The Marginal User Principle for Resource Allocation in Wireless Networks

Daron Acemoglu, Asuman Ozdaglar and R. Srikant

***Abstract*— We consider the problem of resource allocation in a wireless network operated by a single service provider. The motivating model is the downlink in a cellular network where the provider sets the price of entry into the wireless network and then allocates power levels (and transmission rates) to the participating users as a function of the users' channel conditions according to a pre-specified policy. The provider's goal is to design the power allocation policy that maximizes its revenue, recognizing the effects of his decisions on the choice of users to join the network. We show that the power allocation policy chosen by the service provider satisfies the following *marginal user principle:* the network allocates power levels such that the utility of the marginal user, who is indifferent to joining the network or not, is maximized. While the motivation is drawn from power allocation, the marginal user principle also generalizes to other resource allocation problems.**

I. Introduction

Our purpose in this paper is to study the allocation of power and transmission rates in a wireless network from the service provider viewpoint.

A central issue for wireless networks is the allocation of scarce radio resources. The traditional approach to this resource allocation problem is based on a single control objective, such as minimizing total power or maximizing total throughput. The past decade has witnessed the development of a new approach to resource allocation in communication networks. This new approach starts with the utility functions of (potential) users in the network defined over transmission rates, quality of service and potential delays, and develops algorithms for maximizing the sum of utilities of the users. Recent work using this approach include [1]-[10], [12]-[14]. Despite the important insights they have generated, the new utility-based approach does not motivate the system objectives. Why should the objective of the network be the maximization of the sum of users' utilities? Although this may appear as a natural objective from a social point of view, in modern wireless networks resource allocation decisions are made by for-profit service providers and most networks are built and operated for potential profit. To understand how resources are likely to be allocated both in current and future networks, one needs to consider the service provider viewpoint. This paper is an attempt in this direction. We investigate how resources will be allocated in a network operated by a for-profit service provider, and we compare the equilibrium resource allocations to the natural social objective of maximizing the sum of the utilities of potential users.

In our model, a profit-maximizing service provider chooses a rule for power (transmission rate) allocation among multiple users in the downlink of a single cell. The allocation rule specifies how the power resources of the network will be shared among a set of potential users with varying channel gains. The base station measures the channel gains of (participating) users and implements the pre-specified rule to allocate power and transmission rates. In choosing the allocation rule, the service provider recognizes the effect of its allocation rule on the willingness of users to participate and to pay for the right to participate in this network. Although the service provider does not know the exact utility function of each user, it is assumed to have a good understanding of the distribution of the utility functions of potential users. Users, in turn, know their own utility functions (e.g., their own service preferences) and recognize that they will have to transmit under a variety of channel conditions.

The main assumptions of this model are plausible. The assumption of profit-maximizing service provider is natural in this context. Even if in practice service providers may have other objectives besides profit maximization, they must take the profit consequences of their decisions into account, so the profit-maximizing rule is a natural benchmark. It is also plausible to presume that the service provider has a good understanding of the distribution of the utility functions. This could be because of past experience in the same or related markets, or because it has conducted customer surveys. Finally, it is also natural that potential users care about the transmission rates in the network under a variety of different channel gains, for example, because when they join the network, they do not know what their exact channel gains will be at the time of transmission. An alternative interpretation of users' preferences is that once they sign up with the network, they will transmit at various different points in the future under varying channel gains circumstances.

Our investigation reveals some simple lessons about power allocation. We find that profit-maximizing strategy is to choose an allocation rule that maximizes the *expected*

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D. Acemoglu is with the Department of Economics, MIT; daron@mit.edu

A. Ozdaglar is with the Department of Electrical Engineering and Computer Science and the Laboratory for Information and Decision Systems, MIT; asuman@mit.edu

R. Srikant is with the Department of Electrical and Computer Engineering and Coordinated Science Lab, University of Illinois at Urbana-Champaign; rsrikant@uiuc.edu

utility of the *marginal user*. Expected utility here refers to the utility that the user perceives before knowing his and other users' channel gains. Following the economics literature, we assume that the user calculates this expected utility according to the von Neumann-Morgenstern expected utility theory, and using a probability distribution over channel gains. Marginal user refers to the user that is indifferent between participating and not participating in this network given the optimal pricing and allocation strategies of the service provider.

