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Appendix B from Costinot et al., "A Theory of Capital Controls as Dynamic Terms-of-Trade Manipulation" (JPE, vol. 122, no. 1, p. 77)

In continuous time the planning problem of the home government described in Section II.B can be expressed as

$$\max_{c(\cdot)} \int e^{-\rho t} u(c(t)) dt \tag{P$_C$}$$

subject to

$$\int e^{-\rho t} u^{*\prime}(Y - c(t))[y(t) - c(t)] dt = 0.$$

The objective of this appendix is to show that if time is continuous, then proposition 1 generalizes to economies in which $u^{*\prime}(Y - c)(c - y)$ is not a strictly convex function of c. The only assumptions required are those imposed in Section II.A.

Assumption 1. The functions u and u^* are strictly increasing, strictly concave, and twice continuously differentiable with the boundary conditions $\lim_{c\to 0} u'(c) = \lim_{c^*\to 0} u^{*\prime}(c^*) = \infty\, t$.

Assumption 2. The functions $y(t)$ and $y^*(t)$ are bounded away from zero for all t.

Throughout this appendix, for any $\mu > 0$ and any date t, we let

$$\mathcal{C}(t, \mu) \equiv \arg\max_{c \in (0,Y)} u(c) + \mu u^{*\prime}(Y - c)[y(t) - c].$$

To derive proposition 1 in this environment, we first establish four lemmas.

Lemma 1. Suppose that assumptions 1 and 2 hold. Then for any $\mu > 0$ and any date t, $\mathcal{C}(t, \mu) \neq \varnothing$.

Proof. Fix $\mu > 0$ and $t \geq 0$. By assumption 1, we know that $\lim_{c\to 0} u'(c) = \infty$. Thus there must be $m \in (0, Y)$ such that, for all $c \in (0, m)$,

$$u(c) + \mu u^{*\prime}(Y - c)[y(t) - c] < u(m) + \mu u^{*\prime}(Y - m)[y(t) - m]. \tag{B1}$$

By assumption 2, we know that foreign endowments are bounded away from zero. Thus domestic endowments are bounded away from Y. By assumption 1, we therefore have $\lim_{c\to Y} u^{*\prime}(Y - c)[y(t) - c] = -\infty$. Thus there must be $M \in (m, Y)$ such that, for all $c \in (M, Y)$,

$$u(c) + \mu u^{*\prime}(Y - c)[y(t) - c] < u(M) + \mu u^{*\prime}(Y - M)[y(t) - M]. \tag{B2}$$

Since $u(c) + \mu u^{*\prime}(Y - c)[y(t) - c]$ is continuous over $[m, M]$, Weierstrass's extreme value theorem implies the existence of

$$c(t) \in \arg\max_{c \in [m,M]} u(c) + \mu u^{*\prime}(Y - c)[y(t) - c].$$

By inequalities (B1) and (B2), we also have $c(t) \in \mathcal{C}(t, \mu)$. QED

Lemma 2. Suppose that assumptions 1 and 2 hold. Then for any $\mu > 0$ and any pair of dates t and s, if $y(t) > y(s)$, then $c(t) > c(s)$ for all $c(t) \in \mathcal{C}(t, \mu)$ and $c(s) \in \mathcal{C}(s, \mu)$. Similarly, for any date t, if $\mu > \mu'$, then $c(t) < c'(t)$ for all $c(t) \in \mathcal{C}(t, \mu)$ and $c'(t) \in \mathcal{C}(t, \mu')$.

