No Trade and Feasible Joint Posterior Beliefs*

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1 Introduction

An important paper of Arieli et al (2020) provides a characterization (in Theorem 3) of which joint distributions of posteriors are consistent with a common prior in the two state case. They note a "no trade" interpretation of their Theorem and they use a no trade idea to prove one direction of the Theorem. The other direction relies on results in probability theory. To better understand the relation to no trade results, I sketch an alternative proof (for the special case where the prior over posteriors has finite support) where both directions follow from classical no trade results, thus clarifying the connection. This also delivers a many state generalization of Theorem 3 of Arieli et al (2020). And I also take to opportunity to discuss some history of the no trade results.

2 The No Trade Theorem

Let there be $H+1$ agents. Let T_h be a finite set of types of agent h. Let $T = \underset{h=0,H}{\times} T_h$. Let Θ be a finite set. Each agent h has a prior $\pi_h \in \Delta(T \times \Theta)$ which assigns positive probability to each t_h. So the conditional belief $\pi_h(t_{-h}, \theta | t_h)$ is well defined:

$$\pi_h(t_{-h}, \theta | t_h) = \frac{\pi_h((t_h, t_{-h}), \theta)}{\sum_{t'_{-h} \in T_{-h}} \sum_{\theta' \in \Theta} \pi_h((t_h, t'_{-h}), \theta')}$$

for each $h = 0, .., H$, $(t_h, t_{-h}) \in T$ and $\theta \in \Theta$.

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Definition 1 (common prior assumption) *Priors* $(\pi_h)_{h=0}^H$ *satisfy the common prior assumption if there exist, for each* h, $\lambda_h : T_h \to \mathbb{R}_{++}$, *and* $\phi \in \Delta(T \times \Theta)$, *such that*

$$\lambda_h(t_h)\,\pi_h(t_{-h}, \theta | t_h) = \phi((t_h, t_{-h}), \theta)$$

for all $h = 0, ..., H$, $(t_h, t_{-h}) \in T$ *and* $\theta \in \Theta$.

A trade is a collection $(x_h)_{h=0}^H$, where each $x_h : T \times \Theta \to \mathbb{R}$.

Definition 2 *A trade is feasible if*

$$\sum_{h=0}^{H} x_h(t, \theta) \leq 0$$

for all $t \in T$ *and* $\theta \in \Theta$.

Definition 3 *A trade is acceptable if*

$$\sum_{t_{-h} \in T_{-h}} \sum_{\theta \in \Theta} \pi_h(t_{-h}, \theta | t_h)\, x_h((t_h, t_{-h}), \theta) \geq 0$$

for all $h = 0, ..., H$ *and* $t_h \in T_h$, *with strict inequality for some* $h = 0, ..., H$ *and* $t_h \in T_h$.

Definition 4 (no trade) *No trade holds if there does not exist a feasible and acceptable trade.*

Theorem 1 (no trade) *There is no trade if and only if the common prior holds.*

This result first appears as Theorem 1a in chapter 2 of Morris (1991), proved via Farkas' lemma. Note that $\phi((t_h, t_{-h}), \theta)$ is the multiplier of the feasibility constraints for (t, θ), $\lambda_h(t_h)$ is the multiplier of acceptance constraint of type t_h of player h. We can then normalize the sum of $\phi((t_h, t_{-h}), \theta)$ to 1 to make it a probability distribution.

This result provides a converse to the "no trade theorem", establishing that the common prior was necessary as well as sufficient for no trade. This is specifically a converse to the no trade theorem of Sebenius and Geanakoplos (1983), which pointed out that risk neutral agents do not trade if the common prior assumption holds. However, the same logic can be used to provide a converse to other versions of the no trade theorem with risk averse agents, as discussed below.

3 Digression on Terminology and Literature

3.1 Terminology, Interpretation and Formulation

In chapter 2 of my thesis (Morris (1991)) (which was also my job market paper), I followed Harsanyi (1967/68) and others in calling the property in definition 1 "consistent priors" instead of "the common

prior assumption" because I was taking an ex ante perspective and, from an ex ante perspective, this is a weakening of the common prior assumption since it puts no restrictions on agents' beliefs about their own types. However, perhaps starting from the dissertation of Yossi Feinberg (1996), it has become standard to call this property "the common prior assumption" meaning that interim/conditional beliefs are consistent with the existence of a common prior.