This result is intuitive: the service provider would like to maximize participation and the willingness to pay of the users. This basically amounts to choosing the best allocation rule from the point of view of the users. Based on this insight, a naive intuition would have been that the allocation rule would indeed maximize the sum of the utilities of users. This is not the equilibrium allocation rule (or optimal allocation rule from the service provider viewpoint), however. Since the users are potentially heterogeneous, it makes sense for the service provider to maximize the expected utility of the user who is at the margin indifferent between participating and not participating in the network. Individuals who are not participating are irrelevant, while those who are participating but are not marginal (i.e., who are intra-marginal) are already participating, and the service provider does not gain further by increasing their utility.[1] In contrast, by increasing the utility of the marginal user, the service provider can increase the price that it can charge to all users without reducing total demand for participation in the network.

This allocation rule contrasts with some ad hoc rules commonly used in the literature, including proportional fairness rules which maximize the logarithm of the transmission rates to users. Interestingly, our analysis shows that the proportional fairness rule would result as the profit-maximizing allocation rule when the marginal user has a utility function that can be approximated by a logarithmic utility function defined over the transmission rate.

The equilibrium allocation rule in the model is also different from maximizing the sum of utilities of all users. In fact, unless all users have exactly the same utility function, the allocation rule chosen by the service provider will be different from the socially optimal allocation rule. The difference between the equilibrium and the social objective in this case emerges because the service provider is trying to achieve a private objective, profit maximization. Although this objective requires the network to be sufficiently attractive to all users, the service provider ultimately caters to the needs of the marginal user, since this ensures the largest possible demand at a given price. This difference between the choice of the service provider and the socially preferred allocation rules suggests that there may be room for regulation of power and transmission control in communication networks. Naturally, however, such government regulation introduces other potential inefficiencies, and whether regulation would be warranted once these inefficiencies are taken into account is an area we leave for future research.

[1]Expressed differently, intra-marginal users obtain a positive consumer surplus, in the sense that they have a positive net utility of participating in the network. Increasing their utility further would only increase their consumer surplus without providing additional profit to the service provider.

II. Model

A. Preliminaries

We study pricing to allocate resources in a cellular wireless network. We consider the downlink of a single cell, in which there are N potential users (i.e., users which contemplate using the service provided by this particular base station), and denote the set of users by $\mathcal{N} = \{1, \ldots, N\}$. Let p_i be the transmission power allocated by the base station to user i, and assume that the base station has a constraint on its total transmission power given by

$$\sum_{i=1}^{N} p_i \leq P_T. \tag{1}$$

Let h_i represent the channel gain of user i, i.e., $h_i p_i$ is the received power by user i. Then the rate at which the base station transmits to user i, denoted by x_i, is given by

$$x_i = \log\left\{1 + \frac{h_i p_i}{\sigma^2}\right\}, \tag{2}$$

where σ^2 is the background noise level. We assume that the rate is measured in nats per unit time. Note that if $h_i = 0$, then irrespective of the assigned power, user i will not transmit, i.e., $x_i = 0$. Therefore, we adopt the convention that $h_i = 0$ also stands for user i being inactive.

Combining (1) and (2), for a given set of channel gains $\{h_i\}_{i\in\mathcal{N}}$, we obtain the following constraint on the transmission rates $x_1, \ldots, x_N$ assigned to users:

$$\sum_{i\in\mathcal{N},\ h_i>0} \frac{\sigma^2}{h_i}\left(e^{x_i} - 1\right) \leq P_T, \tag{3}$$

and $x_i = 0$ if $h_i = 0$.

The distribution of the channel gain h_i, conditional on $h_i > 0$, depends on the location of the user in the cell and on random shadowing, while the probability of $h_i > 0$ is determined by the probability that a given user will be active. We assume that the $\{h_i\}$ are chosen from some probability distribution in the analysis that follows.

B. Allocation Rules

Our goal is to determine pricing strategies and power allocation rules for profit maximization, i.e., how should a service provider price resources to maximize revenue? In the literature, resource allocation is done to achieve a number of different fairness criteria. For example, the well-known proportional-fairness allocation rule chooses transmission rates which solve the maximization problem

$$\max \sum_{i=1}^{N} \log(1 + x_i)$$

subject to (3) above, for each realization of channel gains across users. This proportional-fairness allocation rule can be viewed as a special case of a class of allocation rules that maximize $v(\mathbf{x})$ subject to (3) above, where $v : \Re^N \to \Re$ is an arbitrary function capturing system objectives.