Proof. Fix $\mu > 0$ and consider a pair of dates t and s such that $y(t) > y(s)$. By definition, if $c(t) \in \mathcal{C}(t, \mu)$ and $c(s) \in \mathcal{C}(s, \mu)$, then

$$u(c(t)) + \mu u^{*\prime}(Y - c(t))[y(t) - c(t)] \geq u(c(s)) + \mu u^{*\prime}(Y - c(s))[y(t) - c(s)],$$
$$u(c(s)) + \mu u^{*\prime}(Y - c(s))[y(s) - c(s)] \geq u(c(t)) + \mu u^{*\prime}(Y - c(t))[y(s) - c(t)].$$

Adding up the two previous inequalities, we obtain after simplification

$$[u^{*\prime}(Y - c(t)) - u^{*\prime}(Y - c(s))][y(t) - y(s)] \geq 0.$$

This implies $u^{*\prime}(Y - c(t)) \geq u^{*\prime}(Y - c(s))$. By assumption 1, u^* is strictly concave. Thus we must have $c(t) \geq c(s)$. To conclude, let us show that we cannot have $c(t) = c(s)$. We proceed by contradiction. If $c(s) \in \mathcal{C}(t, \mu) \cap \mathcal{C}(s, \mu)$, then the following first-order conditions must be satisfied:

$$u'(c(s)) - \mu u^{*\prime}(Y - c(s)) - \mu u^{*\prime\prime}(Y - c(s))[y(s) - c(s)] = 0,$$
$$u'(c(s)) - \mu u^{*\prime}(Y - c(s)) - \mu u^{*\prime\prime}(Y - c(s))[y(t) - c(s)] = 0.$$

This implies $u^{*\prime\prime}(Y - c(s))[y(t) - y(s)] = 0$, which contradicts $y(t) \neq y(s)$. This completes the first part of lemma 2. The second part of lemma 2 can be established in a similar fashion and is omitted. QED

Lemma 3. Suppose that assumptions 1 and 2 hold. Then there exists $\mu > 0$ such that

$$\int e^{-\rho t} u^{*\prime}(Y - c(t))[y(t) - c(t)]dt = 0, \tag{B3}$$

with $c(t) \in \mathcal{C}(t, \mu)$ for all t.

Proof. We proceed in four steps.

Step 1: There exist $\underline{\mu} > 0$ and $\bar{\mu} > \underline{\mu}$ such that

$$\int e^{-\rho t} u^{*\prime}(Y - c(t, \underline{\mu}))[y(t) - c(t, \underline{\mu})]dt < 0, \tag{B4}$$

$$\int e^{-\rho t} u^{*\prime}(Y - c(t, \bar{\mu}))[y(t) - c(t, \bar{\mu})] > 0, \tag{B5}$$

with $c(t, \underline{\mu}) \in \mathcal{C}(t, \underline{\mu})$ and $c(t, \bar{\mu}) \in \mathcal{C}(t, \bar{\mu})$ for all t.

For any t, let us define $\mu(t) \equiv u'(y(t))/u^{*\prime}(Y - y(t))$. We first check that $y(t) \in \mathcal{C}(t, \mu(t))$. Since u is concave, we know that

$$u(c) \leq u(y(t)) + u'(y(t))[c - y(t)]$$

for all c. Since u^* is concave, we also know that

$$\frac{u^{*\prime}(Y - c)}{u^{*\prime}(Y - y(t))} \geq 1$$

if and only if $c \geq y(t)$. The two previous observations imply

$$u(c) \leq u(y(t)) + \frac{u'(y(t))u^{*\prime}(Y-c)}{u^{*\prime}(Y-y(t))}[c-y(t)].$$

Using the definition of $\mu(t)$, this can be rearranged as

$$u(y(t)) \geq u(c) + \mu(t)u^{*\prime}(Y-c)[y(t)-c],$$

which implies $y(t) \in \mathcal{C}(t,\mu(t))$. Now let us define $\underline{\mu} \equiv u'(\bar{y})/u^{*\prime}(Y-\bar{y})$ and $\bar{\mu} \equiv u'(\underline{y})/u^{*\prime}(Y-\underline{y})$ with $\underline{y} \equiv \inf_{t\geq 0} y(t) > 0$ and $\bar{y} \equiv \sup_{t\geq 0} y(t) < Y$. Since u and u^* are strictly concave, we have $\mu(t) \in (\underline{\mu},\bar{\mu})$ for all t. By lemma 2, $y(t) \in \mathcal{C}(t,\mu(t))$ implies that $c(t) > y(t)$ for all $c(t) \in \mathcal{C}(t,\underline{\mu})$ and $c(t) < y(t)$ for all $c(t) \in \mathcal{C}(t,\bar{\mu})$. Since the previous inequalities hold for all t, we have found $\underline{\mu}$ and $\bar{\mu}$ such that

