ONLINE APPENDIX FOR "INTELLECTUAL PROPERTY RIGHTS POLICY, COMPETITION AND INNOVATION"

Daron Acemoglu
Massachusetts Institute of Technology

Ufuk Akcigit
University of Pennsylvania

In these appendices, we provide additional results and the proofs of all the results stated in the text. All cross-references that are not to material in these appendices refer to the main text.

Appendix A: Robustness Checks

Tables A.1-A.4 show results analogous to those shown in Table 1 as we vary λ and γ (we change λ to 1.2 or 1.01, and γ to 0.6 or 0.1). As mentioned in the text, the results are very similar to our baseline results.

Appendix B: Compulsory Versus Bargained License Fees

The analysis in the text characterized the steady-state equilibrium for a given sequence of license fees $\boldsymbol{\zeta}$, implicitly assumed to be determined by IPR policy—i.e., these fees correspond to compulsory licensing fees for intellectual property that has been patented. This corresponds to a world in which once a company patents an innovation, the knowledge embedded in this innovation can be used by its competitors as long as they pay a prespecified license fee. One may also wish to consider an alternative world in which license fees are determined by bilateral bargaining. To characterize the equilibrium in such a world, one must first conduct exactly the same analysis as we have done in this subsection. In other words, one must characterize the equilibrium for a given sequence of license fees, and then taking the license fees agreed by other firms as given, one can consider the bargaining problem between a leader and a follower. In general, there may or may not exist feasible voluntary license fees that the follower and the leader can bargain to (such voluntary agreements may be infeasible even if compulsory licensing is beneficial, since consumers also benefit from licensing).

Figure B.1 plots the value of licensing to a follower in an industry with an n-step gap, $v_0 - v_{-n+1}$, and absolute value of the loss to the leader in the same industry,

E-mail: daron@mit.edu (Acemoglu); uakcigit@econ.upenn.edu (Akcigit)

Journal of the European Economic Association
Preprint prepared on 25 September 2011 using jeea.cls *v1.0.*

TABLE A.1. Robustness check in quick catch-up with $\lambda = 1.01$.

	Full IPR Protection without licensing	Full IPR Protection with licensing	Optimal Uniform IPR with licensing	Optimal State Dependent without licensing	Optimal State Dependent with licensing
λ	1.01	1.01	1.01	1.01	1.01
γ	0.35	0.35	0.35	0.35	0.35
η_1	0	0	0	3.14	0.06
η_2	0	0	0	0.23	0
η_3	0	0	0	0	0
η_4	0	0	0	0	0
η_5	0	0	0	0	0
ζ_1	∞	0.19	0	∞	0.04
ζ_2	∞	0.19	0	∞	0.10
ζ_3	∞	0.19	0	∞	0.10
ζ_4	∞	0.19	0	∞	0.10
ζ_5	∞	0.19	0	∞	0.10
$v_1 - v_0$	0.14	0.19	0.09	0.08	0.10
x^*_{-1}	1.08	1.27	1.29	0.64	1.51
x^*_0	1.67	1.95	1.29	1.25	1.38
μ^*_0	0.25	0	0	0.45	0.01
μ^*_1	0.33	0.50	0.50	0.20	0.45
μ^*_2	0.19	0.25	0.25	0.14	0.23
ω^*	0.99	0.99	0.99	0.99	0.99
Researcher ratio	0.008	0.007	0.007	0.007	0.010
$\ln C(0)$	10.28	11.88	11.90	10.66	12.66
g^*	0.0186	0.0257	0.0258	0.0203	0.0293
Welfare	213.1	247.8	248.3	221.4	265.0

Note: This table provides robustness check results for Table 1 with $\rho = 0.05$, $\lambda = 1.01$, $\gamma = 0.35$ under five different IPR policy regimes. It reports the steady-state equilibrium values of the difference in the values $v_1 - v_0$; the (annual) R&D rate of a follower that is one step behind, x^*_{-1}; the (annual) R&D rate of neck-and-neck competitors, x^*_0; fraction of industries in neck-and-neck competition, μ^*_0; fraction of industries at a technology gap of $n = 1, 2$; the value of “labor share,” ω^*; the ratio of the labor force working in research; initial (annual) consumption, $C(0)$; the annual growth rate, g^*; and the welfare level according to equation (42). It also reports the welfare-maximizing uniform and state-dependent IPR policies with or without licensing. See text for details.

$|v_{n-1} - v_0|$ (with full protection as in column 1 of Tables 2-5).[1] The overall pattern

1. Without licensing, the change in follower's value is $v_{-n+1} - v_{-n}$. Since licensing takes the follower to v_0, the change due to licensing is $v_0 - v_{-n+1}$. Similar reasoning applies to the leader's loss.

TABLE A.2. Robustness check in quick catch-up with $\lambda = 1.2$.

	Full IPR Protection without licensing	Full IPR Protection with licensing	Optimal Uniform IPR with licensing	Optimal State Dependent without licensing	Optimal State Dependent with licensing
λ	1.20	1.20	1.20	1.20	1.20
γ	0.35	0.35	0.35	0.35	0.35
η_1	0	0	0	0.19	0
η_2	0	0	0	0.08	0
η_3	0	0	0	0.05	0
η_4	0	0	0	0.05	0
η_5	0	0	0	0.05	0
ζ_1	∞	25.07	0	∞	0
ζ_2	∞	25.07	0	∞	5.52
ζ_3	∞	25.07	0	∞	10.95
ζ_4	∞	25.07	0	∞	13.64
ζ_5	∞	25.07	0	∞	14.98
$v_1 - v_0$	20.29	25.07	13.86	9.62	14.98
x_{-1}^*	0.06	0.068	0.08	0.02	0.09
x_0^*	0.10	0.12	0.08	0.06	0.09
μ_0^*	0.22	0	0	0.56	0
μ_1^*	0.32	0.45	0.52	0.24	0.45
μ_2^*	0.20	0.26	0.26	0.10	0.24
ω^*	0.81	0.74	0.78	0.92	0.76
Researcher ratio	0.069	0.057	0.070	0.037	0.089
$\ln C(0)$	114.78	116.79	117.17	115.49	117.30
g^*	0.0186	0.0265	0.0282	0.0189	0.0306
Welfare	2303.0	2346.5	2354.7	2317.3	2358.3

Note: This table provides robustness check results for Table 1 with $\rho = 0.05$, $\lambda = 1.2$, $\gamma = 0.35$ under five different IPR policy regimes. It reports the steady-state equilibrium values of the difference in the values $v_1 - v_0$; the (annual) R&D rate of a follower that is one step behind, x_{-1}^*; the (annual) R&D rate of neck-and-neck competitors, x_0^*; fraction of industries in neck-and-neck competition, μ_0^*; fraction of industries at a technology gap of $n = 1, 2$; the value of "labor share," ω^*; the ratio of the labor force working in research; initial (annual) consumption, $C(0)$; the annual growth rate, g^*; and the welfare level according to equation (42). It also reports the welfare-maximizing uniform and state-dependent IPR policies with or without licensing. See text for details.

Note also that Figure B.1 has no value for $n = 0, 1$ since neck-and-neck firms and one-step followers have no surplus to generate through licensing.

TABLE A.3. Robustness check in quick catch-up with $\gamma = 0.1$.

	Full IPR Protection without licensing	Full IPR Protection with licensing	Optimal Uniform IPR with licensing	Optimal State Dependent without licensing	Optimal State Dependent with licensing
λ	1.05	1.05	1.05	1.05	1.05
γ	0.1	0.1	0.1	0.1	0.1
η_1	0	0	0	3.18	0
η_2	0	0	0	0.04	0
η_3	0	0	0	0	0
η_4	0	0	0	0	0
η_5	0	0	0	0	0
ζ_1	∞	3.12	0	∞	0
ζ_2	∞	3.12	0	∞	0.62
ζ_3	∞	3.12	0	∞	1.74
ζ_4	∞	3.12	0	∞	1.82
ζ_5	∞	3.12	0	∞	1.84
$v_1 - v_0$	2.21	3.12	1.64	0.49	1.88
x_{-1}^*	0.27	0.28	0.29	0.19	0.29
x_0^*	0.29	0.31	0.29	0.25	0.29
μ_0^*	0.31	0	0	0.77	0
μ_1^*	0.33	0.50	0.50	0.10	0.49
μ_2^*	0.17	0.25	0.25	0.06	0.25
ω^*	0.94	0.92	0.92	0.98	0.92
Researcher ratio	0.008	0.008	0.008	0.003	0.010
$\ln C(0)$	34.07	36.16	36.20	34.94	36.31
g^*	0.0186	0.0278	0.0280	0.0222	0.0286
Welfare	688.8	734.2	735.1	707.7	737.6

Note: This table provides robustness check results for Table 1 with $\rho = 0.05$, $\lambda = 1.05$, $\gamma = 0.1$ under five different IPR policy regimes. It reports the steady-state equilibrium values of the difference in the values $v_1 - v_0$; the (annual) R&D rate of a follower that is one step behind, x_{-1}^*; the (annual) R&D rate of neck-and-neck competitors, x_0^* ; fraction of industries in neck-and-neck competition, μ_0^*; fraction of industries at a technology gap of $n = 1, 2$; the value of "labor share," ω^*; the ratio of the labor force working in research; initial (annual) consumption, $C(0)$; the annual growth rate, g^*; and the welfare level according to equation (42). It also reports the welfare-maximizing uniform and state-dependent IPR policies with or without licensing. See text for details.