The prior "no trade" literature (showing that the common prior implied no trade) mostly considered environments allowing risk averse agents and I maintained that terminology. I referred to the special case of risk neutral agents as the "no betting" case, probably inspired by Sebenius and Geanakoplos (1983). It has also now become standard to refer to the risk neutral version as no trade, and I follow that new convention in this note.

Some authors in the no trade literature represent asymmetric information by partitions on a fixed state space, while others represent it by type spaces. Obviously, both approaches are equivalent but they make formulations a little different. I used both formulations in Morris (1991) but just the type space formulation in Morris (1994). Feinberg (1996, 2000) and Samet (1998), discussed below, used the partition formulation.

I (foolishly) did not attach a lot of importance to this result at the time. It seemed straightforward. My (ex ante) interpretation was that it showed that differences in prior beliefs do not always lead to trade. The focus of my job market paper (chapter 2 of my thesis) was on extensions, and in particular showing how further natural restrictions on trade (e.g., making agents' signals private and thus requiring incentive compatible elicitation) imposed further restrictions on the set of differences in priors that lead to trade. In the version of my job market paper published in *Econometrica* (Morris (1994)), I was encouraged to focus on the case of risk averse agents (where, among things, I provided a tight converse to Milgrom and Stokey (1982)); the case of risk neutral traders was treated as a special case in the appendix). Thus the above no trade result ended up somewhat buried as part (iv) of Lemma A2 in the appendix, which states that the endowment is interim efficient if and only if beliefs are consistent. "Beliefs are consistent" is - in modern language - the common prior assumption. The "interim efficient" terminology was used to relate the result to results in the body of the paper. In the risk neutral case of Lemma A2, interim efficiency is equivalent to the non-existence of an acceptable trade.

3.2 Literature

The closest precursor to my result - which I apparently did not know about at the time - was Nau and McCardle (1990). They analyzed the implications of rational play in a fixed game if, in addition, there is no arbitrage in the sense that players do now want to make bets with an outside observer

to the game about play in the game. They showed that it implied that the resulting play would correspond to a correlated equilibrium. Their objective was to avoid invoking the common prior and they interpreted their result as providing a a more satisfactory foundation for correlated equilibrium. But their argument implicitly showed an equivalence between Harsanyi consistent priors (i.e., the interim version of the common prior assumption defined above) and a "no arbitrage", or no trade, condition.

There was a vigorous debate in 1990s about how one could interpret the common prior assumption when one's model of the world was in fact interim, i.e., there was no prior stage. See, e.g., Gul (1998) and Aumann (1998). Yossi Feinberg (a student of Bob Aumann) was interested in characterizing the common prior assumption as a property of interim/conditional beliefs only (definately not taking an ex ante perspective). In his thesis (Feinberg (1996)), he proved the above finite no trade result (via the minmax theorem for zero sum games), without knowing about my result but later became aware of it and has always credited it, i.e., in his thesis and published work. His thesis also addressed when the result did or did not hold in infinite type spaces and provided a syntactic characterization. His thesis results were slow to be published (Feinberg (2000)). The above finite no trade result appears in Feinberg (2000) as Theorem 2 (for the two player case) and Theorem 3 (for more than two players), with the emphasis on the syntactic characterization.

Samet (1998) provided an alternative proof of the above finite no trade theorem. He was aware of and inspired by the results of Yossi and I and notes that his main proposition was already proved by Yossi and I. My recollection is that he found the results of Yossi and I mysterious and wanted to understand them better. He provided an elegant geometric argument. In particular, he noted that the common prior assumption (as defined above) has a geometric interpretation as the requirement that the intersection of the convex hulls of agents' posteriors (on the space $T \times \Theta$) is non-empty. And he proved the above no trade result using a generalization of the separating hyperplane theorem he introduced. Although this work was published prior to Feinberg (2000), it was written in response to Feinberg's 1996 thesis and Morris (1991, 1994).

Thus the result is due to Morris (1991) (published as Morris (1994)) and/or Nau and McCardle (1990), was independently discovered by Feinberg (1996) (published as Feinberg (2000)); and, Samet (1998) provides an alternative elegant (geometric) proof. All proofs use different versions of duality.

4 No Trade Variations

We now report a sequence of variations of the above no trade result leading in the direction of Arieli et al (2020) Theorem 3. We first assume (and will maintain throughout this section) that agent 0 is

uniinformed, i.e., $T_0 = \{\varnothing\}$. This will allow us to replace "no trade" conditions with "no money pump" conditions. We will sometimes refer to agents $1, ..., H$ as the regular agents and player 0 as the outside agent.