$$\int e^{-\rho t}u^{*\prime}(Y-c(t,\underline{\mu}))[y(t)-c(t,\underline{\mu})]dt < 0,$$
$$\int e^{-\rho t}u^{*\prime}(Y-c(t,\bar{\mu}))[y(t)-c(t,\bar{\mu})] > 0,$$

with $c(t,\underline{\mu}) \in \mathcal{C}(t,\underline{\mu})$ and $c(t,\bar{\mu}) \in \mathcal{C}(t,\bar{\mu})$ for all t.

Step 2: For any $\mu \in [\underline{\mu},\bar{\mu}]$ and any t, there exist $c^+(t,\mu)$ and $c^-(t,\mu)$ such that

$$c^+(t,\mu) \in \arg\max_{c\in\mathcal{C}(t,\mu)} u^{*\prime}(Y-c)[y(t)-c], \tag{B6}$$

$$c^-(t,\mu) \in \arg\min_{c\in\mathcal{C}(t,\mu)} u^{*\prime}(Y-c)[y(t)-c]. \tag{B7}$$

Take $\underline{c}$ and $\bar{c}$ such that

$$\underline{c} \in \arg\max_{c\in(0,Y)} u(c) + \bar{\mu}u^{*\prime}(Y-c)(\underline{y}-c),$$
$$\bar{c} \in \arg\max_{c\in(0,Y)} u(c) + \underline{\mu}u^{*\prime}(Y-c)(\bar{y}-c).$$

By lemma 1, we know that such $\underline{c}$ and $\bar{c}$ exist. By lemma 2, for any $\mu \in [\underline{\mu},\bar{\mu}]$ and any t, we also must have

$$\mathcal{C}(t,\mu) = \arg\max_{c\in[\underline{c},\bar{c}]} u(c) + \mu u^{*\prime}(Y-c)[y(t)-c].$$

Since $u(c) + \mu u^{*\prime}(Y-c)(y_t-c)$ is continuous in (c,μ), the maximum theorem implies that $\mathcal{C}(t,\mu)$ is compact and, for future reference, upper hemicontinuous in μ. Since $u^{*\prime}(Y-c)[y(t)-c]$ is continuous in c, Weierstrass's extreme value theorem implies the existence of $c^+(t,\mu)$ and $c^-(t,\mu)$ satisfying (B6) and (B7), respectively.

Step 3: There exists $\mu_0 \in [\underline{\mu},\bar{\mu}]$ such that

$$\int e^{-\rho t}u^{*\prime}(Y-c^+(t,\mu_0))[y(t)-c^+(t,\mu_0)]dt > 0, \tag{B8}$$

$$\int e^{-\rho t} u^{*\prime}(Y - c^{-}(t, \mu_0))[y(t) - c^{-}(t, \mu_0)]dt < 0. \quad \text{(B9)}$$

By construction of $c^{+}(\cdot, \mu)$ and $c^{-}(\cdot, \mu)$, for any μ and $c(\cdot, \mu)$ such that $c(t, \mu) \in \mathcal{C}(t, \mu)$ for all t, we have

$$\begin{aligned} &\int e^{-\rho t} u^{*\prime}(Y - c^{+}(t, \mu))[y(t) - c^{+}(t, \mu)]dt \\ &\geq \int e^{-\rho t} u^{*\prime}(Y - c(t, \mu))[y(t) - c(t, \mu)]dt \\ &\geq \int e^{-\rho t} u^{*\prime}(Y - c^{-}(t, \mu))[y(t) - c^{-}(t, \mu)]dt. \end{aligned}$$

Thus inequalities (B4) and (B5) in step 1 imply

$$\int e^{-\rho t} u^{*\prime}(Y - c^{+}(t, \underline{\mu}))[y(t) - c^{+}(t, \underline{\mu})]dt > 0, \quad \text{(B10)}$$

$$\int e^{-\rho t} u^{*\prime}(Y - c^{-}(t, \bar{\mu}))[y(t) - c^{-}(t, \bar{\mu})]dt < 0. \quad \text{(B11)}$$