TABLE A.4. Robustness check in quick catch-up with $\gamma = 0.6$.

	Full IPR Protection without licensing	Full IPR Protection with licensing	Optimal Uniform IPR with licensing	Optimal State Dependent without licensing	Optimal State Dependent with licensing
λ	1.05	1.05	1.05	1.05	1.05
γ	0.6	0.6	0.6	0.6	0.6
η_1	0	0	0	0.61	0.01
η_2	0	0	0	0.18	0
η_3	0	0	0	0.07	0
η_4	0	0	0	0.03	0
η_5	0	0	0	0	0
ζ_1	∞	5.26	5.26	∞	0
ζ_2	∞	5.26	5.26	∞	0.62
ζ_3	∞	5.26	5.26	∞	1.52
ζ_4	∞	5.26	5.26	∞	1.98
ζ_5	∞	5.26	5.26	∞	2.35
$v_1 - v_0$	4.40	5.26	5.26	2.47	2.35
x_{-1}^*	0.13	0.18	0.18	0.03	0.31
x_0^*	0.64	0.85	0.85	0.27	0.25
μ_0^*	0.09	0	0	0.29	0.01
μ_1^*	0.26	0.42	0.42	0.12	0.29
μ_2^*	0.19	0.25	0.25	0.09	0.15
ω^*	0.94	0.94	0.94	0.94	0.93
Researcher ratio	0.073	0.044	0.044	0.084	0.097
$\ln C(0)$	33.19	33.88	33.88	33.91	35.48
g^*	0.0186	0.0198	0.0198	0.0229	0.0303
Welfare	671.3	685.6	685.6	687.3	721.6

Note: This table provides robustness check results for Table 1 with $\rho = 0.05$, $\lambda = 1.05$, $\gamma = 0.6$ under five different IPR policy regimes. It reports the steady-state equilibrium values of the difference in the values $v_1 - v_0$; the (annual) R&D rate of a follower that is one step behind, x_{-1}^*; the (annual) R&D rate of neck-and-neck competitors, x_0^*; fraction of industries in neck-and-neck competition, μ_0^*; fraction of industries at a technology gap of $n = 1, 2$; the value of "labor share," ω^*; the ratio of the labor force working in research; initial (annual) consumption, $C(0)$; the annual growth rate, g^*; and the welfare level according to equation (42). It also reports the welfare-maximizing uniform and state-dependent IPR policies with or without licensing. See text for details.

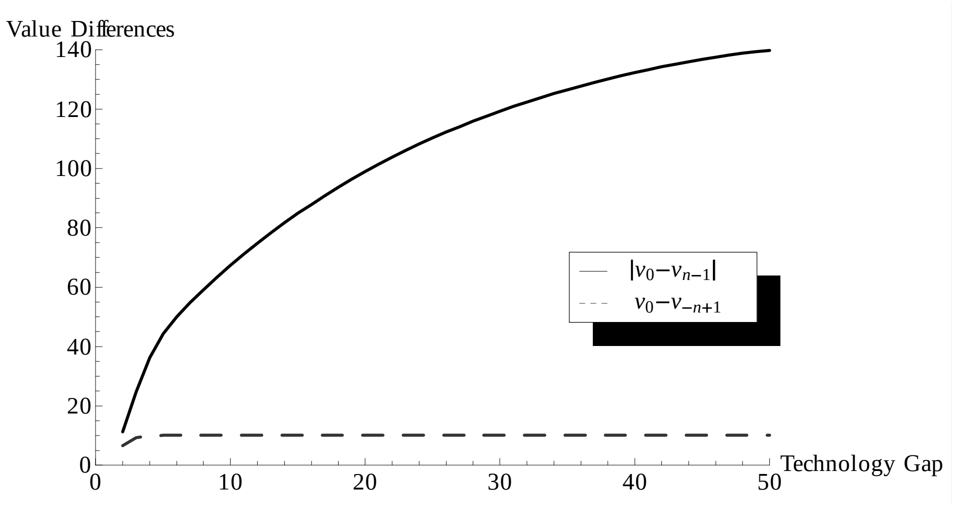

FIGURE B.1. Value differences under full protection and slow catch-up.

is that the latter number is unambiguously greater than the former, which implies that voluntary licensing will not be beneficial in this environment. Therefore, compulsory licensing plays a useful role that bilateral licensing agreements between leaders and followers could not achieve, and is thus a useful policy tool. In addition, our analysis shows that compulsory licensing will be useful for welfare precisely when it is state dependent.

Appendix C: Patent Length, Compulsory Licensing and Infringements Fees under Slow Catch-up

We investigate the slow catch-up environment when all three IPR policies are simultaneously present. We do not repeat the value functions to save space.

Table C.1 first shows our benchmark full protection economy in the first column. The second and third columns report the optimal uniform and state-dependent policies with all three types of policies present. The results are very similar to those reported in Section 7.4 (with only leapfrogging), except that the patent lengths are now set to infinity ($\eta_n = 0$). The optimal IPR policy in this case involves infinitely long patents with prohibitively high compulsory license fees. The only dimension in which IPR protection is not full is because of moderate infringement fees, which permit followers to undertake frontier R&D and leapfrog technology leaders.

Most importantly for our focus, column 3 again shows the benefits of state-dependent IPR policy. This policy again provides greater protection for technology leaders and exploits the trickle-down effect. As a result, initial consumption is approximately twice the level under uniform IPR and innovation incentives are stronger, and the long-run growth rate increases from 2.5% to 3.3%.

TABLE C.1. All three policies in slow catch-up regime.

$\lambda = 1.05,\ \gamma = 0.35$ $B = 0.1$	Full IPR	Optimal Uniform IPR	Optimal State-dependent IPR
$\eta_1 = \eta_2 = \eta_3 = \eta_4 = \eta_5$	0.02	0	0
$\zeta_1 = \zeta_2 = \zeta_3 = \zeta_4 = \zeta_5$	∞	∞	∞
ϑ_1	∞	16.6	0
ϑ_2	∞	16.6	21.7
ϑ_3	∞	16.6	34.6
ϑ_4	∞	16.6	39.4
ϑ_5	∞	16.6	51.3
$v_1 - v_{-3}$	21.4	3.1	4.6
$v_0 - v_{-3}$	10.1	0.8	2.7
x_{-1}^{c*}	0.75	0.16	0.16
x_{-1}^{f*}	0	0.23	0.34
x_0^*	0.99	0.32	0.29
x_1^*	1.10	0.32	0.41
μ_0^*	0.02	0.11	0.36
μ_1^*	0.03	0.41	0.17
μ_2^*	0.03	0.22	0.10
ω^*	0.56	0.94	0.94
Researcher ratio	0.150	0.031	0.065
$\ln C\ (0)$	31.31	35.62	36.38
g^*	0.025	0.027	0.033
Welfare	636.3	723.0	740.5

Note: This table gives the results of the numerical computations with $\rho = 0.05$, $\lambda = 1.05$, $\gamma = 0.35$ under three different IPR policy regimes. It reports the steady-state equilibrium values of the difference in the values $v_1 - v_{-3}$ and $v_0 - v_{-3}$; the (annual) catch-up and frontier R&D rates of a follower that is one step behind, $(x_{-1}^{c*}, x_{-1}^{f*})$; the (annual) R&D rate of neck-and-neck competitors, x_0^*; the (annual) R&D rate of one-step leader, x_1^*; fraction of industries in neck-and-neck competition, μ_0^*; fraction of industries at a technology gap of $n = 1, 2$; the value of "labor share," ω^*; the ratio of the labor force working in research; log of initial (annual) consumption, $\ln C(0)$; the annual growth rate, g^*; and the welfare level according to equation (42). It also reports the welfare-maximizing uniform and state-dependent IPR policies. See text for details.