The question that we are interested in is which function $v(.)$ would be profit maximizing from the point of view of the service provider. It is clear that the problem can be studied either by thinking of the service provider choosing the function $v(.)$, or directly the allocation vector $x \in \Re^N$ as a function of the realization of all users' channel gains, $h \in \Re^N$, which turns out to be more convenient in our analysis.

C. Technology, Preferences and Notation

We first formally define the power allocation rule, or equivalently the rate allocation rule as a function of realizations of the channel gain vector. We assume that the channel gains of the participating users is characterized by a permutation invariant cumulative distribution. This implies *anonymity*, whereby the service provider cannot discriminate among users, except on the basis of their channel gains; so two users with the same channel gain will receive the same rate allocation, given the channel gains of other users. To facilitate the analysis, we introduce the following notation. Let M be the number of participating users and let H_M be a largest-cardinality set in $\Re^M$ such that if $\mathbf{h}, \tilde{\mathbf{h}} \in H_M$, then $\mathbf{h}$ and $\tilde{\mathbf{h}}$ are not permutations of each other. For each $M \in \mathcal{N}$, let $F(\mathbf{h}_M, M)$ be the distribution function defined over $\mathbf{h}_M \in H_M$; i.e., distribution function of the channel gain vector of M participating users.

We define the allocation rule when there are M participating users as a function $x_M : \Re \times H_{M-1} \mapsto \Re$ that, for each $(h, \hat{\mathbf{h}})$, assigns a rate $x(h, \hat{\mathbf{h}})$ to a user, when the channel gain of that user is the scalar h, and the channel gains of the remaining users are given by the $(M-1)$-dimensional vector $\hat{\mathbf{h}} \in H_{M-1}$. This definition of the allocation rule, in particular, the choice of $\Re \times H_{M-1}$ as the domain of the mapping x_M, imposes the anonymity assumption motivated above: the identity of the user and the ordering of the channel gains of the other users is irrelevant; i.e., the individual user with a channel gain h must be assigned the same rate regardless of its identity and the ordering of channel gains among remaining users. This type of restriction is also referred to as symmetry, and it basically rules out allocation rules where users with some characteristics (e.g., different demographics) are assigned different transmission rates even when they have the same channel gain.

We assume that user i has an increasing and concave utility function $u_i(x)$ with $u_i(0) = 0$, which specifies the amount he is willing to pay if he is assigned the deterministic rate of transmission x. Since we have uncertainty in the system regarding channel gains and transmission rates, we use the expected utility theory, which states that user preferences under uncertainty can be represented by an expected utility function (also known as von-Neumann-Morgenstern utility function) U_i, which for M participating users and an allocation rule $x_M(\cdot)$ is given by

$$U_i\left(x_M(\cdot), M\right) = E_{\mathbf{h}_M}\left[u_i(x_M(\mathbf{h}_M))\right]. \tag{4}$$

Two features are worth noting. First, the concavity of $u_i(\cdot)$ is essential in this formulation. In standard economic applications, this corresponds to the *risk aversion* of the user. In this case, we can interpret it either as risk aversion, for example, because users dislike potential variability in transmission rates and service quality, or as flexibility. With the latter interpretation, a more concave utility function implies that the user has little flexibility regarding when he or she can transmit whereas a less concave (closer to linear) utility function would capture greater flexibility. Second, interpreted literally, this formulation corresponds to a situation where each user transmits only once, under a particular channel gain drawn from a distribution. An alternative interpretation may be more appealing, whereby users, after entering the network, will transmit a large number of times under varying channel gains, and therefore $U_i\left(x_M(\cdot), M\right)$ is the "average" payoff they will obtain once they are part of the network. Mathematically, these two interpretations are equivalent. This alternative interpretation would also be complementary to the interpretation of the concavity of the $u_i(\cdot)$ function as capturing the degree of flexibility of users.