4.1 Uninformed Agent

If there is an uninformed outside agent, his prior must be the common prior:

Remark 1 (common prior assumption) *Priors $(\pi_h)_{h=0}^H$ satisfy the common prior assumption if there exist, for each $h = 1, .., H$, $\lambda_h : T_h \to \mathbb{R}_{++}$, such that*

$$\lambda_h(t_h)\,\pi_h(t_{-h}, \theta | t_h) = \pi_0((t_h, t_{-h}), \theta)$$

for all $(t_h, t_{-h}) \in T$ and $\theta \in \Theta$.

We will be interest in trades for the regular players $(x_h)_{h=1}^H$ where gains from trade are 0:

Definition 5 (zero value trade) *$(x_h)_{h=1}^H$ is a zero value trade (for agents $1, .., H$) if*

$$\sum_{t_{-h} \in T_{-h}} \sum_{\theta \in \Theta} \pi_h(t_{-h}, \theta | t_h)\, x_h((t_h, t_{-h}), \theta) = 0$$

for all $h = 1, .., H$ and $t_h \in T_h$.

We say there is no money pump if the outside agent cannot guarantee positive profits from a zero value trade.

Definition 6 *No money pump holds if, for every zero value trade $(x_h)_{h=1}^H$,*

$$\sum_{t \in T} \sum_{\theta \in \Theta} \pi_0(t, \theta) \sum_{h=1}^{H} x_h(t, \theta) \geq 0.$$

The condition requires that the uninformed outside agent cannot make an expected profit offering bets to the agents they are prepared to take.

Lemma 1 (equivalence of no trade and no money pump) *There is no trade if there is no money pump.*

Proof. This is mechanical. We show that the existence of a trade implies a money pump (the other direction is immediate). Suppose $(x_h)_{h=0}^H$ is a feasible and acceptable trade. Then

$$y_h(t_h) = \sum_{t_{-h} \in T_{-h}} \sum_{\theta \in \Theta} \pi_h(t_{-h}, \theta | t_h)\, x_h((t_h, t_{-h}), \theta) \geq 0$$

for each $h = 1, .., H$ and t_h. Let

$$\widehat{x}_h(t, \theta) = x_h(t, \theta) - y_h(t_h)$$

for each $h = 1, .., H$, $t \in T$ and $\theta \in \Theta$. Observe that $(\widehat{x}_h)_{h=1}^H$ is a zero sum trade by construction. Now

$$\begin{aligned}
0 &\leq \sum_{t \in T} \sum_{\theta \in \Theta} \pi_0(t, \theta) x_0(t, \theta), \text{ by acceptability of trade for agent 0} \\
&\leq -\sum_{t \in T} \sum_{\theta \in \Theta} \pi_0(t, \theta) \sum_{h=1}^{H} x_h(t, \theta), \text{ by feasibility} \\
&\leq \sum_{t \in T} \sum_{\theta \in \Theta} \pi_0(t, \theta) \sum_{h=1}^{H} (y_h(t_h) - x_h(t, \theta)), \text{ by non-negativity of } y_h \\
&= -\sum_{t \in T} \sum_{\theta \in \Theta} \pi_0(t, \theta) \sum_{h=1}^{H} \widehat{x}_h(t, \theta), \text{ by definition of } \widehat{x}_h
\end{aligned}$$

The "at least one strict inequality" requirement in acceptability implies that this inequality is strict, so

$$\sum_{t \in T} \sum_{\theta \in \Theta} \pi_0(t, \theta) \sum_{h=1}^{H} \widehat{x}_h(t, \theta) < 0$$

The other direction is immediate. ■

Corollary 1 (No Money Pump) *There is no money pump if and only if priors $(\pi_h)_{h=0}^H$ satisfy the common prior assumption.*

4.2 Θ-Measurable Trades

Now suppose that regular players only bet about the state Θ, but not about others' types:

Definition 7 (Θ-measurability) *A trade $(x_h)_{h=0}^H$ is Θ-measurable if x_h is measurable with respect to (t_h, θ) for all $h = 1, H$, so $x_h : T_h \times \Theta \to \mathbb{R}$.*

Note that we put no restriction on the outside agent's prior in this definition. We will write

$$\begin{aligned}
\pi_h(\theta | t_h) &= \sum_{t_{-h} \in T_{-h}} \pi_h(t_{-h}, \theta | t_h) \text{ for each } h = 1, .., H \\
\text{and } \pi_0(t_h, \theta) &= \sum_{t_{-h} \in T_{-h}} \pi_0((t_h, t_{-h}), \theta)
\end{aligned}$$

We consider a relaxation of the common prior assumption that requires agreement about Θ and T_h bilaterally between the outside agent 0 and each regular agent $h = 1, .., H$.