Appendix D: Additional Robustness Checks

Table D.1 shows that the patterns documented in Table C.1, particularly the gains from state-dependent policy and the major role played by the trickle-down effect, are robust for reasonable changes in parameter values.

TABLE D.1. All three policies in slow catch-up regime robustness checks.

$\kappa = 0.1$	Optimal State-dependent IPR $\gamma = 0.1$ $B = 0.04$	Optimal State-dependent IPR $\gamma = 0.6$ $B = 0.2$	Optimal State-dependent IPR $\lambda = 1.01$ $B = 0.35$	Optimal State-dependent IPR $\lambda = 1.20$ $B = 0.024$
$\eta_1 = \eta_2 = \eta_3 = \eta_4 = \eta_5$	0	0	0	0
$\zeta_1 = \zeta_2 = \zeta_3 = \zeta_4 = \zeta_5$	∞	∞	∞	∞
ϑ_1	0	0	3.6	0
ϑ_2	16.7	12.6	5.5	33.4
ϑ_3	35.6	20.0	8.9	82.3
ϑ_4	44.8	23.0	12.3	100.7
ϑ_5	54.0	32.9	15.8	128.7
$v_1 - v_{-3}$	4.1	4.0	1.3	17.5
$v_0 - v_{-3}$	2.6	1.6	0.8	5.2
x_{-1}^{c*}	0.23	0.06	0.69	0.02
x_{-1}^{f*}	0.28	0.61	0.88	0.10
x_0^*	0.27	0.42	0.97	0.10
x_1^*	0.29	0.98	1.24	0.10
μ_0^*	0.20	0.01	0.10	0.06
μ_1^*	0.47	0.15	0.26	0.48
μ_2^*	0.19	0.09	0.11	0.22
ω^*	0.95	0.90	0.97	0.11
Researcher ratio	0.011	0.161	0.048	0.087
$\ln C\ (0)$	36.08	37.54	13.80	118.35
g^*	0.027	0.044	0.026	0.035
Welfare	732.4	768.3	286.3	2381.2

Note: This table gives the robustness checks of the state-dependent results of Table C.1 with alternative step sizes and R&D elasticity parameters with $\rho = 0.05$. It reports the steady-state equilibrium values of the difference in the values $v_1 - v_{-}3$ and $v_0 - v_{-}3$; the (annual) catch-up and frontier R&D rates of a follower that is one step behind, $(x_{-}1^c *, x_{-}1^f *)$; the (annual) R&D rate of neck-and-neck competitors, x_0^*; the (annual) R&D rate of one-step leader, x_1^*; fraction of industries in neck-and-neck competition, μ_0^*; fraction of industries at a technology gap of $n = 1, 2$; the value of "labor share," ω^*; the ratio of the labor force working in research; log of initial (annual) consumption, $\ln C(0)$; the annual growth rate, g^*; and the welfare level according to equation (42). It also reports the welfare-maximizing uniform and state-dependent IPR policies. See text for details.

In this table, in each column we change one of the two parameters λ and γ (increasing or reducing λ to 1.2 or 1.01, and increasing or reducing γ to 0.6 or 0.1). In each case, we also change the parameter B in equation (45) to ensure the growth rate of the benchmark economy with full IPR protection is the same as in our initial baseline economy, $g^* = 1.86\%$.

To save space, we only show the results from the optimal state-dependent policies. Table D.1 shows that the qualitative patterns in Table C.1 are relatively robust. In all cases, optimal state-dependent IPR is shaped by the trickle-down effect. In all of the various parameterizations we have considered (and with different combinations of policies), there is little protection provided to technology leaders that are one-step ahead, but IPR protection grows as the technology gap increases. This is the typical pattern implied by the trickle-down effect. In addition, in all cases when all three forms of policy are incorporated, optimal IPR policy provides patents of infinite duration and prohibitively high compulsory licensing fees, but deviates from full IPR protection by imposing moderate levels of infringement fees. Most importantly for us, in all cases, these infringements fees are state dependent and provide greater protection to technologically more advanced leaders.

Appendix E: Proofs and Derivations

E.1. Derivation of Optimal R&D Decisions in the Partial Equilibrium Model

Since the costs are linear, optimal R&D decisions imposed that, in equilibrium,

$$v_{n+1} - v_n = \varphi, \text{ for each } n \in \{-2, -1, 0, 1\}. \tag{E.1}$$

Combining this result with equation (1) gives the value of a two-step follower is

$$v_{-2} = \frac{\pi_{-2} + 2\varphi\eta_2}{r}.$$

The previous equation, together with $(E.1)$ implies

$$v_n = v_{-2} + \varphi(n+2) = \frac{\pi_{-2} + 2\varphi\eta_2}{r} + \varphi(n+2), \text{ for each } n \in \{-1, 0, 1, 2\}. \tag{E.2}$$

Now we can use the value of v_2 to solve for x_{-2}^* from equation (1). Similarly, combining $(E.2)$ with (2) gives the value of x_{-1}^*; $(E.2)$ with (3) gives x_0^*. Finally, combining $(E.2)$ with (4) gives the equilibrium value of x_1^*.

E.2. Derivation of Equation (27)

Fix the equilibrium R&D policies of other firms, $x_{-n}^*(t)$, the equilibrium interest and wage rates, $r^*(t)$ and $w^*(t)$, and equilibrium profits $\{\Pi_n^*(t)\}_{n=1}^{\infty}$. Then the value of the firm that is n steps ahead at time t can be written as:

$$V_n(t) = \tag{E.3}$$
$$\max_{x_n(t)} \left\{ \begin{array}{c} \left[\Pi_n^*(t) - w^*(t)\, G(x_n(t))\right] \Delta t + o(\Delta t) \\ +e^{-r^*(t+\Delta t)\Delta t} \left[\begin{array}{c} (x_n(t)\,\Delta t + o(\Delta t))\, V_{n+1}(t+\Delta t) \\ \\ + \left(\eta_n \Delta t + x_{-n}^*(t)\,\Delta t + o(\Delta t)\right) V_0(t+\Delta t) \\ \\ + \left(\begin{array}{c} 1 - x_n(t)\,\Delta t - \eta_n \Delta t \\ -x_{-n}^*(t)\,\Delta t - o(\Delta t) \end{array} \right) V_n(t+\Delta t) \end{array} \right] \end{array} \right\}.$$

The first part of this expression is the flow profits minus R&D expenditures during a time interval of length Δt. The second part is the continuation value after this interval has elapsed. $V_{n+1}(t)$ and $V_0(t)$ are defined as net present discounted values for a leader that is $n+1$ steps ahead and a firm in an industry that is neck-and-neck (i.e., $n = 0$). The second part of the expression uses the fact that in a short time interval Δt, the probability of innovation by the leader is $x_n(t)\,\Delta t + o(\Delta t)$, where $o(\Delta t)$ again denotes second-order terms. This explains the first line of the continuation value. For the remainder of the continuation value, note that the probability that the follower will catch up with the leader is $\left[\eta_n + x_{-n}^*(t)\right] \Delta t + o(\Delta t)$. Finally, the last line applies when no R&D effort is successful and patents continue to be enforced, so that the technology gap remains at n steps. Now, subtract $V_n(t)$ from both sides, divide everything by Δt, and take the limit as $\Delta t \to 0$ to obtain (27).

E.3. Proof of Proposition 3

Equations (24) and (26) imply

$$Y(t) = \frac{w(t)}{\omega(t)} = \frac{Q(t)\,\lambda^{-\sum_{n=0}^{\infty} n\mu_n^*(t)}}{\omega(t)}.$$

Since $\omega(t) = \omega^*$ and $\{\mu_n^*\}_{n=0}^{\infty}$ are constant in steady state, $Y(t)$ grows at the same rate as $Q(t)$. Therefore,

$$g^* = \lim_{\Delta t \to 0} \frac{\ln Q(t+\Delta t) - \ln Q(t)}{\Delta t}.$$

Now note the following: during an interval of length Δt (i) in the fraction μ_n^* of the industries with technology gap $n \geq 1$ the leaders innovate at a rate $x_n^* \Delta t + o(\Delta t)$; (ii) in the fraction μ_0^* of the industries with technology gap of $n = 0$, both firms innovate, so that the total innovation rate is $2x_0^* \Delta t + o(\Delta t)$); and (iii) each innovation increase productivity by a factor λ. Combining these observations, we have

$$\ln Q(t+\Delta t) = \ln Q(t) + \ln \lambda \left[2\mu_0^* x_0^* \Delta t + \sum_{n=1}^{\infty} \mu_n^* x_n^* \Delta t + o(\Delta t) \right].$$

Subtracting $\ln Q(t)$, dividing by Δt and taking the limit $\Delta t \to 0$ gives (39).