Given a price q, M participating users, and an allocation rule $x_M(\cdot)$, the net utility of user i, $i = 1, \ldots, N$, can be expressed as

$$e_i\left(U_i\left(x_M(\cdot), M\right) - q\right),$$

where e_i is a binary participation decision variable, $e_i \in \{0, 1\}$ for user i, such that $e_i = 1$ if user i decides to participate in the network, and $U_i\left(x_M(\cdot), M\right)$ is the expected utility of user i, see Eq. (4). It is clear that the user utility depends on the number of participating users. Hence each user, when deciding whether to participate, needs to form conjectures about the behavior of other users, which they do according to the following user equilibrium definition.

Given a price q, and a class of allocation functions $\{x_M(\cdot)\}_{M \in \mathcal{N}}$, we say that a vector $e = \{e_i\}_{i \in \mathcal{N}} \in \{0, 1\}^N$ is a user equilibrium if $e_i = 1$ only if $U_i\left(x_M(\cdot), M\right) \geq q$ and $M = \sum_{i=1}^{N} e_i$. The optimality condition states that user i will participate only if the expected utility from transmission is greater than the cost of participating in the network. Moreover, the set of optimal solutions should be a fixed point of the best responses, which are functions of M. Note that the user equilibrium notion is similar to the Wardrop equilibirum of transportation networks, see [15], where each user is treated small and does not anticipate the effects of its actions.

The service provider's profit maximization problem can

be written as

$$\max_{q\geq 0,\ \{x_M(\cdot)\}} q\sum_{i=1}^{N} e_i, \tag{5}$$

subject to

$$g_M(\mathbf{k}) \leq P_T, \qquad \forall\ M,\ \forall\ \mathbf{k}\in H_M,$$

where

$$g_M(\mathbf{k}) = \sum_{i\in\{1,\ldots,M\},\ k_i>0} \frac{\sigma^2}{k_i}\left(e^{x_M(h=k_i,\hat{\mathbf{h}}=\mathbf{k_{-i}})}-1\right),$$

and $\{e_i\}_{i\in\mathcal{N}}$ is the user equilibrium defined above.

The model we have outlined corresponds to a dynamic game with the following timing of events:

- The service provider announces an admission price q and a family of allocation rules $\{x_M(\cdot)\}_{M\in\mathcal{N}}$.
- All potential users simultaneously decide whether or not to enter the network.
- The channel gains of all participating users, $\mathbf{h}_M$ is realized, and the pre-specified allocation rule, $x_M(\mathbf{h}_M)$, is implemented.

Characterizing the optimal admission price and allocation rule from the viewpoint of the service provider corresponds to finding the subgame perfect equilibrium of this dynamic game. Here, every different $(q,\{x_M(\cdot)\})$ defines a different subgame. The subgame perfect equilibrium of this game is given by the optimal solution of problem (5) and the corresponding user equilibrium. For our purposes, we can focus on the allocation rule along the equilibrium path and represent the subgame perfect equilibrium as a tuple $\left(q^*, x^*_{M^*}(\cdot), \{e_i^*\}_{i\in\mathcal{N}}, M^*\right)$ that maximizes

$$\max_{q\geq 0,\ x_M(\cdot),\ \{e_i\},M} q\sum_{i=1}^{N} e_i, \tag{6}$$

subject to

$$g_M(\mathbf{k}) \leq P_T, \qquad \forall\ \mathbf{k}\in H_M,$$
$$e_i = 1 \text{ only if } U_i\left(x_M(\cdot),M\right)\geq q,$$
$$\sum_{i=1}^{N} e_i = M.$$

We refer to this problem as the service provider (SP) problem. Also, with some abuse of notation, we refer to $(q^*, x^*_{M^*}(\cdot), M^*)$ as an SP equilibrium. One can also view the above game as a Stackelberg game [11], with the service provider as the leader and the potential users as the followers.

III. Analysis

We now provide explicit analysis and characterization of optimal prices and optimal allocation rules (from the service provider viewpoint). For expositional convenience, we start with a number of cases with special distribution of utilities across users, such as proportional or ordered utilities, building up to the analysis of the general case.