Definition 8 (Θ-consistency) *Priors* $(\pi_h)_{h=0}^H$ *are* Θ*-consistent if there exist, for each* $h = 1, , H$, $\lambda_h : T_h \to \mathbb{R}_{++}$, *such that*

$$\lambda_h(t_h)\,\pi_h(\theta|t_h) = \pi_0(t_h,\theta)$$

for all $h = 1, .., H$, $t_h \in T_h$ *and* $\theta \in \Theta$.

Lemma 2 (no Θ-measurable trade) *There is no* Θ*-measurable trade if and only if priors are* Θ*-consistent.*

Thus the no trade condition is weakened by looking a restricted class of trade and this leads to a relaxation of the common prior assumption. This follows from the same Farkas' lemma logic as Theorem 1. If the outside agent was also restricted to a Θ-measurable trade, this result would be a special case of Theorem 1c in Morris (1991) and Lemma A2(vi) in Morris (1994). The proof can be adapted straightforwardly to this result.

Proof. There is Θ-measurable trade if there do not exist $x_0 : T \times \Theta \to \mathbb{R}$ and, for each h, $x_h : T_h \times \Theta \to \mathbb{R}$, such that

$$\begin{aligned}
\sum_{t\in T,\theta\in\Theta} \pi_0(t,\theta)\,x_0(t,\theta) &\geq 0 \\
\sum_{t_{-h}\in T_{-h},\theta\in\Theta} \pi_h(\theta|t_h)\,x_h(t_h,\theta) &\geq 0 \text{ for each } h = 1, .., H \\
\sum_{h=0}^{H} x_h(t,\theta) &\leq 0
\end{aligned}$$

with strict inequality for one of the acceptability constraints. By Farkas' lemma, this is true if and only if there do not exist $\lambda_0 \in \mathbb{R}_{++}$, $\lambda_h : T_h \to \mathbb{R}_{++}$ for each $h = 1, ..., H$ and $\phi : T \times \Theta \to \mathbb{R}_+$ such that

$$\begin{aligned}
\lambda_h(t_h)\,\pi_h(\theta|t_h) &= \sum_{t_{-h}\in T_{-h}} \phi((t_h,t_{-h}),\theta) \text{ for all } t_h \in T_h \text{ and } \theta \in \Theta \\
\text{and } \lambda_0\pi_0(t,\theta) &= \phi(t,\theta) \text{ for all } t \in T \text{ and } \theta \in \Theta
\end{aligned}$$

Normalizing $\lambda_0 = 1$, this is equivalent to Θ-consistency of priors. ■

There is also an equivalent money pump formulation:

Definition 9 (no Θ-measurable money pump) *No* Θ*-measurable money pump holds if for every* Θ*-measurable, zero value trade* $(x_h)_{h=1}^H$,

$$\sum_{t\in T}\sum_{\theta\in\Theta} \pi_0(t,\theta) \sum_{h=1}^{H} x_h(t_h,\theta) \geq 0.$$

Remark 2 (equivalence of no Θ-measurable trade and no Θ-measurable money pump) *There is no feasible, Θ-measurable and acceptable trade if and only if there is no Θ-measurable money pump.*

Corollary 2 (no Θ-measurable money pump) *There is no Θ-measurable money pump if and only if priors are Θ-consistent.*

4.3 First Order Beliefs

Let $\widetilde{f}_h(t_h) \in \Delta(\Theta)$ be type t_h's first order belief about Θ, i.e.,

$$\widetilde{f}_h(t_h)[\theta] = \sum_{t_{-h} \in T_{-h}} \pi_h(t_{-h}, \theta | t_h)$$

Write $F_h \subseteq \Delta(\Theta)$ be the range of $\widetilde{f}_h$, so $\widetilde{f}_h : T_h \to F_h$; f_h for a typical element of F_h; and $F = F_1 \times .... \times F_H$.

Now if we are interested in the Θ-restricted trades of the previous section, a pair of types with the same first order beliefs, i.e., $\widetilde{f}_h(t'_h) = \widetilde{f}_h(t_h)$, will obviously have the same willingness to accept Θ-restricted trades.