E.4. Proof of Proposition 4

We prove this proposition in four parts.

1. Existence of a steady-state equilibrium.
2. Properties of the sequence of value functions.
3. Properties of the sequence of R&D decisions.
4. Uniqueness of an invariant distribution given R&D policies.

Part 1: Existence of a Steady-State Equilibrium. First, note that each x_n belongs to a compact interval $[0, \bar{x}]$, where $\bar{x}$ is the maximal flow rate of innovation defined in (16) above. Now fix a labor share $\tilde{\omega} \in [0, 1]$ and a sequence $\langle \tilde{\boldsymbol{x}} \rangle$ of (Markovian) steady-state strategies for all other firms in the economy, and consider the dynamic optimization problem of a single firm. Our first result characterizes this problem and shows that given some $z \equiv \langle \tilde{\omega}, \tilde{\boldsymbol{x}} \rangle$, the value function of an individual firm is uniquely determined, while its optimal R&D choices are given by a convex-valued correspondence. In what follows, we denote sets and correspondences by uppercase letters and refer to their elements by lowercase letters, e.g., $x_n(z) \in X_n[z]$.

LEMMA E.1. *Consider a uniform IPR policy $\boldsymbol{\eta}^{uni}$, and suppose that the labor share and the R&D policies of all other firms are given by $z = \langle \tilde{\omega}, \tilde{\boldsymbol{x}} \rangle$. Then the dynamic optimization problem of an individual firm leads to a unique value function $\boldsymbol{v}[z] : \{-1\} \cup \mathbb{Z}_+ \to \mathbb{R}_+$ and optimal R&D policy $\widehat{\boldsymbol{X}}[z] : \{-1\} \cup \mathbb{Z}_+ \rightrightarrows [0, \bar{x}]$ is compact and convex-valued for each $z \in \boldsymbol{Z}$ and upper hemi-continuous in z (where $\boldsymbol{v}[z] \equiv \{v_n[z]\}_{n=-1}^{\infty}$ and $\widehat{\boldsymbol{X}}[z] \equiv \left\{\widehat{X}_n[z]\right\}_{n=-1}^{\infty}$).*

Proof. Fix $z = \left\langle \tilde{\omega}, \{\tilde{x}_n\}_{n=-1}^{\infty} \right\rangle$, and consider the optimization problem of a representative firm, written recursively as:

$$\begin{aligned}
\rho v_n &= \max_{x_n \in [0,\bar{x}]} \left\{ \begin{array}{c} (1 - \lambda^{-n}) - \tilde{\omega} G(x_n) + x_n [v_{n+1} - v_n] \\ + \tilde{x}_{-1} [v_0 - v_n] + \eta [v_0 - v_n] \end{array} \right\} \text{ for } n \in \mathbb{N} \\
\rho v_0 &= \max_{x_0 \in [0,\bar{x}]} \{-\tilde{\omega} G(x_0) + x_0 [v_1 - v_0] + \tilde{x}_0 [v_{-1} - v_0]\} \\
\rho v_{-1} &= \max_{x_{-1} \in [0,\bar{x}]} \{-\tilde{\omega} G(x_0) + x_{-1} [v_0 - v_{-1}] + \eta [v_0 - v_{-1}]\}.
\end{aligned}$$

We now transform this dynamic optimization problem into a form that can be represented as a contraction mapping using the method of “uniformization” (see, for example, Ross, 1996, Chapter 5). Let $\tilde{\boldsymbol{\xi}} = \{\tilde{x}_n\}_{n=-1}^{\infty}$ and $p_{n,n'}\left(\boldsymbol{\xi} \mid \tilde{\boldsymbol{\xi}}\right)$ be the probability that the next state will be n' starting with state n when the firm in question chooses policies $\boldsymbol{\xi} \equiv \{x_n\}_{n=-1}^{\infty}$ and the R&D policy of other firms is given by $\tilde{\boldsymbol{\xi}}$. Using the fact that, because of uniform IPR policy, $x_{-n} = x_{-1}$ for all $n \in \mathbb{N}$, these transition probabilities can be written as:

$p_{-1,0}\left(\xi \mid \tilde{\xi}\right)=\frac{x_{-1}+\eta}{x_n+x_{-1}+\eta}$	$p_{n,0}\left(\xi \mid \tilde{\xi}\right)=\frac{\tilde{x}_{-1}+\eta}{x_n+\tilde{x}_{-1}+\eta}$
$p_{0,-1}\left(\xi \mid \tilde{\xi}\right)=\frac{\tilde{x}_0}{x_0+\tilde{x}_0}$	$p_{n,n+1}\left(\xi \mid \tilde{\xi}\right)=\frac{x_n}{x_n+\tilde{x}_{-1}+\eta}$
$p_{0,1}\left(\xi \mid \tilde{\xi}\right)=\frac{x_0}{x_0+\tilde{x}_0}$	

.

Uniformization involves adding fictitious transitions from a state into itself, which do not change the value of the program, but allow us to represent the optimization problem as a contraction. For this purpose, define the transition rates ψ_n as

$$\psi_n\left(\xi \mid \tilde{\xi}\right)=\begin{cases} x_n+x_{-1}+\eta & \text{for } n\in\{1,2,\ldots\},\\ x_{-1}+\eta & \text{for } n=-1,\\ 2x_n & \text{for } n=0.\end{cases}$$

These transition rates are finite since $\psi_n\left(\xi \mid \tilde{\xi}\right)\leq\psi\equiv 2\bar{x}+\eta<\infty$ for all n, where $\bar{x}$ is the maximal flow rate of innovation defined in (16) in the text (both $\bar{x}$ and η are finite by assumption).

Now following equation (5.8.3) in Ross (1996), we can use these transition rates and define the new transition probabilities (including the fictitious transitions from a state to itself) as:

$$\tilde{p}_{n,n'}\left(\xi \mid \tilde{\xi}\right)\begin{cases}\frac{\psi_n(\xi\mid\tilde{\xi})}{\psi}p_{n,n'}\left(\xi \mid \tilde{\xi}\right) & \text{if } n\neq n',\\ 1-\frac{\psi_n(\xi\mid\tilde{\xi})}{\psi} & \text{if } n=n'.\end{cases}$$

This yields equivalent transition probabilities

$\tilde{p}_{-1,-1}\left(\xi \mid \tilde{\xi}\right)=1-\frac{x_{-1}+\eta}{2\bar{x}+\eta}$	$\tilde{p}_{-1,0}\left(\xi \mid \tilde{\xi}\right)=\frac{x_{-1}+\eta}{2\bar{x}+\eta}$	$\tilde{p}_{0,1}\left(\xi \mid \tilde{\xi}\right)=\frac{x_0}{2\bar{x}+\eta}$
$\tilde{p}_{0,-1}\left(\xi \mid \tilde{\xi}\right)=\frac{\tilde{x}_0}{2\bar{x}+\eta}$	$\tilde{p}_{0,0}\left(\xi \mid \tilde{\xi}\right)=1-\frac{x_0+\tilde{x}_0}{2\bar{x}+\eta}$	$\tilde{p}_{n,n+1}\left(\xi \mid \tilde{\xi}\right)=\frac{x_n}{2\bar{x}+\eta}$
$\tilde{p}_{n,0}\left(\xi \mid \tilde{\xi}\right)=\frac{\tilde{x}_{-1}+\eta}{2\bar{x}+\eta}$	$\tilde{p}_{n,n}\left(\xi \mid \tilde{\xi}\right)=1-\frac{x_n+\tilde{x}_{-1}+\eta}{2\bar{x}+\eta}$	

,

and also defines an effective discount factor β given by

$$\beta\equiv\frac{\psi}{\rho+\psi}=\frac{2\bar{x}+\eta}{\rho+2\bar{x}+\eta}.$$

Also let the per period return function (profit net of R&D expenditures) be

$$\hat{\Pi}_n\left(x_n\right)=\begin{cases}\frac{1-\lambda^{-n}-\tilde{\omega}G(x_n)}{\rho+2\bar{x}+\eta} & \text{if } n\geq 1,\\ \frac{-\tilde{\omega}G(x_n)}{\rho+2\bar{x}+\eta} & \text{otherwise}.\end{cases}\tag{E.4}$$