A. Users with Ordered Utilities

CASE 1: We assume that user i has a utility function

$$u_i(x) = \gamma_i u(x), \tag{7}$$

where γ_i is the utility gain parameter of user i and $u(x)$ is an increasing concave function with $u(0)=0$. Let us assume without loss of generality that

$$\gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_N.$$

Let M denote the number of participating users. In view of the permutation invariant assumption on the distribution function $F(h_M, M)$, the expected utility function for user i, given M participating users and an allocation rule $x_M(\cdot)$ can be expressed as

$$U_i\left(x_M(\cdot),M\right) = \gamma_i U\left(x_M(\cdot),M\right),$$

with

$$U\left(x_M(\cdot),M\right) = \int_{H_M}\left[\frac{1}{M}\sum_{i=1}^{M} u\left(x(h=k_i,\hat{\mathbf{h}}=\mathbf{k_{-i}})\right)\right] dF(\mathbf{k},M), \tag{8}$$

where $\mathbf{k}=(k_i,\mathbf{k_{-i}})\in H_M$ and $\mathbf{k_{-i}}$ denotes the $(M-1)$-dimensional vector without the i^{th} component.

Proposition 1: Let each user have utility function $u_i(x)$ given by Eq. (7). Let $(q^*, x^*_{M^*}(\cdot), M^*)$ be an SP equilibrium. Then $x^*_{M^*}(\cdot)$ can be obtained pointwise, i.e., for each $\mathbf{k}\in H_{M^*}$, the M^* values, $x^*_{M^*}(h=k_i,\hat{\mathbf{h}}=\mathbf{k_{-i}})$, $i=1,\ldots,M^*$, are found by solving the M^*-dimensional optimization problem

$$\max \frac{1}{M^*}\sum_{i=1}^{M^*} u\left(x_{M^*}(h=k_i,\hat{\mathbf{h}}=\mathbf{k_{-i}})\right)$$

subject to

$$x_{M^*}(h=k_i,\hat{\mathbf{h}}=\mathbf{k_{-i}})=0, \qquad \text{if } k_i=0, \tag{9}$$
$$g_{M^*}(\mathbf{k}) \leq P_T,$$

where

$$g_{M^*}(\mathbf{k}) = \sum_{i\in\{1,\ldots,M^*\},\ k_i>0} \frac{\sigma^2}{k_i}\left(e^{x_{M^*}(h=k_i,\hat{\mathbf{h}}=\mathbf{k_{-i}})}-1\right).$$

Moreover, we have $q^* = \gamma_{M^*} U\left(x^*_{M^*}(\cdot),M^*\right)$.

proof: In view of the relation between the γ_i's, it can be seen that $(q^*, x_{M^*}(\cdot), M^*)$ is an optimal solution of

$$\max_{q,x_M(\cdot),M} qM$$

subject to

$$x_M(h=k_i,\hat{\mathbf{h}}=\mathbf{k_{-i}})=0, \qquad \forall\ \mathbf{k}\in H_M \text{with } k_i=0,$$
$$g_M(\mathbf{k}) \leq P_T \qquad \forall\ \mathbf{k}\in H_M,$$
$$\gamma_M \geq \frac{q}{U\left(x_M(\cdot),M\right)}.$$

Since at the optimal solution, the second inequality constraint is satisfied as an equality, it follows that $x^*_{M^*}(\cdot)$ is also an optimal solution of the problem

$$\max_{x_{M^*}(\cdot)} U\left(x_{M^*}(\cdot), M^*\right)$$

subject to

$$x_{M^*}(h = k_i, \hat{\mathbf{h}} = \mathbf{k_{-i}}) = 0, \qquad \forall\ \mathbf{k} \in H_{M^*} \text{with } k_i = 0,$$

$$g_{M^*}(\mathbf{k}) \leq P_T \qquad \forall\ \mathbf{k} \in H_{M^*}.$$

By Eq. (8), this problem has a separable structure and the optimal allocation rule $x^*_{M^*}$ can be obtained pointwise for each $\mathbf{k} \in H_{M^*}$ as stated in the proposition. **Q.E.D.**

The user M^* is the *marginal user*, in the sense that all users with index smaller than M^* will participate in the network *a fortiori* when M^*participates, while those above M^* choose not to participate. In other words, since $q^* = \gamma_{M^*} U\left(x^*_{M^*}(\cdot), M^*\right)$, user $M^*$is indifferent between joining the network or not.