So let us from now on consider the coarsened model where $T_h = F_h$ for each $h = 1, ..., H$ and $\pi^*_h \in \Delta(F \times \Theta)$ is given by

$$\pi^*_h(f, \theta) = \sum_{\left\{(t_1,...,t_H): \widetilde{f}_h(t_h) = f_h \text{ for all } h\right\}} \pi_h(t, \theta)$$

In this coarsened model, the interim beliefs of an agent are the label of the agent. So Θ-consistency on the coarsened type space can be written as:

Definition 10 (Θ-consistency) *Priors $(\pi^*_h)_{h=0}^H$ are Θ-consistent if there exist, for each $h = 1, , H$, $\lambda_h : F_h \to \mathbb{R}_{++}$, such that*

$$\lambda_h(f_h) f_h(\theta) = \pi^*_0(f_h, \theta)$$

for all $h = 1, .., H$, $f_h \in F_h$ and $\theta \in \Theta$.

But note that this condition can be understood as a restriction on $\pi^*_0 \in \Delta(F \times \Theta)$ alone as player h's first order beliefs reveals all we need to know about priors π_h for $h = 1, ..., H$.

Thus we re-state the same property with a different interpretation.

Definition 11 (common prior consistency) *Probability distribution $\pi_0^* \in \Delta(F \times \Theta)$ over first order beliefs and the state space is consistent with the common prior if and only if there exist, for each $h = 1, , H$, $\lambda_h : F_h \to \mathbb{R}_{++}$, such that*

$$\lambda_h(f_h) f_h(\theta) = \pi_0^*(f_h, \theta)$$

for all $h = 1, .., H$, $f_h \in F_h$ and $\theta \in \Theta$.

Remark 3 *"Θ-consistency of $(\pi_h^*)_{h=0}^H$" and "common prior consistency of $\pi_0^* \in \Delta(F \times \Theta)$" are the same property re-interpreted.*

We note that the "zero value trade" condition becomes a little simpler to state:

Definition 12 (Θ-measurable zero value trade) *$(x_h)_{h=1}^H$ is a zero value trade (for agents $1, .., H$) if*

$$\sum_{\theta \in \Theta} f_h(\theta) x_h(f_h, \theta) = 0$$

for all $h = 1, .., H$ and $f_h \in F_h$.

Definition 13 (no Θ-measurable money pump) *No Θ-measurable money pump holds if for every Θ-measurable, zero value trade $(x_h)_{h=1}^H$,*

$$\sum_{f \in F} \sum_{\theta \in \Theta} \pi_0^*(f, \theta) \sum_{h=1}^{H} x_h(f_h, \theta) \geq 0.$$

Now we have a characterization of distributions over the state and first order beliefs consistent with the common prior derived from a variation of the original no trade theorem:

Theorem 2 (Feasible Distributions over First Order Beliefs and the State) *Distribution $\pi_0^* \in \Delta(F \times \Theta)$ over first order belief profile f and state of the world θ is consistent with the common prior if and only if there is no Θ-measurable money pump*

The theorem is just a re-statement of Corollary 2. This is thus a characterization of distributions over the state and first order beliefs consistent with the common prior derived from a variation of the original no trade theorem.

But Arieli et al. (2020) are interested in distributions over first order beliefs only (not the joint distribution over first order beliefs and the state). This will require a new step.

5 Feasible Joint Posterior Beliefs

Arieli et al. (2020) characterize distributions on first order beliefs only. This requires an extra application of a separation argument beyond the no trade theorem. Let $F_1, ..., F_N$ be arbitrary finite collections of posteriors on Θ. Let $\psi \in \Delta(F)$ be a distribution over those posteriors.

Definition 14 *Distribution $\psi \in \Delta(F)$ on first order beliefs alone is consistent with the common prior assumption if there exists $\pi_0^* \in \Delta(F \times \Theta)$ - consistent with the common prior assumption (see definition 11) - whose marginal on F is $\psi \in \Delta(F)$.*

Theorem 3 *Distribution $\psi \in \Delta(F)$ is consistent with the common prior assumption if and only if, for every Θ-measurable, zero value trade $(x_h)_{h=1}^H$, we have*

$$\sum_{f \in F} \psi(f) \left(\max_{\theta} \sum_{h=1}^{H} x_h(f_h, \theta) \right) \geq 0$$

This is a many state characterization of feasible joint distributions on first order beliefs (generalizing the two state version of Arieli at al (2020)). The characterization entails a kind of money pump condition, with a max operator thrown in. Arieli et al (2020) give an interpretation in the two state case.