Using these transformations, the dynamic optimization problem can be written as:

$$\begin{aligned} v_n &= \max_{x_n}\left\{\hat{\Pi}_n\left(x_n\right)+\beta\sum_{n'}\tilde{p}_{n,n'}\left(\xi_n \mid \tilde{\xi}\right)\tilde{v}_{n'}\right\}, \text{for all } n\in\mathbb{Z}, \qquad \text{(E.5)}\\ &\equiv T\tilde{v}_n, \text{for all } n\in\mathbb{Z}.\end{aligned}$$

where $\boldsymbol{v} \equiv \{v_n\}_{n=-1}^{\infty}$ and the second line defines the operator T, mapping from the space of functions $\boldsymbol{V} \equiv \{\boldsymbol{v} : \{-1\} \cup \mathbb{Z}_+ \to \mathbb{R}_+\}$ into itself. T is clearly a contraction mapping. The innovation rates $\{\tilde{x}_n\}_{n=-1}^{\infty}$ are upper hemi-continuous therefore $\tilde{\boldsymbol{p}} : \{-1\} \cup \mathbb{Z}_+ \times \{-1\} \cup \mathbb{Z}_+ \rightrightarrows [0,1]$ is upper-hemicontinuous and forms a multivalued stochastic kernel. Then Proposition 2.2 in Blume (1982) implies that $\tilde{\boldsymbol{p}}$ has the Feller property. Thus, for given $z = \left\langle \tilde{\omega}, \{\tilde{x}_n\}_{n=-1}^{\infty} \right\rangle$, T possesses a unique fixed point $\boldsymbol{v}^* \equiv \{v_n^*\}_{n=-1}^{\infty}$ (e.g., Stokey, Lucas and Prescott, 1989).

Moreover, $x_n \in [0, \bar{x}]$, and v_n for each $n = -1, 0, 1, \ldots$ given by the right-hand side of (E.5) is continuous in x_n, so Berge's Maximum Theorem (Aliprantis and Border, 1999, Theorem 16.31, p. 539) implies that the set of maximizers $\left\{\widehat{X}_n\right\}_{n=-1}^{\infty}$ exists, is nonempty and compact-valued for each z and is upper hemi-continuous in $z = \left\langle \tilde{\omega}, \{\tilde{x}_n\}_{n=-1}^{\infty} \right\rangle$. Moreover, concavity of v_n in x_n for each $n = -1, 0, 1, \ldots$ implies that $\left\{\widehat{X}_n\right\}_{n=-1}^{\infty}$ is also convex-valued for each z, completing the proof. □

Now let us start with an arbitrary $z \equiv \langle \tilde{\omega}, \tilde{\boldsymbol{x}} \rangle \in \boldsymbol{Z} \equiv [0,1] \times [0, \bar{x}]^{\infty}$. From Lemma E.1, this z is mapped into optimal R&D decision sets $\widehat{\boldsymbol{X}}[z]$, where $\hat{x}_n[z] \in \widehat{X}_n[z]$. From R&D policies $\tilde{\boldsymbol{x}}$, we calculate $\boldsymbol{\mu}[\tilde{\boldsymbol{x}}] \equiv \{\mu_n[\tilde{\boldsymbol{x}}]\}_{n=0}^{\infty}$ using equations (35), (36) and (37). Then we can rewrite the labor market clearing condition (38) as

$$\begin{aligned} \omega &= \min\left\{\sum_{n=0}^{\infty} \mu_n \left[\frac{1}{\lambda^n} + G(\tilde{x}_n)\tilde{\omega} + G(\tilde{x}_{-n})\right]\tilde{\omega}; 1\right\}, \\ &\equiv \varphi(\tilde{\omega}, \tilde{\boldsymbol{x}}) \end{aligned} \tag{E.6}$$

where due to uniform IPR, $\hat{x}_{-n} = \hat{x}_{-1}$ for all $n > 0$. Next, define the mapping (correspondence)

$$\Phi[z] \equiv \left(\varphi(z),\ \widehat{\boldsymbol{X}}[z]\right),$$

which maps $\boldsymbol{Z}$ into itself, that is,

$$\Phi : \boldsymbol{Z} \rightrightarrows \boldsymbol{Z}. \tag{E.7}$$

That Φ maps $\boldsymbol{Z}$ into itself follows since $z \in \boldsymbol{Z}$ consists of $\tilde{\boldsymbol{x}} \in [0, \bar{x}]^{\infty}$ and $\tilde{\omega} \in [0,1]$, and the image of z under Φ consists of $\hat{\boldsymbol{x}} \in [0, \bar{x}]^{\infty}$, and moreover, (E.6) is clearly in $[0,1]$ (since the right-hand side is nonnegative and bounded above by 1). Finally, from Lemma E.1, $\widehat{X}_n[z]$ is compact and convex-valued for each $z \in \boldsymbol{Z}$, and also upper hemi-continuous in z, and φ is continuous. Using this construction, we can establish the existence of a steady-state equilibrium as follows.

We first show that the mapping $\Phi : \boldsymbol{Z} \rightrightarrows \boldsymbol{Z}$ constructed in $(E.7)$ has a fixed point, and then establish that when $G'^{-1}\left(\left(1 - \lambda^{-1}\right)/(\rho + \eta)\right) > 0$ this fixed point corresponds to a steady state with $\omega^* < 1$. First, it has already been established that Φ maps $\boldsymbol{Z}$ into itself. We next show that $\boldsymbol{Z}$ is compact in the product topology and is a subset of a locally convex Hausdorff space. The first part follows from the fact that $\boldsymbol{Z}$ can be written as the Cartesian product of compact subsets, $\boldsymbol{Z} = [0,1] \times \prod_{n=-1}^{\infty} [0, \bar{x}]$. Then by Tychonoff's Theorem (e.g., Aliprantis and Border, 1999, Theorem 2.57, p.

52; Kelley, 1955, p. 143), $\boldsymbol{Z}$ is compact in the product topology. Moreover, $\boldsymbol{Z}$ is clearly nonempty and also convex, since for any $z, z' \in \boldsymbol{Z}$ and $\lambda \in [0,1]$, we have $\lambda z + (1-\lambda) z' \in \boldsymbol{Z}$. Finally, since $\boldsymbol{Z}$ is a product of intervals on the real line, it is a subset of a locally convex Hausdorff space (see Aliprantis and Border, 1999, Lemma 5.54, p. 192).

Next, φ is a continuous function from $\boldsymbol{Z}$ into $[0,1]$ and from Lemma E.1, $\widehat{X}_n(z)$ for $n \in \{-1\} \cup \mathbb{Z}_+$ is upper hemi-continuous in z. Consequently, $\Phi \equiv \left\langle \varphi[z],\ \widehat{\boldsymbol{X}}[z] \right\rangle$ has closed graph in z in the product topology. Moreover, each one of $\varphi(z)$ and $\widehat{X}_n(z)$ for $n = -1, 0, \ldots$ is nonempty, compact and convex-valued. Therefore, the image of the mapping Φ is nonempty, compact and convex-valued for each $z \in \boldsymbol{Z}$. The Kakutani-Fan-Glicksberg Fixed Point Theorem implies that if the function Φ maps a convex, compact and nonempty subset of a locally convex Hausdorff space into itself and has closed graph and is nonempty, compact and convex-valued z, then it possesses a fixed point $z^* \in \Phi(z^*)$ (see Aliprantis and Border, 1999, Theorem 16.50 and Corollary 16.51, p. 549–550). This establishes the existence of a fixed point z^* of Φ.

To complete the proof, we need to show that the fixed point, z^*, corresponds to a steady state equilibrium. First, since $\hat{x}_n\left(\omega^*, \{x_n^*\}_{n=-1}^{\infty}\right) = x_n^*$ for $n \in \{-1\} \cup \mathbb{Z}_+$, we have that given a labor share of ω^*, $\{x_n^*\}_{n=-1}^{\infty}$ constitutes an R&D policy vector that is best response to itself, as required by steady-state equilibrium (Definition 3). Next, we need to prove that the implied labor share ω^* leads to labor market clearing. This follows from the fact that the fixed point involves $\omega^* < 1$, since in this case (E.6) will have an interior solution, ensuring labor market clearing. Suppose, to obtain a contradiction, that $\omega^* = 1$. Then, as noted in the text, we must have $\mu_0^* = 1$. From (35), (36), and (37), this implies $x_n^* = 0$ for $n \in \{-1\} \cup \mathbb{Z}_+$. However, we have shown above that this is not possible when $G'^{-1}\left(\left(1-\lambda^{-1}\right)/(\rho+\eta)\right) > 0$. Consequently, (E.6) cannot be satisfied at $\omega^* = 1$, implying that $\omega^* < 1$. When $\omega^* < 1$, the labor market clearing condition (38) is satisfied at ω^* as an equality, so ω^* is an equilibrium given $\{x_n^*\}_{n=-1}^{\infty}$, and thus $z^* = \left(\omega^*, \{x_n^*\}_{n=-1}^{\infty}\right)$ is a steady-state equilibrium as desired.

