(s_1) \leq \frac{\left(w_{c_2}^{1-\sigma} - w_{c_1}^{1-\sigma}\right)\left[p^{\sigma-1}(s_1 - ds) - p^{\sigma-1}(s_2 - ds)\right]}{1-\sigma}.$$

Since λ is weakly submodular and $\sigma < 1$, this implies

$$\left(w_{c_2}^{1-\sigma} - w_{c_1}^{1-\sigma}\right)\left[p^{\sigma-1}(s_1 - ds) - p^{\sigma-1}(s_2 - ds)\right] \geq 0. \tag{19}$$

From Step 2 and Step 3, we know that $p(s_1 - ds) > p(s_2 - ds)$ and $w_{c_2} > w_{c_1}$. Since $1 - \sigma > 0$, this implies $w_{c_2}^{1-\sigma} - w_{c_1}^{1-\sigma} > 0$ and $p^{\sigma-1}(s_1 - ds) - p^{\sigma-1}(s_2 - ds) < 0$, which contradicts Inequality (19). The final part of the proof is identical to the argument in the proof of Proposition 1 in our main paper and omitted. ■