Proof. For $\psi \in \Delta(F)$ and $\xi : F \to \Delta(\Theta)$, write $\psi \circ \xi \in \Delta(F \times \Theta)$ for induced distribution over $F \times \Theta$, so that $\psi \circ \xi(f, \theta) = \psi(f) \xi(\theta|f)$. Recall that $\psi \in \Delta(F)$ is consistent with the common prior assumption if and only if there there exists $\xi : F \to \Delta(\Theta)$ such that $\psi \circ \xi \in \Delta(F \times \Theta)$ is consistent with the common prior assumption, and so (by Theorem 2), every Θ-measurable, zero value trade $(x_h)_{h=1}^H$, satisfies

$$\sum_{f \in F} \psi(f) \sum_{\theta \in \Theta} \xi(\theta|f) \sum_{h=1}^{H} x_h(f_h, \theta) \geq 0$$

The latter requirement can be written as:

$$\max_{\xi: F \to \Delta(\Theta)} \min_{\text{zero value trade } (x_h)_{h=1}^H} \sum_{f \in F} \psi(f) \sum_{\theta \in \Theta} \xi(\theta|f) \sum_{h=1}^{H} x_h(f_h, \theta) \geq 0$$

By the minmax theorem, this is equivalent to requiring:

$$\min_{\text{zero value trade } (x_h)_{h=1}^H} \max_{\xi: F \to \Delta(\Theta)} \sum_{f \in F} \psi(f) \sum_{\theta \in \Theta} \xi(\theta|f) \sum_{h=1}^{H} x_h(f_h, \theta) \geq 0 \tag{1}$$

But

$$\begin{aligned}
&\max_{\xi:F\to\Delta(\Theta)} \sum_{f\in F} \psi(f) \sum_{\theta\in\Theta} \xi(\theta|f) \sum_{h=1}^{H} x_h(f_h,\theta) \\
= &\sum_{f\in F} \psi(f) \max_{\xi\in\Delta(\Theta)} \sum_{\theta\in\Theta} \xi(\theta) \sum_{h=1}^{H} x_h(f_h,\theta) \\
= &\sum_{f\in F} \psi(f) \max_{\theta} \sum_{h=1}^{H} x_h(f_h,\theta) \qquad (2)
\end{aligned}$$

Substituting back (2) back into (1), we obtain

$$\min_{\text{zero value trade } (x_h)_{h=1}^{H}} \sum_{f\in F} \psi(f) \left(\max_{\theta} \sum_{h=1}^{H} x_h(f_h,\theta) \right) \geq 0 \qquad (3)$$

■

5.1 Two States and Theorem 3 of Arieli et al (2020)

Now suppose that $\Theta = \{0,1\}$. Let us identify $f \in \Delta[\{0,1\}]$ with the probability of state 1. Now the two state restriction implies that a Θ-measurable zero value trade must take the form

$$x_h(f,\theta) = \begin{cases} (1-f)\, a_h(f), \text{ if } \theta = 1 \\ -f a_h(f), \text{ if } \theta = 0 \end{cases}$$

for some $(a_h)_{h=1}^{H}$, where $a_h : F_h \to \mathbb{R}$.

Using this representation, observe that

$$\begin{aligned}
\max_{\theta} \sum_{h=1}^{H} x_h(f_h,\theta) &= \max\left(\sum_{h=1}^{H} (1-f_h)\, a_h(f_h), \sum_{h=1}^{H} f_h a_h(f_h) \right) \\
&= \max\left(0, \sum_{h=1}^{H} a_h(f_h) \right) - \sum_{h=1}^{H} f_h a_h(\rho_h)
\end{aligned}$$

So we have:

Corollary 3 *In the two state case, distribution $\psi \in \Delta(F)$ is consistent with the common prior assumption if and only if, for every $(a_h)_{h=1}^{H}$, where $a_h : F_h \to \mathbb{R}$, we have*

$$\sum_{f\in F} \psi(f) \left(\max\left(0, \sum_{h=1}^{H} a_h(f_h) \right) - \sum_{h=1}^{H} f_h a_h(f_h) \right) \geq 0$$

This is Theorem 3 of Arieli et al (2020).

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