Finally, if $\eta > 0$, then (37) implies that $\mu_0^* > 0$. Since $x_0^* > 0$, equation (39) implies $g^* > 0$. Alternatively, if $x_{-1}^* > 0$, then $g^* > 0$ follows from (39). This completes the proof of the existence of a steady-state equilibrium with positive growth.

Part 2: Properties of the Sequence of Value Functions. Let $\{x_n\}_{n=-1}^{\infty}$ be the R&D decisions of the firm and $\{v_n\}_{n=-1}^{\infty}$ be the sequence of values, taking the decisions of other firms and the industry distributions, $\{x_n^*\}_{n=-1}^{\infty}$, $\{\mu_n^*\}_{n=-1}^{\infty}$, ω^* and g, as given. By choosing $x_n = 0$ for all $n \geq -1$, the firm guarantees $v_n \geq 0$ for all $n \geq -1$. Moreover, since flow profit satisfy $\pi_n \leq 1$ for all $n \geq -1$, $v_n \leq 1/\rho$ for all $n \geq -1$, establishing that $\{v_n\}_{n=-1}^{\infty}$ is a bounded sequence, with $v_n \in [0, 1/\rho]$ for all $n \geq -1$.

Proof of $v_1 > v_0$. Suppose, first, $v_1 \leq v_0$, then (34) implies $x_0^* = 0$, and by the symmetry of the problem in equilibrium (30) implies $v_0 = v_1 = 0$. As a result, from (33) we obtain $x_{-1}^* = 0$. Equation (29) implies that when $x_{-1}^* = 0$, $v_1 \geq \left(1-\lambda^{-1}\right)/(\rho+\eta) > 0$, yielding a contradiction and proving that $v_1 > v_0$. □

Proof of $v_{-1} \leq v_0$. Suppose, to obtain a contradiction, that $v_{-1} > v_0$. If $v_1 \leq v_0$, (33) yields $x_{-1}^* = 0$. This implies $v_{-1} = \eta v_0 / (\rho + \eta)$, which contradicts $v_{-1} > v_0$ since $\eta/(\rho + \eta) < 1$. Thus we must have $v_1 > v_0$. The value function of a neck-and-neck firm can be written as:

$$\begin{aligned}\rho v_0 &= \max_{x_0} \left\{ -\omega^* G(x_0) + x_0 [v_1 - v_0] + x_0^* [v_{-1} - v_0] \right\}, \\ &\geq \max_{x_0} \left\{ -\omega^* G(x_0) + x_0 [v_1 - v_0] \right\}, \\ &\geq -\omega^* G\left(x_{-1}^*\right) + x_{-1}^* [v_1 - v_0], \\ &\geq -\omega^* G\left(x_{-1}^*\right) + x_{-1}^* [v_0 - v_{-1}] + \eta [v_0 - v_{-1}], \\ &= \rho v_{-1},\end{aligned} \tag{E.8}$$

which contradicts the hypothesis that $v_{-1} > v_0$ and establishes the claim. □

Proof of $v_n < v_{n+1}$. Suppose, to obtain a contradiction, that $v_n \geq v_{n+1}$. Now (32) implies $x_n^* = 0$, and (29) becomes

$$\rho v_n = (1 - \lambda^{-n}) + x_{-1}^* [v_0 - v_n] + \eta [v_0 - v_n] \tag{E.9}$$

Also from (29), the value for state $n + 1$ satisfies

$$\rho v_{n+1} \geq \left(1 - \lambda^{-n-1}\right) + x_{-1}^* [v_0 - v_{n+1}] + \eta [v_0 - v_{n+1}]. \tag{E.10}$$

Combining the two previous expressions, we obtain

$$\begin{aligned}&(1 - \lambda^{-n}) + x_{-1}^* [v_0 - v_n] + \eta [v_0 - v_n] \\ &\geq 1 - \lambda^{-n-1} + x_{-1}^* [v_0 - v_{n+1}] + \eta [v_0 - v_{n+1}].\end{aligned}$$

Since $\lambda^{-n-1} < \lambda^{-n}$, this implies $v_n < v_{n+1}$, contradicting the hypothesis that $v_n \geq v_{n+1}$, and establishing the desired result, $v_n < v_{n+1}$. Consequently, $\{v_n\}_{n=-1}^{\infty}$ is nondecreasing and $\{v_n\}_{n=0}^{\infty}$ is (strictly) increasing. Since a nondecreasing sequence in a compact set must converge, $\{v_n\}_{n=-1}^{\infty}$ converges to its limit point, v_∞, which must be strictly positive, since $\{v_n\}_{n=0}^{\infty}$ is strictly increasing and has a nonnegative initial value. □

The above results combined complete the proof that values form an increasing sequence.

Part 3: Properties of the Sequence of R&D Decisions.

Proof of $x_{n+1}^* < x_n^*$. From equation (32),

$$\delta_{n+1} \equiv v_{n+1} - v_n < v_n - v_{n-1} \equiv \delta_n \tag{E.11}$$

would be sufficient to establish that $x_{n+1}^* < x_n^*$ whenever $x_n^* > 0$. We next show that this is the case.

Let us write:

$$\bar{\rho} v_n = \max_{x_n} \left\{ \left(1 - \lambda^{-n}\right) - \omega^* G\left(x_n\right) + x_n^* \left[v_{n+1} - v_n\right] + x_{-1}^* v_0 + \eta v_0 \right\}, \quad \text{(E.12)}$$

where $\bar{\rho} \equiv \rho + x_{-1}^* + \eta$. Since x_{n+1}^*, x_n^* and x_{n-1}^* are maximizers of the value functions v_{n+1}, v_n and v_{n-1}, (E.12) implies:

$$\bar{\rho} v_{n+1} = 1 - \lambda^{-n-1} - \omega^* G\left(x_{n+1}^*\right) + x_{n+1}^* \left[v_{n+2} - v_{n+1}\right] + x_{-1}^* v_0 + \eta v_0, \quad \text{(E.13)}$$

$$\bar{\rho} v_n \geq 1 - \lambda^{-n} - \omega^* G\left(x_{n+1}^*\right) + x_{n+1}^* \left[v_{n+1} - v_n\right] + x_{-1}^* v_0 + \eta v_0,$$

$$\bar{\rho} v_n \geq 1 - \lambda^{-n} - \omega^* G\left(x_{n-1}^*\right) + x_{n-1}^* \left[v_{n+1} - v_n\right] + x_{-1}^* v_0 + \eta v_0,$$

$$\bar{\rho} v_{n-1} = 1 - \lambda^{-n+1} - \omega^* G\left(x_{n-1}^*\right) + x_{n-1}^* \left[v_n - v_{n-1}\right] + x_{-1}^* v_0 + \eta v_0.$$

Now taking differences with $\bar{\rho} v_n$ and using the definitions of δ_ns, we obtain

$$\bar{\rho} \delta_{n+1} \leq \lambda^{-n} \left(1 - \lambda^{-1}\right) + x_{n+1}^* \left(\delta_{n+2} - \delta_{n+1}\right)$$
$$\bar{\rho} \delta_n \geq \lambda^{-n+1} \left(1 - \lambda^{-1}\right) + x_{n-1}^* \left(\delta_{n+1} - \delta_n\right).$$

Therefore,

$$\left(\bar{\rho} + x_{n-1}^*\right) \left(\delta_{n+1} - \delta_n\right) \leq -k_n + x_{n+1}^* \left(\delta_{n+2} - \delta_{n+1}\right), \quad \text{(E.14)}$$

where

$$k_n \equiv (\lambda - 1)^2 \lambda^{-n-1} > 0.$$

Now to obtain a contradiction, suppose that $\delta_{n+1} - \delta_n \geq 0$. From (E.14), this implies $\delta_{n+2} - \delta_{n+1} > 0$ since k_n is strictly positive. Repeating this argument successively, we have that if $\delta_{n'+1} - \delta_{n'} \geq 0$, then $\delta_{n+1} - \delta_n > 0$ for all $n \geq n'$. However, we know from Part 2 of the proposition that $\{v_n\}_{n=0}^{\infty}$ is strictly increasing and converges to a constant v_∞. This implies that $\delta_n \downarrow 0$, which contradicts the hypothesis that $\delta_{n+1} - \delta_n \geq 0$ for all $n \geq n' \geq 0$, and establishes that $x_{n+1}^* \leq x_n^*$. To see that the inequality is strict when $x_n^* > 0$, it suffices to note that we have already established (E.11), i.e., $\delta_{n+1} - \delta_n < 0$, thus if equation (32) has a positive solution, then we necessarily have $x_{n+1}^* < x_n^*$.