To show that there exists $\mu_0 \in [\underline{\mu}, \bar{\mu}]$ such that inequalities (B8) and (B9) are satisfied, we proceed by contradiction. Suppose that there does not exist $\mu_0 \in [\underline{\mu}, \bar{\mu}]$ such that the two previous inequalities are satisfied. Then there must exist $\mu_1 \in [\underline{\mu}, \bar{\mu}]$ and $\varepsilon_1 > 0$ such that, for any $\eta > 0$, there exists μ such that $|\mu_1 - \mu| < \eta$ and

$$\begin{aligned} &\int e^{-\rho t} u^{*\prime}(Y - c^{-}(t, \mu_1))[y(t) - c^{-}(t, \mu_1)]dt \\ &- \int e^{-\rho t} u^{*\prime}(Y - c^{+}(t, \mu))[y(t) - c^{+}(t, \mu)]dt > \varepsilon_1. \end{aligned}$$

In step 2, we have already argued that $\mathcal{C}(t, \mu)$ is compact-valued and upper hemicontinuous in μ. So there must be $c(t, \mu) \in \mathcal{C}(t, \mu)$ for all μ and t such that

$$\lim_{\mu \to \mu_1} c(t, \mu) = c(t, \mu_1) \in \mathcal{C}(t, \mu_1). \quad \text{(B12)}$$

For all t, $u^{*\prime}(Y - c)[y(t) - c]$ is continuous in c and uniformly bounded by $\max_{c \in [\underline{c}, \bar{c}]} u^{*\prime}(Y - c)(\bar{y} - c)$. Thus the limit condition (B12) implies

$$\lim_{\mu \to \mu_1} \int e^{-\rho t} u^{*\prime}(Y - c(t, \mu))[y(t) - c(t, \mu)]dt = \int e^{-\rho t} u^{*\prime}(Y - c(t, \mu_1))[y(t) - c(t, \mu_1)]dt.$$

Accordingly, there must be $\varepsilon \in (0, \varepsilon_1)$ and $\eta_1 > 0$ such that if $|\mu_1 - \mu| < \eta_1$, then

$$\int e^{-\rho t} u^{*\prime}(Y - c(t, \mu_1))[y(t) - c(t, \mu_1)]dt - \int e^{-\rho t} u^{*\prime}(Y - c(t, \mu))[y(t) - c(t, \mu)]dt < \varepsilon.$$

By construction of $c^+(\cdot, \mu)$ and $c^-(\cdot, \mu)$, we know that

$$\int e^{-\rho t} u^{*\prime}(Y - c(t, \mu_1))[y(t) - c(t, \mu_1)]dt \geq \int e^{-\rho t} u^{*\prime}(Y - c^-(t, \mu_1))[y(t) - c^-(t, \mu_1)]dt,$$
$$\int e^{-\rho t} u^{*\prime}(Y - c(t, \mu))[y(t) - c(t, \mu)]dt \leq \int e^{-\rho t} u^{*\prime}(Y - c^+(t, \mu))[y(t) - c^+(t, \mu)]dt.$$

The three previous inequalities imply the existence of $\eta_1 > 0$ such that if $|\mu_1 - \mu| < \eta_1$, then

$$\int e^{-\rho t} u^{*\prime}(Y - c^-(t, \mu_1))[y(t) - c^-(t, \mu_1)]dt - \int e^{-\rho t} u^{*\prime}(Y - c^+(t, \mu))[y(t) - c^+(t, \mu)]dt < \varepsilon_1,$$

a contradiction.