That the service provider maximizes the utility of the marginal user is intuitive. If it is possible to increase the utility of the marginal user, $\gamma_{M^*} U\left(x^*_{M^*}(\cdot), M^*\right)$, this will allow the service provider to also increase q^* by the same amount, while still ensuring that $q^* \leq \gamma_m U\left(x^*_{M^*}(\cdot), M^*\right)$ for all $m < M^*$. Therefore, it can increase the price without reducing the number of participants and raise profits. In the optimum, there should be no possibility to raise profits further, and hence the expected utility of the marginal user should be maximized.

A special case of this proposition is when $\gamma_i = \gamma$ for all i. In this case, it can be shown that the SP equilibrium has an identical allocation rule and identical number of participating users to the social optimum (i.e., the allocation and the participation decisions that would be chosen by a planner that maximizes the sum of the expected utilities of all potential users). However, when the γ_i's are different, the SP allocation rule and participation decisions will not be optimal from a social point of view—some users that a social planner would have admitted will typically be excluded by the service provider. It is worth noting that although the allocation rule chosen by the service provider differs from the socially optimal allocation rule, it *does* coincide with the restricted social optimal allocation rule where the system is limited to accept only M^* users. In fact, the allocation rule maximizes the expected utility of all participating users. Although in this case, the allocation rule of the SP equilibrium is socially optimal, conditional on accepting M^* users, we next see that this result is not true in general.

CASE 2: In this case, we assume that user i has an increasing concave utility function $u_i(x)$ that satisfies

$$u_1(x) \geq u_2(x) \geq \cdots \geq u_N(x), \qquad \forall\ x \in [0, \infty). \quad (10)$$

Compared to the assumption in Case 1, this is a fairly weak restriction on the utility functions, requiring that the utility functions do not cross. This essentially amounts to stating that if a particular user values transmission more than another user at some rate, he or she will value transmission at all other rates also more than this user. This assumption will allow us to rank users and define a clear marginal user as in the previous case. The expected utility function for user j, given M participating users and an allocation rule $x_M(\cdot)$, can be written as

$$U_j\left(x_M(\cdot), M\right) = \int_{H_M} \left[\frac{1}{M} \sum_{i=1}^{M} u_j\left(x_M(h = k_i, \hat{\mathbf{h}} = \mathbf{k_{-i}})\right)\right] dF(\mathbf{k}, M). \quad (11)$$

Proposition 2: Let the utility functions $u_i(x)$ satisfy Eq. (10). Let $(q^*, x^*_{M^*}(\cdot), M^*)$ be an SP equilibrium. Then the optimal allocation rule $x^*_{M^*}(\cdot)$ can be obtained pointwise, i.e., for each $\mathbf{k} \in H_{M^*}$, the M^* values, $x^*_{M^*}(h = k_i, \hat{\mathbf{h}} = \mathbf{k_{-i}})$, $i = 1, \ldots, M^*$, are found by solving the M^*-dimensional optimization problem

$$\max \frac{1}{M^*} \sum_{i=1}^{M^*} u_{M^*}\left(x_{M^*}(h = k_i, \hat{\mathbf{h}} = \mathbf{k_{-i}})\right)$$

subject to Eq. (9) and

$$g_{M^*}(\mathbf{k}) \leq P_T.$$

proof: In view of the ordered structure of the utility functions, the proof follows similar steps to those of Proposition 1, and is therefore omitted here. **Q.E.D.**

First, note that, with a similar interpretation to before, the allocation rule maximizes user expected utility, but now, it is not the expected utility of all users, but of the marginal user, M^*.

The intuition for why the utility of the marginal users should be maximized is the same as before. However, the implications are different; the optimum from the point of the service provider does not maximize the sum of utilities of potential users (even conditional on the number of users admitted). In all cases, it maximizes simply the utility of the marginal user, which differs from the intra-marginal users. Consequently, the allocation rule is also very different from a socially optimal rule.

This result helps us clarify the previous results we have obtained: in all cases, the service provider always maximizes the expected utility of the marginal user. In the previous cases, the allocation rule that maximized utility of the marginal user also happens to maximize the utility of all users.

B. Arbitrary Utility Functions

In this case, we assume that each user i has an arbitrary, strictly concave utility function $u_i(x)$ (i.e., we allow

crossing utility functions). Similar to the previous case, the expected utility function for user j is given by Eq. (11).