We next prove that $x_0^* \geq x_{-1}^*$ and then show that under the additional condition $G'^{-1}\left(\left(1 - \lambda^{-1}\right) / (\rho + \eta)\right) > 0$, this inequality is strict. □

Proof of $x_0^* \geq x_{-1}^*$. Equation (30) can be written as

$$\rho v_0 = -\omega^* G\left(x_0^*\right) + x_0^* \left[v_{-1} + v_1 - 2v_0\right]. \quad \text{(E.15)}$$

We have $v_0 \geq 0$ from Part 2 of the proposition. Suppose $v_0 > 0$. Then (E.15) implies $x_0^* > 0$ and

$$v_{-1} + v_1 - 2v_0 > 0 \quad \text{(E.16)}$$
$$v_1 - v_0 > v_0 - v_{-1}.$$

This inequality combined with (34) and (41) yields $x_0^* > x_{-1}^*$. Suppose next that $v_0 = 0$. Inequality (E.16) now holds as a weak inequality and implies that $x_0^* \geq x_{-1}^*$. Moreover, since $G(\cdot)$ is strictly convex and x_0^* is given by (34), (E.15) then implies $x_0^* = 0$ and thus $x_{-1}^* = 0$. □

We now have the following intermediate lemma.

LEMMA E.2. *Suppose that* $G'^{-1}\left(\left(1-\lambda^{-1}\right)/(\rho+\eta)\right) > 0$, *then* $x_0^* > 0$ *and* $v_0 > 0$.

Proof. Suppose, to obtain a contradiction, that $x_0^* = 0$. The first part of the proof then implies that $x_{-1}^* = 0$. Then (29) implies

$$\rho v_1 \geq 1 - \lambda + \eta\left[v_0 - v_1\right].$$

Equation (30) together with $x_0^* = 0$ gives $v_0 = 0$, and hence

$$v_1 - v_0 \geq \frac{1-\lambda^{-1}}{\rho+\eta}.$$

Combined with this inequality, (34) implies

$$\begin{aligned} x_0^* &\geq \max\left\{G'^{-1}\left(\frac{1-\lambda^{-1}}{\omega^*(\rho+\eta)}\right), 0\right\}, \\ &\geq \max\left\{G'^{-1}\left(\frac{1-\lambda^{-1}}{\rho+\eta}\right), 0\right\}, \end{aligned}$$

where the second inequality follows from the fact that $\omega^* \leq 1$. The assumption that $G'^{-1}\left(\left(1-\lambda^{-1}\right)/(\rho+\eta)\right) > 0$ then implies $x_0^* > 0$, thus leading to a contradiction and establishing that $x_0^* > 0$. Strict convexity of $G(\cdot)$ together with $x_0^* > 0$ then implies $v_0 > 0$. □

Proof of $x_0^* > x_{-1}^*$ when $G'^{-1}\left(\left(1-\lambda^{-1}\right)/(\rho+\eta)\right) > 0$. Given Lemma E.2, $G'^{-1}\left(\left(1-\lambda^{-1}\right)/(\rho+\eta)\right) > 0$ implies that $x_0^* > 0$. Then $(E.15)$ implies

$$v_1 - v_0 > v_0 - v_{-1}$$

and as a result $x_0^* > x_{-1}^*$. □

Proof of $x_0^* > x_1^*$. To prove that $x_0^* > x_1^*$, let us write the value functions v_2, v_1 and v_0 as in (E.13):

$$\begin{aligned} \bar{\rho} v_2 &= 1 - \lambda^{-2} - \omega^* G\left(x_2^*\right) + x_2^*\left[v_3 - v_2\right] + x_{-1}^* v_0 + \eta v_0, \\ \bar{\rho} v_1 &\geq 1 - \lambda^{-1} - \omega^* G\left(x_2^*\right) + x_2^*\left[v_2 - v_1\right] + x_{-1}^* v_0 + \eta v_0, \\ \bar{\rho} v_1 &\geq 1 - \lambda^{-1} - \omega^* G\left(x_0^*\right) + x_0^*\left[v_2 - v_1\right] + x_{-1}^* v_0 + \eta v_0, \\ \bar{\rho} v_0 &= -\omega^* G\left(x_0\right) + x_0^*\left[v_1 - v_0\right] + \eta v_0 + x_{-1}^* v_0 + x_0^*\left[v_{-1} - v_0\right]. \end{aligned}$$

Now taking differences with $\bar{\rho}v_n$ and using the definitions of δ_ns as in (E.11), we obtain

$$\begin{aligned}
\bar{\rho}\delta_2 &\leq \lambda^{-1}\left(1-\lambda^{-1}\right) + x_2^*\left(\delta_3-\delta_2\right), \qquad \text{(E.17)}\\
\bar{\rho}\delta_1 &\geq \left(1-\lambda^{-1}\right) + x_0^*\left(\delta_2-\delta_1\right) + x_{-1}^*\left[v_0-v_0\right] - x_0^*\left[v_{-1}-v_0\right],\\
\bar{\rho}\delta_1 &\geq \left(1-\lambda^{-1}\right) + x_0^*\left(\delta_2-\delta_1\right) - x_0^*\left[v_{-1}-v_0\right],\\
\bar{\rho}\delta_1 &\geq \left(1-\lambda^{-1}\right) + x_0^*\left(\delta_2-\delta_1\right) - x_0^*\left[v_{-1}-v_0\right].
\end{aligned}$$

Next recall from Part 2 that $v_{-1} - v_0 \leq 0$. Moreover, the first part of the first part of the proof has established that $x_{-1}^* - x_0^* \leq 0$. Therefore $\left[x_{-1}^* - x_0\right]\left[v_{-1} - v_0\right] \geq 0$, and the last inequality then implies

$$\bar{\rho}\delta_1 \geq \left(1-\lambda^{-1}\right) + x_0^*\left(\delta_2-\delta_1\right).$$

Now combining this inequality with the first inequality of (E.17), we obtain

$$\left(\bar{\rho}+x_0^*\right)\left(\delta_2-\delta_1\right) \leq -\left(1-\lambda^{-1}\right)^2 + x_2^*\left(\delta_3-\delta_2\right). \qquad \text{(E.18)}$$

Part 2 has already established $\delta_2 > \delta_3$, so that the right-hand side is strictly negative, therefore, we must have $\delta_2 - \delta_1 < 0$, which implies that $x_0^* > x_1^*$ and completes the proof. □

The above results together complete the proof of Part 3.

Part 4: Uniqueness of the Invariant Distribution.

LEMMA E.3. *Consider a uniform IPR policy $\boldsymbol{\eta}^{uni}$ and a corresponding steady-state equilibrium $\langle \boldsymbol{\mu}^*, \boldsymbol{v}, \boldsymbol{x}^*, \omega^*, g^* \rangle$. Then, there exists $n^* \in \mathbb{N}$ such that $x_n^* = 0$ for all $n \geq n^*$.*

Proof. The first-order condition of the maximization of the value function (29) implies

$$G'\left(x_n\right) \geq \frac{v_{n+1}-v_n}{\omega^*} \text{ and } x_n \geq 0,$$

with complementary slackness. $G'(0)$ is strictly positive by assumption. If $\left(v_{n+1}-v_n\right)/\omega^* < G'(0)$, then $x_n = 0$. The second part of the proposition implies that $\{v_n\}_{n=-1}^{\infty}$ is a convergent and thus a Cauchy sequence, which implies that there exists $\exists n^* \in \mathbb{N}$ such that $v_{n+1} - v_n < \omega^* G'(0)$ for all $n \geq n^*$. □