Step 4: There exists $c(\cdot, \mu_0)$ such that $c(t, \mu_0) \in \mathcal{C}(t, \mu_0)$ for all t and

$$\int e^{-\rho t} u^{*\prime}(Y - c(t, \mu_0))[y(t) - c(t, \mu_0)]dt = 0. \tag{B13}$$

Let

$$H(T) \equiv \int^T e^{-\rho t} u^{*\prime}(Y - c^-(t, \mu_0))[y(t) - c^-(t, \mu_0)]dt + \int_T e^{-\rho t} u^{*\prime}(Y - c^+(t, \mu_0))[y(t) - c^+(t, \mu_0)]dt.$$

By step 3, there must exist $\underline{T} < \overline{T}$ such that $H(\underline{T}) > 0 > H(\overline{T})$. Since H is continuous in T, the intermediate value theorem implies the existence of T_0 such that $H(T_0) = 0$. Now let us construct $c(\cdot, \mu_0)$ such that $c(t, \mu_0) \equiv c^-(t, \mu_0)$ for all $t < T_0$ and $c(t, \mu_0) \equiv c^+(t, \mu_0)$ for all $t \geq T_0$. By construction, $c(\cdot, \mu_0)$ satisfies equation (B13) with $c(t, \mu_0) \in \mathcal{C}(t, \mu_0)$. QED

Lemma 4. Suppose that there exist $\mu > 0$ and $c(\cdot, \mu)$ such that

i. $c(t, \mu) \in \mathcal{C}(t, \mu)$ for all t,
ii. $\int e^{-\rho t}\{u^{*\prime}(Y - c(t, \mu))[y(t) - c(t, \mu)]\}dt = 0$.

Then any solution $c^0(\cdot)$ of (P_C) must be such that $c^0(t) \in \mathcal{C}(t, \mu)$ for almost all t.

Proof. Suppose that $c^0(\cdot)$ is a solution of

$$\max_{c(\cdot)} \int e^{-\rho t} u(c(t))dt$$

subject to

$$\int e^{-\rho t}\{u^{*\prime}(Y - c(t))[y(t) - c(t)]\}dt = 0.$$

By condition ii, we must therefore have

$$\int e^{-\rho t}u(c^0(t))dt \geq \int e^{-\rho t}u(c(t,\mu))dt.$$

Since

$$\int e^{-\rho t}\{u^{*\prime}(Y - c^0(t))[y(t) - c^0(t)]\}dt = \int e^{-\rho t}\{u^{*\prime}(Y - c(t,\mu))[y(t) - c(t,\mu)]\}dt = 0,$$

this further implies

$$\begin{aligned}&\int e^{-\rho t}\{u(c^0(t)) + \mu u^{*\prime}(Y - c^0(t))[y(t) - c^0(t)]\}dt\\ \geq &\int e^{-\rho t}\{u(c(t,\mu)) + \mu u^{*\prime}(Y - c(t,\mu))[y(t) - c(t,\mu)]\}dt.\end{aligned}$$

By condition i, we know that

$$\begin{aligned}&\int e^{-\rho t}\{u(c(t,\mu)) + \mu u^{*\prime}(Y - c(t,\mu))[y(t) - c(t,\mu)]\}dt\\ &= \max_{c(\cdot)}\int e^{-\rho t}\{u(c(t)) + \mu u^{*\prime}(Y - c(t))[y(t) - c(t)]\}dt.\end{aligned}$$

Thus the previous inequality implies

$$c^0(\cdot) \in \arg\max \int e^{-\rho t}\{u(c(t)) + \mu u^{*\prime}(Y - c(t))[y(t) - c(t)]\}dt,$$

which requires $c^0(t) \in \mathcal{C}(t,\mu)$ for almost all t. QED

We are now ready to establish proposition 1.

PROPOSITION 1 (Procyclical consumption). Suppose that assumptions 1 and 2 hold. Then for any solution $c(\cdot)$ of (P_C) and almost all pairs of dates t and s, if $y(t) > y(s)$, then $c(t) > c(s)$.

Proof. By lemmas 1 and 3, the conditions of lemma 4 are satisfied. Thus if $c(\cdot)$ is a solution of the planning problem (P_C), we must have $c(t) \in \mathcal{C}(t,\mu)$ for almost all t. By lemma 2, we know that $y(t) > y(s)$ implies $c(t) > c(s)$ for all $c(t) \in \mathcal{C}(t,\mu)$ and $c(s) \in \mathcal{C}(s,\mu)$. Proposition 1 derives from the two previous observations. QED