The SP problem in this case can be written as

$$\max_{M,\ S(M),\ q,\ x_M(\cdot)} qM \tag{12}$$

subject to

$$x_M(h = k_i, \hat{\mathbf{h}} = \mathbf{k_{-i}}) = 0, \qquad \forall\ \mathbf{k} \in H_M \text{with } k_i = 0,$$

$$g_M(\mathbf{k}) \leq P_T, \qquad \forall\ \mathbf{k} \in H_M,$$

$$U_i\left(x_M(\cdot), M\right) \geq q, \qquad \forall\ i \in S(M),$$

where $S(M)$ is a subset of users with cardinality equal to M (i.e., the set of participating users, not necessarily $\{1, \ldots, M\}$ in this case). We have the following proposition. The proof is omitted due to lack of space.

Proposition 3: Let $(M^*, S(M^*), q^*, x^*_{M^*}(\cdot))$ be an optimal solution of problem (12). Then there exists an m^* and a possibly empty set $R_{m^*} \subset S(M^*)$ such that, $x^*_{M^*}(\cdot)$ is an optimal solution of

$$\max_{x_{M^*}(\cdot)} U_{m^*}\left(x_{M^*}(\cdot)\right)$$

subject to Eq. (9) and

$$U_m\left(x_{M^*}(\cdot)\right) = U_{m^*}\left(x_{M^*}(\cdot)\right), \qquad \forall\ m \in R_{m^*},$$

$$g_{M^*}(\mathbf{k}) \leq P_T, \forall\ \mathbf{k} \in H_{M^*},$$

and $q^* = U_m\left(x^*_{M^*}(\cdot)\right)$ for all $m \in R_{m^*} \cup \{m^*\}$. If $R_{m^*} = \emptyset$, then $x^*_{M^*}(\cdot)$ can be found pointwise, i.e., for each $\mathbf{k} \in H_{M^*}$, the M^* values, $x^*_{M^*}(h = k_i, \hat{\mathbf{h}} = \mathbf{k_{-i}})$, $i = 1, \ldots, M^*$, are found by solving the M^*-dimensional optimization problem

$$\max \frac{1}{M^*} \sum_{i=1}^{M^*} u_{m^*}\left(x_{M^*}(h = k_i, \hat{\mathbf{h}} = \mathbf{k_{-i}})\right)$$

subject to Eq. (9) and

$$g_{M^*}(\mathbf{k}) \leq P_T.$$

Since the utility functions may cross in this case, there may be more than one marginal user class (i.e., users with different utility functions), which is given by the set $R_{m^*} \cup \{m^*\}$. $R_{m^*} = \emptyset$ corresponds to the case when there is a single marginal user. Even when $R_{m^*} \neq \emptyset$, the rate allocation rule is determined using the utility functions of marginal users, so the marginal user principle still applies. For the special case when H_{M^*} is a singleton[2], the optimal allocations $x^*_{M^*}(h = k_i, \hat{\mathbf{h}} = \mathbf{k_{-i}})$, $i = 1, \ldots, M^*$, are found by solving the problem

$$\max \frac{1}{M^*} \sum_{i=1}^{M^*} u_{m^*}\left(x_{M^*}(h = k_i, \hat{\mathbf{h}} = \mathbf{k_{-i}})\right)$$

[2]This would happen when h_i takes a finite number of values, and there is a large number of users such that every channel gain realization of admitted users is a permutation of the same vector

subject to Eq. (9) and

$$\frac{1}{M^*} \sum_{i=1}^{M^*} u_m\left(x_{M^*}(h = k_i, \hat{\mathbf{h}} = \mathbf{k_{-i}})\right)$$
$$= \frac{1}{M^*} \sum_{i=1}^{M^*} u_{m^*}\left(x_{M^*}(h = k_i, \hat{\mathbf{h}} = \mathbf{k_{-i}})\right),\ \forall\ m \in R_{m^*},$$

$$g_{M^*}(\mathbf{k}) \leq P_T.$$

IV. Conclusions

In this paper, we have studied a power allocation problem where a service provider sets an entry price and announces a power allocation strategy and then users decide to join the network or not. It is shown that the optimal power allocation scheme maximizes the utility of the marginal user. This is in contrast to the widely-studied social welfare maximization and fairness criteria. The marginal user principle highlighted by our analysis is not restricted to the power allocation problem in wireless networks. Similar results hold in other resource allocation problems in communication networks.

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