An immediate consequence of Lemma E.3, combined with (35) is that $\mu_n = 0$ for all $n \geq n^*$ (since there is no innovation in industries with technology gap greater than n^*). Thus the law of motion of an industry can be represented by a finite Markov chain. Moreover, because after an innovation by a follower, all industries jump to the neck-and-neck state, this Markov chain is irreducible (and aperiodic), thus converges to a unique steady-state distribution of industries. More formally, there exists n^* such that $x^*_{n^*} = 0$ and $x^*_n = 0$ for all $n > n^*$. Combined with the fact $G'^{-1}\left(\left(1-\lambda^{-1}\right)/\left(\rho+\eta\right)\right) > 0$ and that either $\eta > 0$ or $x^*_{-1} > 0$, this implies that the states $n > n^*$ are transient and can be ignored. Consequently, $\{\mu^*_n\}^{\infty}_{n=0}$ forms a finite and irreducible Markov chain over the states $n = 0, 1, \ldots, n^*$. To see this, let $n^* = \min_{n\in\{0,\ldots,n^{**}\}}\{n \in \mathbb{N}{:}v_{n+1} - v_n \leq \omega^* G'(0)\}$. Such an n^* exists, since the set $\{0, \ldots, n^{**}\}$ is finite and nonempty because of the assumption that $G'^{-1}\left(\left(1-\lambda^{-1}\right)/\left(\rho+\eta\right)\right) > 0$. Then by construction $x^*_n > 0$ for all $n < n^*$ and $x^*_{n^*} = 0$ as desired. Now denoting the probability of being in state $\tilde{n}$ starting in state n after τ periods by $P^{\tau}(n, \tilde{n})$, we have that $\lim_{\tau\to\infty} P^{\tau}(n, \tilde{n}) = 0$ for all $\tilde{n} > n^*$ and for all n. Thus we can focus on the finite Markov chain over the states $n = 0, 1, \ldots, n^*$, and $\{\mu^*_n\}^{n^*}_{n=0}$ is the limiting (invariant) distribution of this Markov chain. Given $\{x^*_n\}^{n^*}_{n=-1}$, $\{\mu^*_n\}^{n^*}_{n=0}$ is uniquely defined. Moreover, the underlying Markov chain is irreducible (since $x^*_n > 0$ for $n = 0, 1, \ldots, n^* - 1$, so that all states communicate with $n = 0$ or $n = 1$). Therefore, by Theorem 11.2 in Stokey, Lucas and Prescott (1989, p. 62) there exists a unique stationary distribution $\{\mu^*_n\}^{\infty}_{n=0}$.

E.5. Proof of Proposition 5

We prove this proposition using two crucial lemmas.

LEMMA E.4. *Consider the state-dependent IPR policy $\boldsymbol{\eta}$, and suppose that $\langle \boldsymbol{\mu}^*, \boldsymbol{v}, \boldsymbol{x}^*, \omega^*, g^* \rangle$ is a steady-state equilibrium. Then there exists a state $n^* \in \mathbb{N}$ such that $\mu^*_n = 0$ for all $n \geq n^*$.*

Proof. There are two cases to consider. First, suppose that $\{v_n\}_{n\in\mathbb{Z}_+}$ is strictly increasing. Then it follows from the proof of Lemma E.3 that there exists a state $n^* \in \mathbb{N}$ such that $x^*_n = 0$ for all $n \geq n^*$, and as in the proof of Part 4 of Proposition 4, states $n \geq n^*$ are transient (i.e., $\lim_{\tau\to\infty} P^{\tau}(n, \tilde{n}) = 0$ for all $\tilde{n} > n^*$ and for all n), so $\mu^*_n = 0$ for all $n \geq n^*$.

Second, in contrast to the first case, suppose that there exists some $n^{**} \in \mathbb{Z}_+$ such that $v_{n^{**}} \geq v_{n^{**}+1}$. Then, let $n^* = \min_{n\in\{0,\ldots,n^{**}\}}\{n \in \mathbb{N}{:}v_{n+1} - v_n \leq \omega^* G'(0)\}$, which is again well defined. Then, optimal R&D decision (32) immediately implies that $x^*_n > 0$ for all states with $n < n^*$, and since $x^*_{n^*} = 0$, all states $n > n^*$ are transient and $\lim_{\tau\to\infty} P^{\tau}(n, \tilde{n}) = 0$ for all $\tilde{n} > n^*$ and for all n, completing the proof. □

LEMMA E.5. *Consider the state-dependent IPR policy $\boldsymbol{\eta}$ and suppose that the labor share and the R&D policies of all other firms are given by $z = \langle \tilde{\omega}, \tilde{\boldsymbol{x}} \rangle$. Then the dynamic optimization problem of an individual firm leads to a unique value function $\boldsymbol{v}[z] : \mathbb{Z} \to \mathbb{R}_+$ and optimal R&D policy $\widehat{\boldsymbol{X}}[z] : \mathbb{Z} \rightrightarrows [0, \bar{x}]$ are compact and convex-valued for each $z \in \boldsymbol{Z}$ and upper hemi-continuous in z (where $\boldsymbol{v}[z] \equiv \{v_n[z]\}^{\infty}_{n=-1}$, $\widehat{\boldsymbol{X}}[z] \equiv \left\{\widehat{X}_n[z]\right\}^{\infty}_{n=-1}$).*

Proof. The proof follows closely that of Lemma E.1. In particular, again using uniformization, the maximization problem of an individual firm can be written as a contraction mapping similar to (E.5) there. The finiteness of the transition probabilities follows, since $\psi_n\left(\boldsymbol{\xi} \mid \tilde{\boldsymbol{\xi}}\right) \leq \psi \equiv 2\bar{x} + \max_n\{\eta_n\} < \infty$ (this is a consequence of the fact that $\bar{x}$ defined in (16) is finite and $\max_n\{\eta_n\}$ is finite, since each $\eta_n \in \mathbb{R}_+$ and by assumption, there exists $\bar{n} < \infty$ such that $\eta_n = \eta_{\bar{n}}$). This contraction mapping uniquely determines the value function $\boldsymbol{v}[z] : \mathbb{Z} \to \mathbb{R}_+$.

Berge's Maximum Theorem (Aliprantis and Border, 1999, Theorem 16.31, p. 539) again implies that each of $\widehat{X}_n(z)$ for $n \in \mathbb{Z}$ is upper hemi-continuous in $z = \langle \tilde{\omega}, \tilde{\boldsymbol{x}} \rangle$, and moreover, since v_n for $n \in \mathbb{Z}$ is concave in x_n, the maximizer of $\boldsymbol{v}[z]$, $\widehat{\boldsymbol{X}} \equiv \left\{\widehat{X}_n\right\}_{n=-\infty}^{\infty}$, are nonempty, compact and convex-valued. □

Now using the previous two lemmas, we can establish the existence of a steady-state equilibrium. This part of the proof follows that of Proposition 4 closely. Fix $z = \left\langle \tilde{\omega}, \{\tilde{x}_n\}_{n=-\infty}^{\infty} \right\rangle$, and define $\boldsymbol{Z} \equiv [0,1] \times \prod_{n=-\infty}^{\infty} [0, \bar{x}]$. Again by Tychonoff's Theorem, $\boldsymbol{Z}$ is compact in the product topology. Then consider the mapping $\Phi : \boldsymbol{Z} \rightrightarrows \boldsymbol{Z}$ constructed as $\Phi \equiv \left(\varphi, \widehat{\boldsymbol{X}}\right)$, where φ is given by (E.6) and $\widehat{\boldsymbol{X}}$ is defined in Lemma E.5. Clearly Φ maps $\boldsymbol{Z}$ into itself. Moreover, as in the proof of Proposition 4, $\boldsymbol{Z}$ is nonempty, convex, and a subset of a locally convex Hausdorff space. The proof of Lemma E.5 then implies that Φ has closed graph in the product topology and is nonempty, compact and convex-valued in z. Consequently, the Kakutani-Fan-Glicksberg Fixed Point Theorem again applies and implies that Φ has a fixed point $z^* \in \Phi(z^*)$. The argument that the fixed point z^* corresponds to a steady-state equilibrium is identical to that in Proposition 4, and follows from the fact that within argument identical to that of Lemma E.2, $G'^{-1}\left(\left(1-\lambda^{-1}\right) / \left(\rho + \eta_1\right)\right) > 0$ implies $x_0^* > 0$. The result that $\omega^* < 1$ then follows immediately. Finally, as in the proof of Proposition 4, either $\eta_1 > 0$ or $x_{-1}^* > 0$ is sufficient for $g^* > 0$.

References

- **Aliprantis, Charalambos D. and Kim C. Border (1999).** *Infinite Dimensional Analysis: Hitchhiker's Guide*. Springer-Verlag, New York.
- **Blume, Lawrence E. (1982).** "New Techniques for the Study of Stochastic Equilibrium Processes." *Journal of Mathematical Economics*, 9, 61-70.
- **Kelley, John L. (1955).** *General Topology*. Springer-Verlag, New York.
- **Ross, Sheldon (1996).** *Stochastic Processes.* John Wiley & Sons, Inc., New York.
- **Stokey, Nancy and Robert Lucas with E. Prescott (1989).** *Recursive Methods in Economic Dynamics*. Harvard University Press, Cambridge, MA.